Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 2689–2699
Research Article

ISSN : 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Inclusion relationships for certain subclasses of analytic functions involving linear operator

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Communicated by Y.-Z. Chen

Abstract

Based on a linear operator, some new subclasses of analytic and univalent functions are introduced. The object of the present paper is to derive inclusion relationships for these classes. Some applications of the inclusion results are also obtained.

Keywords: Analytic functions, univalent functions, subordination, Hadamard product, linear operator.
2010 MSC: 30C45, 30C50.

1. Introduction

In this paper, we denote $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1},$$

which are analytic and univalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For functions f and g given by

$$g(z) = z + \sum_{k=1}^{\infty} b_{k+1} z^{k+1},$$

analytic in $\mathbb{U}$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=1}^{\infty} a_{k+1} b_{k+1} z^{k+1}.$$

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doi:10.22436/jnsa.010.05.35

Received 2017-03-06

We say f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{U}$ with $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{U}$, then we have $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Denote by $\mathcal{P}$ the class of all positive real part functions $p(z)$, which satisfy the conditions $\Re\{p(z)\} > 0$ $(z \in \mathbb{U})$ and $p(0) = 1$. Denote by $\mathcal{Q}$ the class of functions $\phi(z) \in \mathcal{P}$ such that $\phi(\mathbb{U})$ is convex and symmetrical with respect to the real axis. Let $\phi \in \mathcal{Q}$, Ma et al. [5] introduced the following subclasses $\mathcal{S}^*(\phi), \mathcal{K}(\phi)$ and $\mathcal{C}(\phi, \psi)$ defined by

$$\mathcal{S}^*(\phi) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \phi(z), z \in \mathbb{U} \right\},$$
$$\mathcal{K}(\phi) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in \mathbb{U} \right\},$$

and

$$\mathcal{C}(\phi, \psi) = \left\{ f : f \in \mathcal{A}, g(z) \in \mathcal{S}^*(\phi) \quad \text{and} \quad \tfrac{zf'(z)}{g(z)} \prec \phi(z), z \in \mathbb{U} \right\}.$$

For details, one can refer literatures [3, 10]. For $a \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$ where $\mathbb{Z}_0^- = \{\ldots, -2, -1, 0\}$, Saitoh [9] introduced a linear operator $\mathcal{L}(a, c) : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{L}(a, c)f(z) = \varphi(a, c; z) * f(z), \quad (f \in \mathcal{A},\ z \in \mathbb{U}),$$

where $\varphi(a, c; z)$ is the incomplete beta function defined by

$$\varphi(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad (z \in \mathbb{U}), \tag{1.1}$$

where $(a)_k = a(a+1)\cdots(a+k-1), k \in \mathbb{N}, a \in \mathbb{C}$. The operator $\mathcal{L}(a, c)$ is an extension of the Carlson-Shaffer operator [1]. Recently, Cho et al. [2] introduced a family of linear operators $\mathcal{J}^\lambda(a, c) : \mathcal{A} \to \mathcal{A}$ as follows

$$\mathcal{J}^\lambda(a, c)f(z) = \varphi^{(+)}(a, c; z) * f(z), \quad (a, c \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ z \in \mathbb{U}), \tag{1.2}$$

where $\varphi^{(+)}(a, c; z)$ is the function defined in terms of the Hadamard product by the following relation

$$\varphi(a, c; z) * \varphi^{(+)}(a, c; z) = \frac{z}{(1-z)^{\lambda+1}}, \quad (\lambda > -1,\ z \in \mathbb{U}), \tag{1.3}$$

where $\varphi(a, c; z)$ is given by (1.1). We can obtain from (1.1), (1.2), (1.3) that

$$\varphi^{(+)}(a, c; z) = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k (c)_k}{(1)_k (a)_k} a_{k+1} z^{k+1}, \quad (z \in \mathbb{U}),$$

and

$$\mathcal{J}^\lambda(a, c) = z + \sum_{k=1}^{\infty} \frac{(\lambda+1)_k (c)_k}{(1)_k (a)_k} a_{k+1} z^{k+1}, \quad (z \in \mathbb{U}). \tag{1.4}$$

In this paper, we define a new linear operator $\mathcal{N}_\xi^\lambda(a, c) : \mathcal{A} \to \mathcal{A}$ as follows

$$\mathcal{N}_\xi^\lambda(a, c)f(z) = (1-\xi)\mathcal{J}^\lambda(a, c)f(z) + \xi z\big(\mathcal{J}^\lambda(a, c)f(z)\big)', \quad (z \in \mathbb{U}). \tag{1.5}$$

From (1.4) and (1.5), we conclude that

$$\mathcal{N}_\xi^\lambda(a, c)f(z) = z + \sum_{k=1}^{\infty} \frac{(1+\xi k)(\lambda+1)_k (c)_k}{(1)_k (a)_k} a_{k+1} z^{k+1}, \quad (z \in \mathbb{U}),$$

and some identities

$$z\big(\mathcal{N}_\xi^\lambda(a,c)f(z)\big)' = (\lambda+1)\mathcal{N}_\xi^{\lambda+1}(a,c)f(z) - \lambda\mathcal{N}_\xi^\lambda(a,c)f(z), \tag{1.6}$$
$$z\big(\mathcal{N}_\xi^\lambda(a+1,c)f(z)\big)' = a\mathcal{N}_\xi^\lambda(a,c)f(z) + (1-a)\mathcal{N}_\xi^\lambda(a+1,c)f(z).$$

We observe that

(1) $\mathcal{N}_0^1(2,1)f(z) = f(z)$;

(2) $\mathcal{N}_0^1(1,1)f(z) = zf'(z)$;

(3) $\mathcal{N}_0^n(a,a)f(z) = \mathfrak{D}_n f(z)$ $(n > -1)$, see [4];

(4) $\mathcal{N}_0^{1-\mu}(1-\mu,2)f(z) = \Omega_z^\mu f(z)$ $(\mu < 2)$, see [8];

(5) $\mathcal{N}_0^\delta(\delta+2,1)f(z) = \mathfrak{F}_\delta f(z)$ $(\delta > -1)$, see [3].

With the operator $\mathcal{N}_\xi^\lambda(a,c)$, we introduce some function classes for some η $(0 \leqslant \eta < 1)$, $\gamma(\geqslant 0)$ and for some $\phi, \psi \in \mathcal{Q}$ as follows

$$\mathcal{S}_{\lambda,\xi}^{a,c}(\eta;\phi) = \left\{ f : f \in \mathcal{A},\ \frac{1}{1-\eta}\left(\frac{z\big(\mathcal{N}_\xi^\lambda(a,c)f(z)\big)'}{\mathcal{N}_\xi^\lambda(a,c)f(z)} - \eta\right) \prec \phi(z) \right\}, \tag{1.7}$$

$$\mathcal{C}_{\lambda,\xi}^{a,c}(\eta;\phi,\psi) = \left\{ f : f \in \mathcal{A},\ \mathcal{N}_\xi^\lambda(a,c)g(z) \in \mathcal{S}^*(\psi), \frac{1}{1-\eta}\left(\frac{z\big(\mathcal{N}_\xi^\lambda(a,c)f(z)\big)'}{\mathcal{N}_\xi^\lambda(a,c)g(z)} - \eta\right) \prec \phi(z) \right\}, \tag{1.8}$$

and

$$\begin{aligned}\mathcal{R}_{\lambda,\xi}^{a,c}(\eta,\gamma;\phi,\psi) = &\Big\{ f : f \in \mathcal{A},\ \mathcal{N}_\xi^\lambda(a,c)g(z) \in \mathcal{S}^*(\psi),\\ &\frac{1}{1-\eta}\left((1-\gamma)\frac{\mathcal{N}_\xi^\lambda(a,c)f(z)}{\mathcal{N}_\xi^\lambda(a,c)g(z)} + \gamma\frac{\big(\mathcal{N}_\xi^\lambda(a,c)f(z)\big)'}{\big(\mathcal{N}_\xi^\lambda(a,c)g(z)\big)'} - \eta\right) \prec \phi(z) \Big\}.\end{aligned} \tag{1.9}$$

In particular, for $\eta = 0$ in (1.7), we denote $\mathcal{S}_{\lambda,\xi}^{a,c}(0;\phi) = \mathcal{S}_{\lambda,\xi}^{a,c}(\phi)$. Based on differential subordination properties, we derive some inclusion relationships of above classes.

In order to derive our main results, we shall need the following lemmas.

Lemma 1.1 ([6])**.** *Let the function* $h(z)$ *be convex in* $\mathbb{U}$ *with* $\mathfrak{R}\{\beta h(z)+\gamma\} > 0$. *If the function* $p(z)$ *is analytic in* $\mathbb{U}$ *with* $p(0) = h(0) = 1$, *then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma 1.2 ([6])**.** *Let the function* $h(z)$ *be convex in* $\mathbb{U}$ *and let* $P : \mathbb{U} \to \mathbb{C}$ *with* $\mathfrak{R}\{P(z)\} > 0$ $(z \in \mathbb{U})$. *If the function* $p(z)$ *is analytic in* $\mathbb{U}$ *with* $p(0) = h(0) = 1$, *then*

$$p(z) + P(z) \cdot zp'(z) \prec h(z) \Rightarrow p(z) \prec h(z).$$

2. Main results

In this section, we state and prove our general results involving the function classes given by Section 1.

Theorem 2.1. *Let* $\lambda \geqslant 0$, $\mathfrak{R}\{a\} > 1-\eta$, $\mathcal{N}_\xi^\lambda(a,c)f(z) \neq 0$ $(z \in \mathbb{U} \setminus \{0\})$. *Then*

$$\mathcal{S}_{\lambda+1,\xi}^{a,c}(\eta;\phi) \subset \mathcal{S}_{\lambda,\xi}^{a,c}(\eta;\phi) \subset \mathcal{S}_{\lambda,\xi}^{a+1,c}(\eta;\phi). \tag{2.1}$$

Proof. First of all, let us suppose $f(z) \in \mathcal{S}^{a,c}_{\lambda+1,\xi}(\eta;\phi)$. Then from definition of this class, we have

$$\frac{1}{1-\eta}\left(\frac{z\left(\mathcal{N}^{\lambda+1}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda+1}_{\xi}(a,c)f(z)}-\eta\right) \prec \phi(z). \tag{2.2}$$

Denote

$$p(z) = \frac{1}{1-\eta}\left(\frac{z\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}-\eta\right). \tag{2.3}$$

We can easily check that $p(z)$ is univalent in $\mathbb{U}$ with $p(0) = 1$. From (1.6) and (2.3), we give another identities that

$$[(1-\eta)p(z)+\eta]\mathcal{N}^{\lambda}_{\xi}(a,c)f(z) = z\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)',$$

and

$$\begin{aligned}(1-\eta)p(z)+\eta+\lambda &= \frac{z\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}+\lambda = \frac{z\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)'+\lambda\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}\\ &= \frac{(\lambda+1)\mathcal{N}^{\lambda+1}_{\xi}(a,c)f(z)}{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}.\end{aligned} \tag{2.4}$$

Taking logarithmic and differentiating both sides of (2.4) with respect to z, we have

$$\frac{(1-\eta)p'(z)}{(1-\eta)p(z)+\eta+\lambda} = \frac{\left(\mathcal{N}^{\lambda+1}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda+1}_{\xi}(a,c)f(z)} - \frac{\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}. \tag{2.5}$$

Then from (2.3) and (2.5), we get

$$\frac{z\left(\mathcal{N}^{\lambda+1}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda+1}_{\xi}(a,c)f(z)} = (1-\eta)p(z)+\eta+\frac{(1-\eta)zp'(z)}{(1-\eta)p(z)+\eta+\lambda}.$$

Hence, we have

$$\frac{1}{1-\eta}\left(\frac{z\left(\mathcal{N}^{\lambda+1}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda+1}_{\xi}(a,c)f(z)}-\eta\right) = p(z)+\frac{zp'(z)}{(1-\eta)p(z)+\eta+\lambda}. \tag{2.6}$$

Thus from (2.2), (2.6), we obtain that

$$p(z)+\frac{zp'(z)}{(1-\eta)p(z)+\eta+\lambda} \prec \phi(z). \tag{2.7}$$

Because $\phi(z) \in \Omega$ is positive real part function, and

$$\Re\left\{(1-\eta)\phi(z)+\eta+\lambda\right\} > \eta+\lambda > 0,$$

from Lemma 1.1 and (2.7), it follows that $p(z) \prec \phi(z)$, that is, $f(z) \in \mathcal{S}^{a,c}_{\lambda,\xi}(\eta;\phi)$.

Next, we will show that $\mathcal{S}^{a,c}_{\lambda,\xi}(\eta;\phi) \subset \mathcal{S}^{a+1,c}_{\lambda,\xi}(\eta;\phi)$. Let $f(z) \in \mathcal{S}^{a,c}_{\lambda,\xi}(\eta;\phi)$, then from (1.7) we have

$$\frac{1}{1-\eta}\left(\frac{z\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}-\eta\right) \prec \phi(z). \tag{2.8}$$

Denote

$$q(z) = \frac{1}{1-\eta}\left(\frac{z\left(\mathcal{N}^{\lambda}_{\xi}(a+1,c)f(z)\right)'}{\mathcal{N}^{\lambda}_{\xi}(a+1,c)f(z)}-\eta\right). \tag{2.9}$$

It is easy to check that $q(z)$ is univalent in $\mathbb{U}$ with $q(0) = 1$. From (2.9), we give another identities that

$$[(1-\eta)q(z)+\eta]\mathcal{N}_\xi^\lambda(a+1,c)f(z) = z\big(\mathcal{N}_\xi^\lambda(a+1,c)f(z)\big)',$$

and

$$\begin{aligned}(1-\eta)q(z)+\eta+a-1 &= \frac{z\big(\mathcal{N}_\xi^\lambda(a+1,c)f(z)\big)'}{\mathcal{N}_\xi^\lambda(a+1,c)f(z)}+a-1\\ &= \frac{z\big(\mathcal{N}_\xi^\lambda(a+1,c)f(z)\big)'+(a-1)\mathcal{N}_\xi^\lambda(a+1,c)f(z)}{\mathcal{N}_\xi^\lambda(a+1,c)f(z)}\\ &= \frac{a\mathcal{N}_\xi^\lambda(a,c)f(z)}{\mathcal{N}_\xi^\lambda(a+1,c)f(z)}.\end{aligned} \tag{2.10}$$

Taking logarithmic and differentiating both sides of (2.10) with respect to z, we have

$$\frac{(1-\eta)q'(z)}{(1-\eta)q(z)+\eta+a-1} = \frac{\big(\mathcal{N}_\xi^\lambda(a,c)f(z)\big)'}{\mathcal{N}_\xi^\lambda(a,c)f(z)} - \frac{\big(\mathcal{N}_\xi^\lambda(a+1,c)f(z)\big)'}{\mathcal{N}_\xi^\lambda(a+1,c)f(z)}. \tag{2.11}$$

Then from (2.9) and (2.11), we get

$$\frac{z\big(\mathcal{N}_\xi^\lambda(a,c)f(z)\big)'}{\mathcal{N}_\xi^\lambda(a,c)f(z)} = (1-\eta)q(z)+\eta+\frac{(1-\eta)zq'(z)}{(1-\eta)q(z)+\eta+a-1}.$$

Hence, it follows that

$$\frac{1}{1-\eta}\left(\frac{z\big(\mathcal{N}_\xi^\lambda(a,c)f(z)\big)'}{\mathcal{N}_\xi^\lambda(a,c)f(z)}-\eta\right) = q(z)+\frac{zq'(z)}{(1-\eta)q(z)+\eta+a-1}. \tag{2.12}$$

Thus from (2.8), (2.12) we obtain that

$$q(z)+\frac{zq'(z)}{(1-\eta)q(z)+\eta+a-1} \prec \phi(z). \tag{2.13}$$

Because of $\phi(z) \in \mathcal{Q}$ and

$$\Re\Big\{(1-\eta)\phi(z)+\eta+a-1\Big\} > \Re\{\eta+a-1\} > 0,$$

from Lemma 1.1 and (2.13), it follows that $f(z) \in \mathcal{S}_{\lambda,\xi}^{a+1,c}(\eta;\phi)$. Therefore, the theorem is proved. □

If $\eta = 0$, in view of (1.7) and (2.1), we get the following inclusion relation

$$\mathcal{S}_{\lambda+1,\xi}^{a,c}(\phi) \subset \mathcal{S}_{\lambda,\xi}^{a,c}(\phi). \tag{2.14}$$

Theorem 2.2. *Let* $\lambda \geqslant 0$, $\mathcal{N}_\xi^\lambda(a,c)g(z) \neq 0$ $(z \in \mathbb{U}\setminus\{0\})$. *Then*

$$\mathcal{C}_{\lambda+1,\xi}^{a,c}(\eta;\phi,\psi) \subset \mathcal{C}_{\lambda,\xi}^{a,c}(\eta;\phi,\psi) \subset \mathcal{C}_{\lambda,\xi}^{a+1,c}(\eta;\phi,\psi).$$

Proof. Suppose $f(z) \in \mathcal{C}_{\lambda+1,\xi}^{a,c}(\eta;\phi,\psi)$. Then from (1.8), the subordination is satisfied

$$\frac{1}{1-\eta}\left(\frac{z\big(\mathcal{N}_\xi^{\lambda+1}(a,c)f(z)\big)'}{\mathcal{N}_\xi^{\lambda+1}(a,c)g(z)}-\eta\right) \prec \phi(z), \tag{2.15}$$

where $\mathcal{N}_{\xi}^{\lambda+1}(a,c)g(z) \in \mathcal{S}^*(\psi)$. In view of (2.14), we get

$$\frac{z\big(\mathcal{N}_{\xi}^{\lambda+1}(a,c)g(z)\big)'}{\mathcal{N}_{\xi}^{\lambda+1}(a,c)g(z)} \prec \psi(z) \Rightarrow \frac{z\big(\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)\big)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)} \prec \psi(z), \tag{2.16}$$

i.e.,

$$\mathcal{N}_{\xi}^{\lambda+1}(a,c)g(z) \prec \mathcal{S}^*(\psi) \Rightarrow \mathcal{N}_{\xi}^{\lambda}(a,c)g(z) \prec \mathcal{S}^*(\psi).$$

Denote

$$u(z) = \frac{1}{1-\eta}\left(\frac{z\big(\mathcal{N}_{\xi}^{\lambda}(a,c)f(z)\big)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)} - \eta\right). \tag{2.17}$$

We can easily check that $u(z)$ is univalent in $\mathbb{U}$ with $u(0) = 1$. From (2.17), we give another identity that

$$[(1-\eta)u(z)+\eta]\mathcal{N}_{\xi}^{\lambda}(a,c)g(z) = z\big(\mathcal{N}_{\xi}^{\lambda}(a,c)f(z)\big)'. \tag{2.18}$$

Using the formula

$$z\Big(\mathcal{N}_{\xi}^{\lambda}(a,c)f(z)\Big)' = \mathcal{N}_{\xi}^{\lambda}(a,c)(zf'(z)),$$

and differentiating both sides of (2.18) with respect to z, we get

$$(1-\eta)u'(z)\mathcal{N}_{\xi}^{\lambda}(a,c)g(z) + [(1-\eta)u(z)+\eta]\big(\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)\big)' = \big(\mathcal{N}_{\xi}^{\lambda}(a,c)(zf'(z))\big)',$$

which yields that

$$(1-\eta)zu'(z) + [(1-\eta)u(z)+\eta]\frac{z\big(\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)\big)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)} = \frac{z\big(\mathcal{N}_{\xi}^{\lambda}(a,c)(zf'(z))\big)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)}. \tag{2.19}$$

If we apply (1.6), (2.17), (2.19), then

$$\begin{aligned}
\frac{z\big(\mathcal{N}_{\xi}^{\lambda+1}(a,c)f(z)\big)'}{\mathcal{N}_{\xi}^{\lambda+1}(a,c)g(z)} &= \frac{(\lambda+1)\mathcal{N}_{\xi}^{\lambda+1}(a,c)(zf'(z))}{(\lambda+1)\mathcal{N}_{\xi}^{\lambda+1}(a,c)g(z)} \\
&= \frac{z\big(\mathcal{N}_{\xi}^{\lambda}(a,c)(zf'(z))\big)' + \lambda\mathcal{N}_{\xi}^{\lambda}(a,c)(zf'(z))}{z\big(\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)\big)' + \lambda\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)} \\
&= \frac{\frac{z\left(\mathcal{N}_{\xi}^{\lambda}(a,c)(zf'(z))\right)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)} + \frac{\lambda\mathcal{N}_{\xi}^{\lambda}(a,c)(zf'(z))}{\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)}}{\frac{z\left(\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)\right)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)} + \lambda} \\
&= (1-\eta)\frac{zu'(z)}{\frac{z\left(\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)\right)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)} + \lambda} + (1-\eta)u(z) + \eta.
\end{aligned}$$

Hence

$$\frac{1}{1-\eta}\left(\frac{z\big(\mathcal{N}_{\xi}^{\lambda+1}(a,c)f(z)\big)'}{\mathcal{N}_{\xi}^{\lambda+1}(a,c)g(z)} - \eta\right) = u(z) + \frac{zu'(z)}{Q(z)+\lambda}, \tag{2.20}$$

where

$$Q(z) = \frac{z\big(\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)\big)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)g(z)} \prec \psi(z), \quad (\psi(z) \in \mathcal{Q}),$$

from (2.16). We know that

$$\Re\left\{\frac{1}{Q(z)+\lambda}\right\} = \Re\left\{\frac{\overline{Q(z)+\lambda}}{(Q(z)+\lambda)(\overline{Q(z)+\lambda})}\right\}$$

$$= \frac{1}{|Q(z)+\lambda|^2}\Re\{Q(z)+\lambda\} > \frac{\lambda}{|Q(z)+\lambda|^2} \geqslant 0.$$

Applying Lemma 1.2 and (2.15), (2.20), it follows that $u(z) \prec \phi(z)$, that is, $f(z) \in \mathcal{C}^{a,c}_{\lambda,\xi}(\eta;\phi,\psi)$.

The second part of the theorem is similar to Theorem 2.1. We omit it. □

Theorem 2.3. *Let* $\gamma \geqslant 0$, $\mathcal{N}^{\lambda}_{\xi}(a,c)g(z) \neq 0 \ \ (z \in \mathbb{U}\setminus\{0\})$. *Then*

$$\mathcal{R}^{a,c}_{\lambda,\xi}(\eta,\gamma;\phi,\psi) \subset \mathcal{R}^{a,c}_{\lambda,\xi}(\eta,0;\phi,\psi).$$

Proof. If $\gamma = 0$, the result is obvious. Now let us consider the case of $\gamma > 0$. Let $f(z) \in \mathcal{R}^{a,c}_{\lambda,\xi}(\eta,\gamma;\phi,\psi)$. Then from (1.9), we have

$$\frac{1}{1-\eta}\Big((1-\gamma)\frac{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}{\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)} + \gamma\frac{\big(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\big)'}{\big(\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)\big)'} - \eta\Big) \prec \phi(z), \tag{2.21}$$

where $\mathcal{N}^{\lambda}_{\xi}(a,c)g(z) \in \mathcal{S}^*(\psi)$, i.e.,

$$\frac{z\big(\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)\big)'}{\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)} \prec \psi(z), \quad (\psi(z) \in \mathcal{Q}).$$

Denote

$$s(z) = \frac{1}{1-\eta}\Big(\frac{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}{\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)} - \eta\Big). \tag{2.22}$$

The function $s(z)$ is univalent in $\mathbb{U}$ with $u(0) = 1$. From (2.22), we get

$$[(1-\eta)s(z)+\eta]\mathcal{N}^{\lambda}_{\xi}(a,c)g(z) = \mathcal{N}^{\lambda}_{\xi}(a,c)f(z). \tag{2.23}$$

Differentiating both sides of (2.23) with respect to z, we get

$$(1-\eta)s'(z)\mathcal{N}^{\lambda}_{\xi}(a,c)g(z) + [(1-\eta)s(z)+\eta]\big(\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)\big)' = \big(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\big)',$$

which yields that

$$(1-\eta)s(z)+\eta + \frac{(1-\eta)s'(z)\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)}{\big(\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)\big)'} = \frac{\big(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\big)'}{\big(\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)\big)'}. \tag{2.24}$$

If we apply (2.23), (2.24), we find that

$$(1-\gamma)\frac{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}{\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)} + \gamma\frac{\big(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\big)'}{\big(\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)\big)'} = (1-\eta)s(z)+\eta+(1-\eta)\gamma\frac{zs'(z)}{P(z)}, \tag{2.25}$$

where

$$P(z) = \frac{z\big(\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)\big)'}{\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)} \prec \psi(z),$$

and

$$\Re\{P(z)\} > 0.$$

Hence, from (2.21) and (2.25) we have

$$s(z) + \gamma\frac{zs'(z)}{P(z)} \prec \phi(z). \tag{2.26}$$

We know that

$$\Re\left\{\frac{\gamma}{P(z)}\right\} = \Re\left\{\frac{\gamma\overline{P(z)}}{P(z)\overline{P(z)}}\right\} = \frac{\gamma}{|P(z)|^2}\Re\{P(z)\} > 0.$$

Applying Lemma 1.2 and (2.26), it follows that $s(z) \prec \phi(z)$, i.e., $f(z) \in \mathcal{R}^{a,c}_{\lambda,\xi}(\eta,0;\phi,\psi)$. □

3. Applications based on above classes

In this section, we define another subclasses of above classes introduced in Section 1, and apply Nunokawa's Lemma to derive further inclusion relationships.

For $0 < \alpha \leqslant 1$, and $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ in the classes $\mathcal{S}^{a,c}_{\lambda,\xi}(\eta;\phi)$, $\mathcal{C}^{a,c}_{\lambda,\xi}(\eta;\phi,\psi)$, $\mathcal{R}^{a,c}_{\lambda,\xi}(\eta,\gamma;\phi,\psi)$, we denote $\mathcal{SS}^{a,c}_{\lambda,\xi}(\eta,\alpha)$, $\mathcal{SC}^{a,c}_{\lambda,\xi}(\eta,\alpha,\psi)$, $\mathcal{SR}^{a,c}_{\lambda,\xi}(\eta,\gamma,\alpha,\psi)$ defined by the followings

$$\mathcal{SS}^{a,c}_{\lambda,\xi}(\eta,\alpha) = \left\{ f : f \in \mathcal{A},\ \ \left| \arg\left(\frac{z\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)} - \eta \right) \right| < \frac{\pi\alpha}{2} \right\}, \tag{3.1}$$

$$\mathcal{SC}^{a,c}_{\lambda;\xi}(\eta,\alpha,\psi) = \left\{ f : f \in \mathcal{A},\ \ \mathcal{N}^{\lambda}_{\xi}(a,c)g(z) \in \mathcal{S}^*(\psi), \left| \arg\left(\frac{z\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)} - \eta \right) \right| < \frac{\pi\alpha}{2} \right\},$$

and

$$\begin{aligned}\mathcal{SR}^{a,c}_{\lambda;\xi}(\eta,\gamma,\alpha,\psi) = &\Big\{ f : f \in \mathcal{A},\ \ \mathcal{N}^{\lambda}_{\xi}(a,c)g(z) \in \mathcal{S}^*(\psi),\\ &\left| \arg\left((1-\gamma)\frac{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)}{\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)} + \gamma\frac{\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)'}{\left(\mathcal{N}^{\lambda}_{\xi}(a,c)g(z)\right)'} - \eta \right) \right| < \frac{\pi\alpha}{2} \Big\}.\end{aligned}$$

Lemma 3.1 ([7, Nunokawa's Lemma]). *Let the function* $p(z)$ *given by*

$$p(z) = 1 + \sum_{k=m}^{\infty} p_k z^k, \quad (p_m \neq 0),$$

be analytic in $\mathbb{U}$ *with* $p(z) \neq 0\ \ (z \in \mathbb{U})$. *If there exists a point* z_0 $(|z_0| < 1)$ *such that*

$$|\arg\{p(z)\}| < \frac{\pi\alpha}{2}, \quad (|z| < |z_0|),$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\alpha}{2},$$

for some $\alpha > 0$, *then*

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2il\arg\{p(z_0)\}}{\pi},$$

for some

$$l \geqslant \frac{m(a + a^{-1})}{2} > m,$$

where $[p(z_0)]^{1/\alpha} = \pm ia \quad (a > 0)$.

Theorem 3.2. *Let* $\lambda \geqslant 0$, $\Re\{a\} > 1 - \eta$, $\mathcal{N}^{\lambda}_{\xi}(a,c)f(z) \neq 0\ \ (z \in \mathbb{U}\setminus\{0\})$. *Then*

$$\mathcal{SS}^{a,c}_{\lambda+1,\xi}(\eta,\sigma) \subset \mathcal{SS}^{a,c}_{\lambda,\xi}(\eta,\alpha),$$

where

$$\sigma = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{m\alpha\cos(\frac{\pi\delta}{2})}{r + m\alpha\sin(\frac{\pi\delta}{2})} \right), \tag{3.2}$$

for some $r > 0, m \in \mathbb{N}$ *and* $0 \leqslant \delta < \alpha$.

Proof. Suppose $f(z) \in \mathcal{SS}^{a,c}_{\lambda+1,\xi}(\eta,\sigma)$. Then from (3.1), we have

$$\left| \arg\left(\frac{z\left(\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)\right)'}{\mathcal{N}^{\lambda}_{\xi}(a,c)f(z)} - \eta \right) \right| < \frac{\pi\sigma}{2}, \quad (0 \leqslant \eta < 1,\ 0 < \sigma \leqslant 1). \tag{3.3}$$

It follows that there exists a function $\phi(z)$ given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\sigma} \in \mathcal{Q},$$

such that

$$\frac{1}{1-\eta}\left(\frac{z\left(\mathcal{N}_{\xi}^{\lambda+1}(a,c)f(z)\right)'}{\mathcal{N}_{\xi}^{\lambda+1}(a,c)f(z)} - \eta\right) \prec \phi(z). \tag{3.4}$$

Let function $p(z)$ defined by

$$p(z) = \frac{1}{1-\eta}\left(\frac{z\left(\mathcal{N}_{\xi}^{\lambda}(a,c)f(z)\right)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)f(z)} - \eta\right). \tag{3.5}$$

Based on analysis in Theorem 2.1 and also from (3.4), (3.5), we know that $p(z) \prec \phi(z)$. Because $\phi(z) \in \mathcal{Q}$, we have $p(z) \neq 0$. Therefore, the argument $\arg\{p(z)\}$ is well-defined.

Now, let us prove Theorem 3.2 by contradiction. Suppose there exists a point $z_0 (|z_0| < 1)$ such that

$$|\arg\{p(z)\}| < \frac{\pi\alpha}{2}, \quad (|z| < |z_0|),$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\alpha}{2},$$

for some $\alpha > 0$, then applying Lemma 3.1 we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2il \arg\{p(z_0)\}}{\pi}, \tag{3.6}$$

for some

$$l \geqslant \frac{m(a + a^{-1})}{2} \geqslant m,$$

where $[p(z_0)]^{1/\alpha} = \pm ia \quad (a > 0)$. From (2.6), we get

$$\begin{aligned}
\arg\left(\frac{z\left(\mathcal{N}_{\xi}^{\lambda+1}(a,c)f(z_0)\right)'}{\mathcal{N}_{\xi}^{\lambda+1}(a,c)f(z_0)} - \eta\right) &= \arg\left(p(z_0) + \frac{z_0 p'(z_0)}{(1-\eta)p(z_0) + \eta + \lambda}\right) \\
&= \arg\left(p(z_0)\left(1 + \frac{z_0 p'(z_0)}{p(z_0)[(1-\eta)p(z_0) + \eta + \lambda]}\right)\right) \\
&= \arg\{p(z_0)\} + \arg\left(1 + \frac{z_0 p'(z_0)}{p(z_0)[(1-\eta)p(z_0) + \eta + \lambda]}\right).
\end{aligned}$$

We denote

$$(1-\eta)p(z_0) + \eta + \lambda = \frac{z\left(\mathcal{N}_{\xi}^{\lambda}(a,c)f(z_0)\right)'}{\mathcal{N}_{\xi}^{\lambda}(a,c)f(z_0)} + \lambda \doteq re^{\pm i\frac{\pi\delta}{2}},$$

for some $r > 0$ and $0 \leqslant \delta < \alpha$.

In case of $\arg\{p(z_0)\} = \frac{\pi\alpha}{2}$, from (3.6) we have

$$\begin{aligned}
\arg\left(\frac{z\left(\mathcal{N}_{\xi}^{\lambda+1}(a,c)f(z_0)\right)'}{\mathcal{N}_{\xi}^{\lambda+1}(a,c)f(z_0)} - \eta\right) &= \frac{\pi\alpha}{2} + \arg\left(1 + \frac{il\alpha}{r}e^{\mp i\frac{\pi\delta}{2}}\right) \\
&\geqslant \frac{\pi\alpha}{2} + \tan^{-1}\left(\frac{l\alpha\cos(\frac{\pi\delta}{2})}{r + l\alpha\sin(\frac{\pi\alpha}{2})}\right).
\end{aligned}$$

It is obvious that the function $G(l)$ given by

$$G(l) = \frac{l\alpha\cos(\frac{\pi\delta}{2})}{r + l\alpha\sin(\frac{\pi\alpha}{2})},$$

is increasing. Hence, applying (3.2), we get

$$\begin{aligned}\arg p(z_0) &= \arg\Big(\frac{z\big(\mathcal{N}_\xi^{\lambda+1}(a,c)f(z_0)\big)'}{\mathcal{N}_\xi^{\lambda+1}(a,c)f(z_0)} - \eta\Big)\\ &\geqslant \frac{\pi\alpha}{2} + \tan^{-1}\Big(\frac{m\alpha\cos(\frac{\pi\delta}{2})}{r + m\alpha\sin(\frac{\pi\alpha}{2})}\Big) = \frac{\pi\sigma}{2}.\end{aligned}$$

In case of $\arg\{p(z_0)\} = -\frac{\pi\alpha}{2}$, with the same method we have

$$\begin{aligned}\arg p(z_0) &= \arg\Big(\frac{z\big(\mathcal{N}_\xi^{\lambda+1}(a,c)f(z_0)\big)'}{\mathcal{N}_\xi^{\lambda+1}(a,c)f(z_0)} - \eta\Big)\\ &= -\frac{\pi\alpha}{2} + \arg\big(1 - \frac{il\alpha}{r}e^{\mp i\frac{\pi\delta}{2}}\big)\\ &\leqslant -\frac{\pi\alpha}{2} - \tan^{-1}\Big(\frac{m\alpha\cos(\frac{\pi\delta}{2})}{r + m\alpha\sin(\frac{\pi\alpha}{2})}\Big) = -\frac{\pi\sigma}{2}.\end{aligned}$$

Therefore, we obtain a contradiction of the condition (3.3). So there is no point $z_0 \in \mathbb{U}$ such that $|\arg\{p(z_0)\}| = \frac{\pi\alpha}{2}$, that is, $|\arg\{p(z)\}| < \frac{\pi\alpha}{2}$ for all $z \in \mathbb{U}$. Therefore, $f(z) \in \mathcal{SS}_{\lambda,\xi}^{a,c}(\eta, \alpha)$. Therefore, the theorem is proved. □

With the same method, we can obtain the following Theorem 3.3 and Theorem 3.4.

Theorem 3.3. *Let* $\lambda \geqslant 0$, $\mathcal{N}_\xi^{\lambda}(a,c)g(z) \neq 0 \ \ (z \in \mathbb{U}\setminus\{0\})$. *Then*

$$\mathcal{SC}_{\lambda+1,\xi}^{a,c}(\eta, \sigma_1, \psi) \subset \mathcal{SC}_{\lambda,\xi}^{a,c}(\eta, \alpha, \psi),$$

where

$$\sigma_1 = \alpha + \frac{2}{\pi}\tan^{-1}\Big(\frac{m\alpha\cos(\frac{\pi\delta_1}{2})}{r_1 + m\alpha\sin(\frac{\pi\delta_1}{2})}\Big),$$

for some $r_1 > 0,\ m \in \mathbb{N}$ *and* $0 \leqslant \delta_1 < \alpha$.

Theorem 3.4. *Let* $\gamma \geqslant 0, \mathcal{N}_\xi^{\lambda}(a,c)g(z) \neq 0 \ \ (z \in \mathbb{U}\setminus\{0\})$. *Then*

$$\mathcal{SR}_{\lambda,\xi}^{a,c}(\eta, \gamma, \sigma_2, \psi) \subset \mathcal{SR}_{\lambda,\xi}^{a,c}(\eta, 0, \alpha, \psi),$$

where

$$\sigma_2 = \alpha + \frac{2}{\pi}\tan^{-1}\Big(\frac{m\alpha\cos(\frac{\pi\delta_2}{2})}{r_2 + m\alpha\sin(\frac{\pi\delta_2}{2})}\Big),$$

for some $r_2 > 0, m \in \mathbb{N}$ *and* $0 \leqslant \delta_2 < \alpha$.

Acknowledgment

This work is partially supported by NNSF of China (No. 61673006), research fund from Yangtze University (No. 2015CQN77) and research fund from Yangtze University College of Engineering and Technology (No. 15J0802). We would also thank the referees for their significant suggestions which improve the structure of the paper and the editors for handling this paper.

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