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Semicontinuity of approximate solution mappings for parametric generalized weak vector equilibrium problems

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Abstract

In this paper, we first introduce a new set-valued mapping by the scalar approximate solution mapping of a parametric generalized weak vector equilibrium problem and obtain some of its properties. By one of obtained properties, we establish the lower semicontinuity the approximate solution mapping to a parametric generalized weak vector equilibrium problem without the assumptions about monotonicity and approximate solution mappings. Simultaneously, under some suitable conditions, we obtain the upper semicontinuity of the approximate solution mapping to a generalized parametric weak vector equilibrium problem. Our main results improve and extend the corresponding ones in the literature.

Keywords: Parametric generalized weak vector equilibrium problems, lower semicontinuity, upper semicontinuity, approximate solution mappings.
2010 MSC: 49K40, 90C31, 91B50.

1. Introduction

It is well-known that the vector equilibrium problem provides a unified model of several problems, for example the vector optimization problem, the vector variational inequality problem, the vector complementarity problem and the vector saddle point problem. In the literature, existence results for various types of vector equilibrium problems have been investigated intensively, e.g., see [5, 9, 11, 10, 12, 13, 24] and the references therein.

When dealing with the (semi) continuity of the solution maps to parametric vector equilibrium problems, the scalarization approach has been shown to be a very effective and powerful method. By using a scalarization method, Cheng and Zhu [8] obtained the upper semicontinuity and lower semicontinuity of the solution mapping to a parametric weak vector variational inequality in finite-dimensional Euclidean spaces. By using the ideas of Cheng and Zhu [8], Gong [14] established the continuity of the solution mapping to a parametric weak vector equilibrium problem with vector-valued mappings. By using a new proof method different from the ones in [8, 14], Chen et al. [7] established the lower semicontinuity and continuity of the solution mapping to a parametric generalized vector equilibrium problem involving

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set-valued mappings . By virtue of a density result and a scalarization approach, Gong and Yao [15] first discussed the lower semicontinuity of the set of efficient solutions to parametric vector equilibrium problems. Chen and Li [6] discussed the lower semicontinuity and continuity results of the solution sets to a parametric strong vector equilibrium problem and a parametric weak vector equilibrium problem without the uniform compactness assumption. By virtue of a key assumption that includes the information about the solution set, Li and Fang [21] established the lower semicontinuity of the solution mappings to a parametric weak vector equilibrium problem with vector-valued mappings. Under the assumption of the f-property, Xu and Li [28] obtained the lower semicontinuity of the solution mapping to a parametric generalized strong vector equilibrium problem by using a scalarization method. The obtained results extend and generalize the corresponding ones in [14] and [6]. Under the assumptions which do not contain any information about solution mappings, Wang and Li [26] established the lower semicontinuity of the solution mapping to a parametric generalized vector equilibrium problem by using a scalarization method. The obtained results improve the corresponding ones in [14, 15, 7, 21, 28]. Wang et al. [27] establish the lower semicontinuity and upper semicontinuity of the solution set to a parametric generalized strong vector equilibrium problem by using a scalarization method and a density result.

On the other hand, exact solutions of the problems may not exist in many practical problems because the data of the problems are not sufficiently regular. Moreover, these mathematical models are usually solved by numerical methods (iterative procedures or heuristic algorithms) which produce approximations to the exact solutions. So it is impossible to obtain an exact solution of many practical problems. Naturally, investigating approximate solutions of parametric equilibrium problems is very interesting in both practical applications and computations. However, to the best of our knowledge, there are only a few papers concerning the stability of approximate solution mappings for parametric variational inequality or parametric equilibrium problems. Khanh and Luu [18] obtained the semicontinuity of the approximate solution mappings of parametric multivalued quasivariational inequalities in topological vector spaces. Kimura and Yao [19] established the existence results for two types of approximate generalized vector equilibrium problems, and further obtained the semicontinuity of approximate solution mappings. Anh and Khanh [1] obtained Hausdorff semicontinuity (or Berge semicontinuity) of two kinds of approximate solution mappings to parametric generalized vector quasiequilibrium problems. By using a scalarization method, Li and Li [22] have investigated the Hausdorff continuity (or Berge continuity) of the approximate solution mapping for a parametric scalar equilibrium problem. By using a scalarization method, they also obtained a sufficient condition of the lower semicontinuity of the approximate solution mapping for a parametric vector equilibrium problem. By using the monotonicity of the approximate solution mappings, Li et al. [23] established the Lipschitz continuity of the approximate solution mappings for a parametric scalar equilibrium problem.

Motivated by the work reported in [17, 23, 25, 26, 27], the aim of this paper is to establish the lower semicontinuity and the upper semicontinuity of the approximate solution mapping to a parametric generalized weak vector equilibrium problem (in short, PGWVEP). By a scalarization method and introducing a new set-valued mapping, we establish the lower semicontinuity of the approximate solution mapping to PGWVEP without the assumptions about monotonicity and approximate solution mappings. We also establish the upper semicontinuity of the approximate solution mapping to PGWVEP. Our main proof methods are new and different from the ones used in the literature.

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts and some of their properties. In Section 3, we discuss the lower semicontinuity of the approximate solution mapping to PGWVEP. In Section 4, we establish the upper semicontinuity of the approximate solution mapping to PGWVEP.

2. Preliminaries and notations

Throughout this paper, let X and Y be real normed and Hausdorff topological vector spaces. We also assume that C is a pointed closed convex cone in Y with its interior $\mathrm{int}C \neq \emptyset$. Let Y^* be the topological dual space of Y and let C^* be the dual cone of cone C, defined by

$$C^* = \{f \in Y^* : f(c) \geqslant 0,\ \forall c \in C\}.$$

Denote the quasi-interior of C by $C^\sharp$, i.e.,

$$C^\sharp = \{f \in Y^* : f(c) > 0,\ \forall c \in C \setminus \{0\}\}.$$

Since $\mathrm{int}C \neq \emptyset$, the dual cone C^* has a weak* compact base. Let $c_0 \in \mathrm{int}C$ be a fixed point and

$$B^*_{c_0} = \{f \in C^* | f(c_0) = 1\} \quad \text{and} \quad B^\sharp_{c_0} = \{f \in C^\sharp | f(c_0) = 1\}.$$

Then $B^*_{c_0} = \{f \in C^* | f(c_0) = 1\}$ is a weak* compact base of C^*.

We denote by B_Y the closed unit ball in Y. We also assume that 0_X and 0_Y denote the origins of X and Y, respectively. Let E be a nonempty subset of X and let $F : E \times E \to 2^Y$ be a nonempty set-valued mapping. We consider the following generalized weak vector equilibrium problem (in short, GWVEP) of finding $x \in E$ such that

$$F(x, y) \bigcap (-\mathrm{int}C) = \emptyset, \quad \forall y \in E.$$

Let Z be a real topological space. When the mapping F is perturbed by a parameter μ which varies over a subset Λ of Z, we consider the following parametric generalized weak vector equilibrium problem (in short, PGWVEP) of finding $x \in E$ such that

$$F(x, y, \mu) \bigcap (-\mathrm{int}C) = \emptyset, \quad \forall y \in E,$$

where $F : E \times E \times \Lambda \subset X \times X \times Z \to 2^Y \setminus \{\emptyset\}$ is a set-valued mapping.

For each $\mu \in \Lambda$ and $t \in R_+$, let $S(\mu, t)$ denote the approximate solution mapping of PGWVEP corresponding to (μ, t), i.e.,

$$S(\mu, t) = \{x \in E : [F(x, y, \mu) + tc_0] \bigcap (-\mathrm{int}C) = \emptyset,\ \forall y \in E\},$$

where $c_0 \in \mathrm{int}C$. For each $\mu \in \Lambda, t \in R_+$ and $f \in B^*_{c_0}$, we denote by $S_f(\mu, t)$ the f-approximate solution mapping of PGWVEP corresponding to (μ, t), i.e.,

$$S_f(\mu, t) := \{x \in E : f(z) + t \geqslant 0,\ \forall z \in F(x, y, \mu),\ \forall y \in E\}.$$

Now, we recall some concepts and properties which will be useful in the sequel.

Definition 2.1 ([2]). Let G be a set-valued map from X to Y.

(i) G is said to be lower semicontinuous (in short, l.s.c.) at $x_0 \in X$, if for any sequence $\{x_n\}$ with $x_n \to x_0$ and $y_0 \in G(x_0)$, there exists a sequence $\{y_n\} \subseteq G(x_n)$ such that $y_n \to y_0$.
It could be phrased as follows:
G is said to be l.s.c. at $x_0 \in X$, if for any $y_0 \in G(x_0)$ and any neighborhood $W(y_0)$ of y_0, there exists a neighborhood $V(x_0)$ of x_0 such that

$$G(x) \bigcap W(y_0) \neq \emptyset,\ \forall x \in V(x_0).$$

G is said to be lower semicontinuous if G is l.s.c. at every point $x \in X$.

(ii) G is said to be upper semicontinuous (in short, u.s.c.) at $x_0 \in X$, if for any neighborhood $W(G(x_0))$ of $G(x_0)$, there exists a neighborhood $W(x_0)$ of x_0 such that

$$G(x) \subseteq W(G(x_0)), \quad \forall x \in W(x_0).$$

G is said to be upper semicontinuous if G is u.s.c. at every point $x \in X$.

Definition 2.2 ([25]). Let E be a convex subset of X and $G : E \to 2^Y$ be a set-valued map with $G(x) \neq \emptyset$, for all $x \in E$. G is said to be convex on E, if for any $x_1, x_2 \in E$ and $\lambda \in (0, 1)$,

$$\lambda G(x_1) + (1 - \lambda)G(x_2) \subseteq G[\lambda x_1 + (1 - \lambda)x_2].$$

Definition 2.3 ([4]). Let E be a convex subset of X, C be a cone of Y and $G : E \to 2^Y$ be a set-valued mapping with $G(x) \neq \emptyset$, for all $x \in E$. G is said to be C-convex on E, if for any $x_1, x_2 \in E$ and $\lambda \in (0, 1)$,

$$\lambda G(x_1) + (1 - \lambda)G(x_2) \subseteq G[\lambda x_1 + (1 - \lambda)x_2] + C.$$

Definition 2.4 ([17]). Let E be a convex subset of X, C be a cone of Y and $G : E \to 2^Y$ be a set-valued mapping with $G(x) \neq \emptyset$, for all $x \in E$. G is said to be C-concave on E, if for any $x_1, x_2 \in E$ and $\lambda \in (0, 1)$,

$$G[\lambda x_1 + (1 - \lambda)x_2] \subseteq \lambda G(x_1) + (1 - \lambda)G(x_2) + C.$$

Definition 2.5 ([20]). Let E be a subset of X, D be a cone of Z and $G : E \to 2^Z$ be a set-valued map with $G(x) \neq \emptyset$, for all $x \in E$. G is said to be D-subconvexlike on E, if there exists $\theta \in \text{int}D$ such that for any $x_1, x_2 \in E$, $\lambda \in (0, 1)$ and $\varepsilon > 0$,

$$\varepsilon\theta + \lambda G(x_1) + (1 - \lambda)G(x_2) \subseteq G(E) + D.$$

Definition 2.6 ([17]). Let P and Q be two topological vector spaces. Let D be a nonempty subset of P. A set-valued mapping $H : P \to 2^Q$ is said to be uniformly continuous on D, if for any neighborhood V of $0_Q \in Q$, there exists a neighborhood U of $0_P \in P$ such that for any $x_1, x_2 \in D$ with $x_1 - x_2 \in U$,

$$H(x_1) \subseteq H(x_2) + V.$$

Lemma 2.7 ([3]). *For each neighborhood U of 0_X, there exists a balanced open neighborhood U_1 of 0_X such that*

$$U_1 + U_1 \subset U.$$

Lemma 2.8 ([3]). *The union $\Gamma = \bigcup_{i \in I} \Gamma_i$ of a family of l.s.c. set-valued mappings Γ_i from a topological space X into a topological space Y also is an l.s.c. set-valued mapping from X into Y, where I is an index set.*

Lemma 2.9 ([16]). *Let G be a set-valued map from X to Y and $u_0 \in X$. If $G(u_0)$ is compact, then G is u.s.c. at u_0 if and only if for any sequence $\{u_n\} \subset X$ with $\{u_n\} \to u_0$ and for any $y_n \in G(u_n)$, there exist $y_0 \in G(u_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$.*

The following lemma plays an important role in the proof of the lower semicontinuity of the solution mapping $S(\cdot, \cdot)$.

Lemma 2.10 (see[25]). *Let E be a convex subset of X and $G : E \to 2^Y$ be a set-valued map with $G(x) \neq \emptyset$, for all $x \in E$. If G is convex on E and $x_0 \in \text{int}E$, then G is l.s.c. at x_0.*

3. Lower semicontinuity

In this section, we first introduce a new set-valued mapping $H^f_{\mu_0}(\cdot)$ by the f-approximate solution mapping $S_f(\cdot, \cdot)$ of PGWVEP and establish its lower semicontinuity. Then by the lower semicontinuity of $H^f_{\mu_0}(\cdot)$, we obtain the lower semicontinuity of $S_f(\cdot, \cdot)$. Finally, we discuss the lower semicontinuity of $S(\cdot, \cdot)$ of PGWVEP.

Let $\mu_0 \in \Lambda, c_0 \in \text{int}C$ and $f \in B^*_{c_0}$. We define a new set-valued $H^f_{\mu_0} : R_+ \to 2^X$ by

$$H^f_{\mu_0}(t) = S_f(\mu_0, t), \quad \forall t \in R_+.$$

Lemma 3.1. *Let* $\mu_0 \in \Lambda$ *and* $f_0 \in B^*_{c_0}$. *Let* E *be a nonempty convex subset of* X *and* $\mathrm{dom}(H^{f_0}_{\mu_0}) \neq \emptyset$. *If for any* $y \in E$, $F(\cdot, y, \mu_0)$ *is* C*-concave on* E, *then* $\mathrm{dom}(H^{f_0}_{\mu_0})$ *is convex and* $H^{f_0}_{\mu_0}(\cdot)$ *is convex on* $\mathrm{dom}(H^{f_0}_{\mu_0})$.

Proof. Take any $t_1, t_2 \in \mathrm{dom}(H^{f_0}_{\mu_0})$, $x_1 \in H^{f_0}_{\mu_0}(t_1)$, $x_2 \in H^{f_0}_{\mu_0}(t_2)$ and $\lambda \in [0, 1]$. Then by the definition of $H^{f_0}_{\mu_0}$, for any $y \in E$, for any $z_1 \in F(x_1, y, \mu_0)$ and $z_2 \in F(x_2, y, \mu_0)$, we have

$$f_0(z_1) + t_1 \geqslant 0,$$

and

$$f_0(z_2) + t_2 \geqslant 0.$$

Therefore, by the linearity of f_0, for any $y \in E, z_1 \in F(x_1, y, \mu_0), z_2 \in F(x_2, y, \mu_0)$, we have

$$\lambda[f_0(z_1) + t_1] + (1-\lambda)[f_0(z_2) + t_2] = [f_0(\lambda z_1 + (1-\lambda)z_2] + [\lambda t_1 + (1-\lambda)t_2] \geqslant 0. \tag{3.1}$$

Since, for any $y \in E$, $F(\cdot, y, \mu_0)$ is C-concave on E,

$$F(\lambda x_1 + (1-\lambda)x_2, y, \mu_0) \subseteq \lambda F(x_1, y, \mu_0) + (1-\lambda)F(x_2, y, \mu_0) + C, \quad \forall y \in E.$$

Thus, for any $z \in F(\lambda x_1 + (1-\lambda)x_2, y, \mu_0)$, there exist $z_1 \in F(x_1, y, \mu_0)$, $z_2 \in F(x_2, y, \mu_0)$ and $c \in C$ such that

$$z = \lambda z_1 + (1-\lambda)z_2 + c.$$

Then it follows from (3.1) and $f_0 \in B^*_{c_0}$ that

$$f_0(z) + [\lambda t_1 + (1-\lambda)t_2] \geqslant 0, \quad \forall z \in F(\lambda x_1 + (1-\lambda)x_2, y, \mu_0), \quad \forall y \in E. \tag{3.2}$$

Note that $x_1, x_2 \in E$. Since E is convex, $\lambda x_1 + (1-\lambda)x_2 \in E$. Thus, by (3.2) we have

$$\lambda x_1 + (1-\lambda)x_2 \in H^{f_0}_{\mu_0}(\lambda t_1 + (1-\lambda)t_2).$$

Therefore, $\mathrm{dom}(H^{f_0}_{\mu_0})$ is convex and

$$\lambda H^{f_0}_{\mu_0}(t_1) + (1-\lambda)H^{f_0}_{\mu_0}(t_2) \subseteq H^{f_0}_{\mu_0}(\lambda t_1 + (1-\lambda)t_2).$$

So $H^{f_0}_{\mu_0}(\cdot)$ is convex on $\mathrm{dom}(H^{f_0}_{\mu_0})$ and the proof is complete. □

Remark 3.2. Since concave functions must be cone-concave and the converse may not hold, Lemma 3.1 improves and generalizes [22, Lemma 3.3].

Lemma 3.3. *Let* $\mu_0 \in \Lambda$, $f_0 \in B^*_{c_0}$ *and* $t_0 \in \mathrm{intdom}(H^{f_0}_{\mu_0})$. *Let* E *be a convex subset of* X. *If for any* $y \in E$, $F(\cdot, y, \mu_0)$ *is* C*-concave on* E, *then* $H^{f_0}_{\mu_0}(\cdot)$ *is l.s.c. at* t_0.

Proof. By Lemma 3.1, $\mathrm{dom}(H^{f_0}_{\mu_0})$ is convex and $H^{f_0}_{\mu_0}(\cdot)$ is convex on $\mathrm{dom}(H^{f_0}_{\mu_0})$. So, by Lemma 2.10, it follows from $t_0 \in \mathrm{int}(\mathrm{dom}H^{f_0}_{\mu_0})$ that $H^{f_0}_{\mu_0}(\cdot)$ is l.s.c. at t_0, and the proof is complete. □

Theorem 3.4. *Let* $\mu_0 \in \Lambda$, $f_0 \in B^*_{c_0}$ *and* $t_0 \in \mathrm{intdom}(H^{f_0}_{\mu_0})$. *Suppose that the following conditions are satisfied:*

(i) E *is a nonempty convex subset of* X *and for any* $y \in E$, $F(\cdot, y, \mu_0)$ *is* C*-concave on* E;

(ii) $F(\cdot, \cdot, \cdot)$ *is uniformly continuous on* $E \times E \times N(\mu_0)$, *where* $N(\mu_0)$ *is a neighborhood of* μ_0.

Then $S_{f_0}(\cdot, \cdot)$ *is l.s.c. at* (μ_0, t_0).

Proof. To prove the result by contradiction, suppose that $S_{f_0}(\cdot, \cdot)$ is not l.s.c. at (μ_0, t_0). Then there exist $x_0 \in S_{f_0}(\mu_0, t_0)$ and a neighborhood W_0 of 0_X, for any neighborhoods $U(\mu_0)$ and $U(t_0)$ of μ_0 and t_0, respectively, there exist $\mu \in U(\mu_0)$ and $t \in V(t_0)$ such that

$$(\{x_0\} + W_0) \bigcap S_{f_0}(\mu, t) = \emptyset.$$

Hence, there exist sequences $\{\mu_n\}$ with $\mu_n \to \mu_0$ and $\{t_n\}$ with $t_n \to t_0$ such that

$$(\{x_0\} + W_0) \bigcap S_{f_0}(\mu_n, t_n) = \emptyset, \quad \forall n. \tag{3.3}$$

For the above W_0, it follows from Lemma 2.7 that there exists a balanced neighborhood W_1 of 0_X such that

$$W_1 + W_1 \subset W_0. \tag{3.4}$$

By condition (i) and Lemma 3.3, we get that $H^{f_0}_{\mu_0}(\cdot)$ is l.s.c. at t_0. Thus, for the above $x_0 \in S_{f_0}(\mu_0, t_0) = H^{f_0}_{\mu_0}(t_0)$ and W_1, there exists a balanced neighborhood $V_1(t_0)$ of t_0 such that

$$(\{x_0\} + W_1) \bigcap H^{f_0}_{\mu_0}(t) = (\{x_0\} + W_1) \bigcap S_{f_0}(\mu_0, t) \neq \emptyset, \forall t \in V_1(t_0).$$

Let $t^{'} \in V_1(t_0)$ be fixed with $t_0 - t^{'} > 0$. Then

$$(\{x_0\} + W_1) \bigcap H^{f_0}_{\mu_0}(t^{'}) = (\{x_0\} + W_1) \bigcap S_{f_0}(\mu_0, t^{'}) \neq \emptyset.$$

Take

$$x_1 \in (\{x_0\} + W_1) \bigcap S_{f_0}(\mu_0, t^{'}). \tag{3.5}$$

Since $t_0 - t^{'} > 0$ and $c_0 \in \mathrm{int}C$, there exists $\delta_0 > 0$ such that

$$\delta_0 B_Y + \{(t_0 - t^{'})c_0\} \subset C. \tag{3.6}$$

For the above $\delta_0 B_Y$, it follows from Lemma 2.7 that there exists $\delta_1 > 0$ such that

$$\delta_1 B_Y + \delta_1 B_Y \subset \delta_0 B_Y. \tag{3.7}$$

Since $t_n \to t_0$, there exists a natural number N_0 such that

$$t_n c_0 \in \{t_0 c_0\} + \delta_1 B_Y, \quad \forall n > N_0.$$

Therefore, combining with (3.6) and (3.7), we get that

$$\delta_1 B_Y + \{t_n c_0\} \subset C + \{t^{'} c_0\}, \quad \forall n > N_0. \tag{3.8}$$

Since $F(\cdot,\cdot,\cdot)$ is uniformly continuous on $E \times E \times N(\mu_0)$, for the above $\delta_1 B_Y$, there exist two neighborhoods $W_1(0_X)$ and $\bar{W}_1(0_X)$ of 0_X and a neighborhood $V(0_Z)$ of 0_Z, for any $(\bar{x}_1, \bar{y}_1, \bar{\mu}_1), (\bar{x}_2, \bar{y}_2, \bar{\mu}_2) \in E \times E \times N(\mu_0)$ with $\bar{x}_1 - \bar{x}_2 \in W_1(0_X), \bar{y}_1 - \bar{y}_2 \in \bar{W}_1(0_X)$ and $\bar{\mu}_1 - \bar{\mu}_2 \in V(0_Z)$, we have

$$F(\bar{x}_1, \bar{y}_1, \bar{\mu}_1) \subset \delta_1 B_Y + F(\bar{x}_2, \bar{y}_2, \bar{\mu}_2). \tag{3.9}$$

By (3.5), we can see that $x_1 \in E$. Naturally, $[\{x_1\} + W_1 \bigcap W_1(0_X)] \bigcap E \neq \emptyset$. We take

$$x_2 \in [\{x_1\} + W_1 \bigcap W_1(0_X)] \bigcap E. \tag{3.10}$$

We show that $x_2 \in S_{f_0}(\mu_{n_0}, t_{n_0})$. It follows from $\mu_n \to \mu_0$ that there exists μ_{n_0} with $n_0 > N_0$ such that

$$\mu_{n_0} \in N(\mu_0) \bigcap (\{\mu_0\} + V(0_Z)). \tag{3.11}$$

By (3.5), we have

$$f_0(z) + t' \geqslant 0, \quad \forall z \in F(x_1, y, \mu_0), \quad \forall y \in E. \tag{3.12}$$

For any $y' \in E$, we can see that there exists $y_0 \in E$ such that $y' - y_0 \in \bar{W}_1(0_X)$. By (3.10), $x_2 - x_1 \in W_1(0_X)$. Then it follows from (3.11) and (3.9) that

$$F(x_2, y', \mu_{n_0}) \subset \delta_1 B_Y + F(x_1, y_0, \mu_0).$$

Thus, it follows from $n_0 > N_0$ and (3.8) that

$$F(x_2, y', \mu_{n_0}) + \{t_{n_0} c_0\} \subset C + \{t' c_0\} + F(x_1, y_0, \mu_0), \quad \forall y' \in E.$$

Combining with (3.12) and $f_0 \in B^*_{c_0}$ we have $f_0(z) + t_{n_0} \geqslant 0$, for all $z \in F(x_2, y', \mu_{n_0})$, $y' \in E$. So

$$x_2 \in S_{f_0}(\mu_{n_0}, t_{n_0}). \tag{3.13}$$

It follows from (3.4), (3.5) and (3.10) that $x_2 \in \{x_0\} + W_0$. Thus, combine with (3.13), we have

$$[\{x_0\} + W_0] \bigcap S_{f_0}(\mu_{n_0}, t_{n_0}) \neq \emptyset,$$

which contradicts (3.3). So $S_{f_0}(\cdot, \cdot)$ is l.s.c. at (μ_0, t_0), and this completes the proof. □ □

Remark 3.5. Li and Fang [21, Lemma 3.1] and Chen and Huang [5, Lemma 3.1] used a key assumption which includes the solution set information to obtain the lower semicontinuity of $S_f(\cdot)$. Under the assumption of cone-strict monotonicity, Chen et al. [7, Lemma 3.2] obtained the lower semicontinuity of $S_f(\cdot)$. The main advantage of Theorem 3.4 is that it does not require any information on the approximate solution set and monotonicity.

Lemma 3.6. *Let $t > 0$, $c_0 \in \mathrm{int}C$ and $\mu \in \Lambda$. If for each $x \in E$, $F(x, \cdot, \mu)$ is C-subconvexlike on E, then*

$$S(\mu, t) = \bigcup_{f \in B^*_{c_0}} S_f(\mu, t).$$

Proof. Since $F(x, \cdot, \mu)$ is C-subconvexlike on E, by Definition 2.5, $F(x, \cdot, \mu) + \{tc_0\}$ is also C-subconvexlike on E. It follows from the proof similar to [26, Lemma 3.2] that

$$S(\mu, t) = \bigcup_{f \in C^* \setminus \{0_{Y^*}\}} S_f(\mu, t).$$

Obviously, $\bigcup_{f \in B^*_{c_0}} S_f(\mu, t) \subset \bigcup_{f \in C^* \setminus \{0_{Y^*}\}} S_f(\mu, t)$. So it suffices to prove that

$$\bigcup_{f \in C^* \setminus \{0_{Y^*}\}} S_f(\mu, t) \subset \bigcup_{f \in B^*_{c_0}} S_f(\mu, t). \tag{3.14}$$

Let $x_0 \in \bigcup_{f \in C^* \setminus \{0_{Y^*}\}} S_f(\mu, t)$. Then there exists $f_0 \in C^* \setminus \{0_{Y^*}\}$ such that

$$f_0(z + tc_0) \geqslant 0, \quad \forall z \in F(x_0, y, \mu_0), \quad \forall y \in E. \tag{3.15}$$

Since $c_0 \in \mathrm{int}C$ and $f_0 \in C^* \setminus \{0_{Y^*}\}$, $f_0(c_0) > 0$. It follows from (3.15) that

$$[\frac{1}{f_0(c_0)} f_0](z) + t \geqslant 0, \quad \forall z \in F(x_0, y, \mu_0), \quad \forall y \in E.$$

Naturally, $f_1 := \frac{1}{f_0(c_0)} f_0 \in B^*_{c_0}$ and

$$f_1(z) + t \geqslant 0, \quad \forall z \in F(x_0, y, \mu_0), \quad \forall y \in E.$$

Then $x_0 \in S_{f_1}(\mu, t)$. So (3.14) holds, and the proof is complete. □

Remark 3.7. Since cone-convex mapping must be cone-subconvexlike and the converse may not hold, Lemma 3.6 improves and generalizes [22, Lemma 4.1].

Now we establish the lower semicontinuity of $S(\cdot,\cdot)$.

Theorem 3.8. *Let $\mu_0 \in \Lambda$. Suppose that the following conditions are satisfied:*

(i) *the assumptions* (i) *and* (ii) *of Theorem 3.4 are fulfilled;*

(ii) *for every $f \in B^*_{c_0}$, $t_0 \in \mathrm{int}(\mathrm{dom}H^f_{\mu_0})$;*

(iii) *for any $x \in E$, $F(x,\cdot,\mu_0)$ is C-subconvexlike on E.*

Then $S(\cdot,\cdot)$ is l.s.c. at (μ_0, t_0).

Proof. Since $F(x,\cdot,\mu)$ is C-subconvexlike on E for any $x \in E$, by Lemma 3.6, we have

$$S(\mu_0, t_0) = \bigcup_{f \in B^*_{c_0}} S_f(\mu_0, t_0).$$

By Theorem 3.4, for any $f \in B^*_{c_0}$, $S_f(\cdot,\cdot)$ is l.s.c. at (μ_0, t_0). So it follows from Lemma 2.8 that $S(\cdot,\cdot)$ is l.s.c. at (μ_0, t_0), and this completes the proof. □

Remark 3.9. Under the assumptions of cone-strict monotonicity and cone-convexity, Gong [14, Theorem 4.1] obtained the lower semicontinuity of the solutions set for the parameterized weak vector equilibrium problem. The main advantages of Theorem 3.8 are that it uses cone-subconvexlike mappings instead of cone-convex mppings and removes the assumption of the cone-strict monotonicity.

Remark 3.10. Since cone-convex mappings must be cone-subconvexlike and the converse may not hold, Theorem 3.8 improves and generalizes the l.s.c. of [22, Theorem 4.1].

We give an example to illustrate Theorem 3.8.

Example 3.11. Let $X = R, Y = R^2, C = R^2_+ = \{(y_1, y_2) : y_1 \geqslant 0, y_2 \geqslant 0\}, D = [0,2] \times [0,2], \Lambda = [-2,2], E = [0,2], \forall \mu \in \Lambda$. $F : [-2,2] \times [-2,2] \times \Lambda \to 2^Y$ is defined by

$$F(x, y, \mu) = \{(-x^2 + y^2 - \mu, -x + y - \mu^2)\} + D.$$

Take $\mu_0 = 0$, $N(\mu_0) = [-1, 1]$, $c_0 = (3, 3) \in \mathrm{int}C$ and $t_0 = 2$. Then it is easy to see that all assumptions of Theorem 3.8 are fulfilled, by Theorem 3.8, $S(\cdot,\cdot)$ is l.s.c. at (μ_0, t_0).

4. Upper semicontinuity

In this section, we discuss the upper semicontinuity of the approximate solution mapping of PGWVEP under the assumptions which do not contain any information about monotonicity and approximate solution mappings.

Theorem 4.1. *Let $(\mu_0, t_0) \in \mathrm{dom}S$ and E be compact. If for any $y \in E$, $F(\cdot, y, \cdot)$ is l.s.c. on $E \times \Lambda$, then $S(\cdot,\cdot)$ is u.s.c. at (μ_0, t_0).*

Proof. To prove the result by contradiction, suppose that $S(\cdot,\cdot)$ is not u.s.c. at (μ_0, t_0). Then there exists an open neighborhood U_0 of $S(\mu_0, t_0)$ for any neighborhoods $U(\mu_0)$ and $U(t_0)$ of μ_0 and t_0, respectively, there exist $\bar{\mu} \in U(\mu_0)$ and $\bar{t} \in U(t_0)$ such that

$$S(\bar{\mu}, \bar{t}) \not\subseteq U_0.$$

It follows from the arbitrariness of $U(\mu_0)$ and $U(t_0)$ that there exist sequences $\{\mu_n\}$ with $\mu_n \to \mu_0$ and $\{t_n\}$ with $t_n \to t_0$ such that

$$S(\mu_n, t_n) \not\subseteq U_0, \quad \forall n,$$

and then for all n, there exists

$$x_n \in S(\mu_n, t_n), \tag{4.1}$$

such that

$$x_n \notin U_0. \tag{4.2}$$

Naturally, $x_n \in E$ for all n. Since E is compact, there exist $x_0 \in E$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$.

We next prove that $x_0 \in S(\mu_0, t_0)$. In fact if $x_0 \notin S(\mu_0, t_0)$, then there exists $y_0 \in E$ such that

$$[F(x_0, y_0, \mu_0) + t_0 c_0] \bigcap (-\mathrm{int}C) \neq \emptyset.$$

Therefore there exists $z_0 \in F(x_0, y_0, \mu_0)$ such that

$$z_0 + t_0 c_0 \in -\mathrm{int}C. \tag{4.3}$$

Naturally, $(x_{n_k}, \mu_{n_k}) \to (x_0, \mu_0)$. Since, for any $y \in E$, $F(\cdot, y, \cdot)$ is l.s.c. on $E \times \Lambda$, there exists $z_{n_k} \in F(x_{n_k}, y_0, \mu_{n_k})$ such that

$$z_{n_k} \to z_0. \tag{4.4}$$

Since $-\mathrm{int}C$ is an open set and $t_n \to t_0$, it follows from (4.3) and (4.4) that there exists a natural number k_0 such that

$$z_{n_k} + t_{n_k} c_0 \in -\mathrm{int}C, \quad \forall k > k_0.$$

Thus

$$[F(x_{n_k}, y_0, \mu_{n_k}) + t_{n_k} c_0] \bigcap (-\mathrm{int}C) \neq \emptyset, \quad \forall k > k_0,$$

which contradicts (4.1). So

$$x_0 \in S(\mu_0, t_0).$$

Since U_0 is an open neighborhood of $S(\mu_0, t_0)$ and $x_{n_k} \to x_0$, there exists a natural number $\bar{k}$ such that

$$x_{n_k} \in U_0, \quad \forall k > \bar{k},$$

which contradicts (4.2). Thus $S(\cdot, \cdot)$ is u.s.c. at (μ_0, t_0), and the proof of the theorem is complete. □

Remark 4.2. Under the assumptions of cone-strictly monotone and cone-convexity, Gong [14, Theorem 3.1] obtained the upper semicontinuity of the solutions set for the parameterized weak vector equilibrium problem. The main advantage of Theorem 4.1 is that it does not require any information on the cone-strictly monotone and the cone-convexity.

Remark 4.3. Theorem 4.1 discusses the upper semicontinuity of the approximate solution mapping involving set-valued mappings, which is more general than the corresponding result in [22, Theorem 4.1] in the case that F is a set-valued mapping.

We give an example to illustrate Theorem 4.1.

Example 4.4. Let $X = R, Y = R^2, C = R^2_+ = \{(y_1, y_2) : y_1 \geqslant 0, y_2 \geqslant 0\}$, $\Lambda = [0, 2]$, $E(\mu) = [\mu, 3]$, $\forall \mu \in \Lambda$. $F : [0, 3] \times [0, 3] \times \Lambda \to 2^Y$ is defined by

$$F(x, y, \mu) = [x + y - \mu, 10] \times [x - y + \mu, 10].$$

Take $c_0 = (1, 1) \in \mathrm{int}C$. Then it is easy to see that all assumptions of Theorem 4.1 are fulfilled, and $\mathrm{dom}S = [0, 2] \times R_+$ and

$$S(\mu, t) = \begin{cases} [\mu - t, 3], & \mu \in [\frac{3}{2}, 2], \\ [3 - \mu - t, 3], & \mu \in [0, \frac{3}{2}]. \end{cases}$$

Naturally, $S(\cdot, \cdot)$ is u.s.c. on $\mathrm{dom}S$.

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