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The split variational inequality problem and its algorithm iteration

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Abstract

The split variational inequality problem under a nonlinear transformation has been considered. An iterative algorithm is presented to solve this split problem. Strong convergence results are obtained.

Keywords: Split variational inequality, iterative algorithm, nonlinear transformation.
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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with its inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $\mathcal{C} \subset \mathcal{H}$ be a nonempty closed convex set. Let $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ and $\psi : \mathcal{C} \to \mathcal{C}$ be three nonlinear operators. Now we consider the following variational inequality of finding $u \in \mathcal{C}$ such that

$$\langle \mathcal{A}u, \psi(v) - \psi(u)\rangle \geqslant 0, \quad \forall v \in \mathcal{C}. \tag{1.1}$$

Denote the set of the solutions of (1.1) by $VI(\mathcal{A}, \psi, \mathcal{C})$.

If $\psi \equiv I$, then (1.1) reduces to the variational inequality of finding $u \in \mathcal{C}$ such that

$$\langle \mathcal{A}u, v - u\rangle \geqslant 0, \quad \forall v \in \mathcal{C}. \tag{1.2}$$

Denote the set of the solutions of (1.2) by $VI(\mathcal{A}, \mathcal{C})$.

Variational inequalities have played a critical and significant part in the study of several unrelated problems arising in physics, finance, economics, network analysis, elasticity, optimization, water resources, medical images and structural analysis. For some related work, please refer to Ceng et al. [5, 6], Cianciaruso et al. [12], Glowinski [13], Iusem [16], Korpelevič [17], Noor [1], Qin and Cho [19], Qin and Yao [20], Yao et al. [27, 29, 31, 32], Zegeye et al. [34], and Zhang et al. [35].

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Recently, the split feasibility problem has been studied extensively, see [2–4, 7–9, 24–26, 28, 33]. The split feasibility problem is formulated as finding $\hat{\nu}$ such that

$$\hat{\nu} \in \mathcal{C} \quad \text{and} \quad \psi(\hat{\nu}) \in \mathcal{Q}, \tag{1.3}$$

where $\mathcal{C}$ and $\mathcal{Q}$ are two closed convex subsets of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $\psi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. Such problems arise in the field of intensity-modulated radiation therapy.

Special cases.

(i) If $\mathcal{C} = \mathrm{Fix}(S)$ and $\mathcal{Q} = \mathrm{Fix}(T)$, then (1.3) reduces to the split common fixed point problem which was first introduced by Censor and Segal [11]. The reader can refer to He and Du [14, 15] and Yao et al. [30].

(ii) If $\mathcal{C} = \mathrm{VI}(\mathcal{A}, \mathcal{D})$ and $\mathcal{Q} = \mathrm{VI}(\mathcal{B}, \mathcal{E})$, then (1.3) reduces to the split variational inequality problem which was studied in [10].

In the present manuscript, we focus on the following split variational inequality problem of finding a point $\hat{\nu}$ such that

$$\hat{\nu} \in \mathrm{VI}(\mathcal{A}, \psi, \mathcal{C}) \quad \text{and} \quad \psi(\hat{\nu}) \in \mathrm{VI}(\mathcal{B}, \mathcal{C}). \tag{1.4}$$

Remark 1.1. In the existing literature, the split problem that requires to find a point of an operator in one space whose image under a linear transformation is a point of another operator in the image space. However, in (1.4), the transformation ψ is nonlinear.

In order to solve (1.4), we introduce a new iterative algorithm. Under some mild assumptions, we show the strong convergence of the presented algorithm.

2. Preliminaries

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$.

Definition 2.1. An operator $\vartheta : \mathcal{C} \to \mathcal{C}$ is said to be L-*Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|\vartheta(u) - \vartheta(u^\dagger)\| \leqslant L\|u - u^\dagger\|$$

for all $u, u^\dagger \in \mathcal{C}$.

Definition 2.2. An operator $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ is said to be

(1) *monotone* if

$$\langle \mathcal{A}u - \mathcal{A}u^\dagger, u - u^\dagger \rangle \geqslant 0, \quad \forall u, u^\dagger \in \mathcal{C};$$

(2) *strongly monotone* if there exists a constant $\delta > 0$ such that

$$\langle \mathcal{A}u - \mathcal{A}u^\dagger, u - u^\dagger \rangle \geqslant \delta\|u - u^\dagger\|^2, \quad \forall u, u^\dagger \in \mathcal{C};$$

(3) *inverse strongly monotone* if there exists $\eta > 0$ such that

$$\langle \mathcal{A}u - \mathcal{A}u^\dagger, u - u^\dagger \rangle \geqslant \eta\|\mathcal{A}u - \mathcal{A}u^\dagger\|^2, \quad \forall u, u^\dagger \in \mathcal{C}.$$

Let $\psi : \mathcal{C} \to \mathcal{C}$ be a nonlinear operator.

Definition 2.3. An operator $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ is said to be η-*inverse strongly* ψ-*monotone* iff

$$\langle \mathcal{A}u - \mathcal{A}u^\dagger, \psi(u) - \psi(u^\dagger) \rangle \geqslant \eta\|\mathcal{A}u - \mathcal{A}u^\dagger\|^2$$

for all $u, u^\dagger \in \mathcal{C}$ and for some $\eta > 0$.

Let $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The effective domain of $\mathcal{A}$ is denoted by $\mathrm{dom}(\mathcal{A})$, that is, $\mathrm{dom}(\mathcal{A}) = \{x \in H : \mathcal{A}x \neq \emptyset\}$.

Definition 2.4. A set-valued operator $\mathcal{A}$ is said to be *monotone* on $\mathcal{H}$ iff

$$\langle x - y, u - v \rangle \geqslant 0$$

for all $x, y \in \mathrm{dom}(\mathcal{A})$, $u \in \mathcal{A}x$ and $v \in \mathcal{A}y$.

A monotone operator $\mathcal{A}$ on $\mathcal{H}$ is said to be maximal iff its graph is not strictly contained in the graph of any other monotone operator on $\mathcal{H}$.

For every point $u \in \mathcal{H}$, there exists a unique nearest point in $\mathcal{C}$, denoted by $\mathrm{proj}_{\mathcal{C}}[u]$ such that

$$\|u - \mathrm{proj}_{\mathcal{C}}[u]\| \leqslant \|u - u^{\dagger}\|$$

for all $u^{\dagger} \in \mathcal{C}$.

The mapping $\mathrm{proj}_{\mathcal{C}}$ is called the metric projection of $\mathcal{H}$ onto $\mathcal{C}$. It is well-known that $\mathrm{proj}_{\mathcal{C}}$ is a typical firmly nonexpansive mapping and is characterized by the following property

$$\langle u - \mathrm{proj}_{\mathcal{C}}[u], u^{\dagger} - \mathrm{proj}_{\mathcal{C}}[u] \rangle \leqslant 0, \quad \forall u \in \mathcal{H},\ u^{\dagger} \in \mathcal{C}. \tag{2.1}$$

Lemma 2.5 ([22])**.** *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ be a ξ-inverse strongly monotone mapping. Then,*

$$\|(I - \delta\mathcal{A})u - (I - \delta\mathcal{A})u^{\dagger}\|^2 \leqslant \|x - y\|^2 + \delta(\delta - 2\xi)\|\mathcal{A}u - \mathcal{A}u^{\dagger}\|^2, \quad \forall u, u^{\dagger} \in \mathcal{C}, \tag{2.2}$$

where $\delta > 0$ is a constant.

Lemma 2.6 ([23])**.** *Let $\{\varpi_n\} \subset [0, \infty)$, $\{\mu_n\} \subset (0, 1)$, and $\{\rho_n\}$ be three sequences such that*

$$\varpi_{n+1} \leqslant (1 - \mu_n)\varpi_n + \rho_n, \quad \forall n \geqslant 1.$$

Assume the following restrictions are satisfied

(i) $\sum_{n=1}^{\infty} \mu_n = \infty$;
(ii) $\overline{\lim}_{n\to\infty} \frac{\rho_n}{\mu_n} \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\rho_n| < \infty$.

Then $\lim_{n\to\infty} \varpi_n = 0$.

Lemma 2.7 ([18])**.** *Let $\{w_n\}$ be a sequence of real numbers. Assume $\{w_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \leqslant w_{n_k+1}$ for all $k \geqslant 0$. For every $n \geqslant N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \leqslant n : w_{n_i} < w_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geqslant N_0$

$$\max\{w_{\tau(n)}, w_n\} \leqslant w_{\tau(n)+1}.$$

3. Main results

In this section, we study the split variational inequality problem and the convergence analysis of its iterative algorithm.

Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{C} \subset \mathcal{H}$ be a nonempty closed convex set. Let $\psi : \mathcal{C} \to \mathcal{C}$ be a weakly continuous and δ-strongly monotone mapping such that its range $R(\psi) = \mathcal{C}$. Let $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ be an η-inverse strongly ψ-monotone mapping. Let $\mathcal{B} : \mathcal{C} \to \mathcal{H}$ be a ξ-inverse strongly monotone mapping.

Remark 3.1. For $u, v \in \mathcal{C}$, since ψ is δ-strongly monotone, we get

$$\delta\|u-v\|^2 \leqslant \langle \psi(u)-\psi(v), u-v\rangle \leqslant \|\psi(u)-\psi(v)\|\|u-v\|.$$

Then,

$$\|\psi(u)-\psi(v)\| \geqslant \delta\|u-v\|, \quad \forall u, v \in \mathcal{C}. \tag{3.1}$$

In view of the assumptions on the operator ψ, we know ψ^{-1} is single-valued and Lipschitz continuous.

Problem 3.2. The split variational inequality problem is to find $\hat{u}$ such that

$$\hat{u} \in VI(\mathcal{A}, \psi, \mathcal{C}) \quad \text{and} \quad \psi(\hat{u}) \in VI(\mathcal{B}, \mathcal{C}). \tag{3.2}$$

Denote the set of solutions of (3.2) by Λ, i.e., $\Lambda = VI(\mathcal{A}, \psi, \mathcal{C}) \cap \psi^{-1}(VI(\mathcal{B}, \mathcal{C}))$. Throughout, we assume $\Lambda \neq \emptyset$.

Algorithm 3.3. For given initial value $x_0 \in \mathcal{C}$, define the sequence $\{x_n\}$ by the following form

$$\begin{cases} u_n = \text{proj}_{\mathcal{C}}[\psi(x_n) - \zeta_n \mathcal{A} x_n], \\ v_n = \text{proj}_{\mathcal{C}}[\eta_n \mu \vartheta(x_n) + (1-\eta_n)(u_n - \delta_n \mathcal{B} u_n)], \\ \psi(x_{n+1}) = \xi_n \psi(x_n) + (1-\xi_n) v_n, \quad n \geqslant 0, \end{cases} \tag{3.3}$$

where $\vartheta : \mathcal{C} \to \mathcal{H}$ is an L-Lipschitz continuous mapping, $\{\eta_n\}$, $\{\xi_n\}$, and $\{\delta_n\}$ are three real number sequences in $[0,1]$, $\{\zeta_n\}$ is a real number sequence in $(0,\infty)$, and $0 < \mu < \delta/L$ is a constant.

Remark 3.4. The solution of the variational inequality of finding $x^* \in \Lambda$ such that

$$\langle \mu\vartheta(x^*) - \psi(x^*), \psi(x) - \psi(x^*)\rangle \leqslant 0, \quad \forall x \in \Lambda \tag{3.4}$$

is unique. As a matter of fact, assume that both x^* and $\tilde{x}$ solve (3.4). Then,

$$\langle \mu\vartheta(x^*) - \psi(x^*), \psi(\tilde{x}) - \psi(x^*)\rangle \leqslant 0 \quad \text{and} \quad \langle \mu\vartheta(\tilde{x}) - \psi(\tilde{x}), \psi(x^*) - \psi(\tilde{x})\rangle \leqslant 0.$$

Adding up the above two inequalities, we deduce

$$\langle \mu\vartheta(\tilde{x}) - \psi(\tilde{x}) - \mu\vartheta(x^*) + \psi(x^*), \psi(x^*) - \psi(\tilde{x})\rangle \leqslant 0.$$

It follows that

$$\|\psi(x^*) - \psi(\tilde{x})\|^2 \leqslant \mu\langle \vartheta(x^*) - \vartheta(\tilde{x}), \psi(x^*) - \psi(\tilde{x})\rangle \leqslant \mu\|\vartheta(x^*) - \vartheta(\tilde{x})\|\|\psi(x^*) - \psi(\tilde{x})\|.$$

This together with (3.1) implies that

$$\delta\|x^* - \tilde{x}\| \leqslant \|\psi(x^*) - \psi(\tilde{x})\| \leqslant \mu\|\vartheta(x^*) - \vartheta(\tilde{x})\| \leqslant \mu L\|x^* - \tilde{x}\|.$$

We get $x^* = \tilde{x}$ immediately because of $\mu L < \delta$. So, the variational inequality (3.4) has a unique solution denoted by $\hat{u}$.

Proposition 3.5. *Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\mathcal{A} : \mathcal{C} \to \mathcal{H}$ be an η-inverse strongly ψ-monotone mapping. Then,*

$$\|(\psi(x) - \zeta\mathcal{A}x) - (\psi(y) - \zeta\mathcal{A}y)\|^2 \leqslant \|\psi(x) - \psi(y)\|^2 + \zeta(\zeta - 2\eta)\|\mathcal{A}x - \mathcal{A}y\|^2, \quad \forall x, y \in \mathcal{C}. \tag{3.5}$$

Proof. In fact,

$$\begin{aligned} \|(\psi(x) - \zeta\mathcal{A}x) - (\psi(y) - \zeta\mathcal{A}y)\|^2 &= \|\psi(x) - \psi(y)\|^2 - 2\zeta\langle \mathcal{A}x - \mathcal{A}y, \psi(x) - \psi(y)\rangle + \zeta^2\|\mathcal{A}x - \mathcal{A}y\|^2 \\ &\leqslant \|\psi(x) - \psi(y)\|^2 - 2\zeta\eta\|\mathcal{A}x - \mathcal{A}y\|^2 + \zeta^2\|\mathcal{A}x - \mathcal{A}y\|^2 \\ &\leqslant \|\psi(x) - \psi(y)\|^2 + \zeta(\zeta - 2\eta)\|\mathcal{A}x - \mathcal{A}y\|^2. \end{aligned}$$

□

Theorem 3.6. *Assume the following conditions are satisfied:*

(C1): $\lim_{n\to\infty}\eta_n = 0$ *and* $\sum_n \eta_n = \infty$;
(C2): $0 < \underline{\lim}_{n\to\infty}\xi_n \leqslant \overline{\lim}_{n\to\infty}\xi_n < 1$ *and* $\delta \in (L\mu, 2\eta)$;
(C3): $0 < \underline{\lim}_{n\to\infty}\zeta_n \leqslant \overline{\lim}_{n\to\infty}\zeta_n < 2\eta$ *and* $0 < \underline{\lim}_{n\to\infty}\delta_n \leqslant \overline{\lim}_{n\to\infty}\delta_n < 2\xi$.

Then the sequence $\{x_n\}$ *generated by* (3.3) *converges strongly to* $\hat{u} \in \Lambda$ *which solves the variational inequality* (3.4).

Proof. Note that $\hat{u} \in VI(\mathcal{A}, \psi, \mathcal{C})$ and $\psi(\hat{u}) \in VI(\mathcal{B}, \mathcal{C})$. Using (2.1), we deduce $\psi(\hat{u}) = \mathrm{proj}_{\mathcal{C}}[\psi(\hat{u}) - \zeta_n\mathcal{A}\hat{u}] = \mathrm{proj}_{\mathcal{C}}[\psi(\hat{u}) - \delta_n\mathcal{B}\psi(\hat{u})]$ for all $n \geqslant 0$. First, from (2.2), we have

$$\begin{aligned}\|(u_n - \delta_n\mathcal{B}u_n) - (\psi(\hat{u}) - \delta_n\mathcal{B}\psi(\hat{u}))\|^2 &\leqslant \|u_n - \psi(\hat{u})\|^2 + \delta_n(\delta_n - 2\xi)\|\mathcal{B}u_n - \mathcal{B}\psi(\hat{u})\|^2 \\ &\leqslant \|u_n - \psi(\hat{u})\|^2.\end{aligned} \tag{3.6}$$

By (3.5), we get

$$\begin{aligned}\|u_n - \psi(\hat{u})\|^2 &= \|\mathrm{proj}_{\mathcal{C}}[\psi(x_n) - \zeta_n\mathcal{A}x_n] - \mathrm{proj}_{\mathcal{C}}[\psi(\hat{u}) - \zeta_n\mathcal{A}\hat{u}]\|^2 \\ &\leqslant \|(\psi(x_n) - \zeta_n\mathcal{A}x_n) - (\psi(\hat{u}) - \zeta_n\mathcal{A}\hat{u})\|^2 \\ &\leqslant \|\psi(x_n) - \psi(\hat{u})\|^2 + \zeta_n(\zeta_n - 2\eta)\|\mathcal{A}x_n - \mathcal{A}\hat{u}\|^2 \\ &\leqslant \|\psi(x_n) - \psi(\hat{u})\|^2\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\|\psi(x_{n+1}) - \zeta_{n+1}\mathcal{A}x_{n+1} - (\psi(x_n) - \zeta_{n+1}\mathcal{A}x_n)\|^2 \\ &\quad\leqslant \|\psi(x_{n+1}) - \psi(x_n)\|^2 + \zeta_{n+1}(\zeta_{n+1} - 2\eta)\|\mathcal{A}x_{n+1} - \mathcal{A}x_n\|^2.\end{aligned}$$

From (3.1), (3.3), (3.6), and (3.7), we have

$$\begin{aligned}\|v_n - \psi(\hat{u})\| &= \|\mathrm{proj}_{\mathcal{C}}[\eta_n\mu\vartheta(x_n) + (1-\eta_n)(u_n - \delta_n\mathcal{B}u_n)] - \mathrm{proj}_{\mathcal{C}}[\psi(\hat{u}) - \delta_n\mathcal{B}\psi(\hat{u})]\| \\ &\leqslant \|\eta_n(\mu\vartheta(x_n) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})) + (1-\eta_n)((u_n - \delta_n\mathcal{B}u_n) - (\psi(\hat{u}) - \delta_n\mathcal{B}\psi(\hat{u})))\| \\ &\leqslant \eta_n\|\mu\vartheta(x_n) - \mu\vartheta(\hat{u})\| + \eta_n\|\mu\vartheta(\hat{u}) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})\| \\ &\quad + (1-\eta_n)\|(u_n - \delta_n\mathcal{B}u_n) - (\psi(\hat{u}) - \delta_n\mathcal{B}\psi(\hat{u}))\| \\ &\leqslant \eta_n\mu L\|x_n - \hat{u}\| + \eta_n\|\mu\vartheta(\hat{u}) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})\| + (1-\eta_n)\|u_n - \psi(\hat{u})\| \\ &\leqslant \eta_n\mu L/\delta\|\psi(x_n) - \psi(\hat{u})\| + \eta_n\|\mu\vartheta(\hat{u}) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})\| + (1-\eta_n)\|\psi(x_n) - \psi(\hat{u})\| \\ &= [1 - (1 - \mu L/\delta)\eta_n]\|\psi(x_n) - \psi(\hat{u})\| + \eta_n\|\mu\vartheta(\hat{u}) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})\| \\ &\leqslant [1 - (1 - \mu L/\delta)\eta_n]\|\psi(x_n) - \psi(\hat{u})\| + \eta_n(\|\mu\vartheta(\hat{u}) - \psi(\hat{u})\| + 2\xi\|\mathcal{B}\psi(\hat{u})\|).\end{aligned} \tag{3.8}$$

By combination of (3.6), (3.7), and (3.8), we obtain

$$\begin{aligned}&\|v_n - \psi(\hat{u})\|^2 \\ &\leqslant \|\eta_n(\mu\vartheta(x_n) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})) + (1-\eta_n)((u_n - \delta_n\mathcal{B}u_n) - (\psi(\hat{u}) - \delta_n\mathcal{B}\psi(\hat{u})))\|^2 \\ &\leqslant \eta_n\|\mu\vartheta(x_n) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})\|^2 + (1-\eta_n)\|(u_n - \delta_n\mathcal{B}u_n) - (\psi(\hat{u}) - \delta_n\mathcal{B}\psi(\hat{u}))\|^2 \\ &\leqslant \eta_n\|\mu\vartheta(x_n) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})\|^2 + (1-\eta_n)[\|u_n - \psi(\hat{u})\|^2 \\ &\quad + \delta_n(\delta_n - 2\xi)\|\mathcal{B}u_n - \mathcal{B}\psi(\hat{u})\|^2] \\ &\leqslant \eta_n\|\mu\vartheta(x_n) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})\|^2 + (1-\eta_n)[\|\psi(x_n) - \psi(\hat{u})\|^2 \\ &\quad + \delta_n(\delta_n - 2\xi)\|\mathcal{B}u_n - \mathcal{B}\psi(\hat{u})\|^2 + \zeta_n(\zeta_n - 2\eta)\|\mathcal{A}x_n - \mathcal{A}\hat{u}\|^2] \\ &\leqslant \eta_n\|\mu\vartheta(x_n) - \psi(\hat{u}) + \delta_n\mathcal{B}\psi(\hat{u})\|^2 + (1-\eta_n)\|\psi(x_n) - \psi(\hat{u})\|^2.\end{aligned} \tag{3.9}$$

According to (3.3) and (3.8), we have

$$\|\psi(x_{n+1}) - \psi(\hat{u})\| \leqslant \xi_n\|\psi(x_n) - \psi(\hat{u})\| + (1-\xi_n)\|v_n - \psi(\hat{u})\|$$

$$\begin{aligned}&\leqslant \xi_n\|\psi(x_n)-\psi(\hat{u})\|+(1-\xi_n)[1-(1-\mu L/\delta)\eta_n]\|\psi(x_n)-\psi(\hat{u})\|\\&\quad+(1-\xi_n)\eta_n(\|\mu\vartheta(\hat{u})-\psi(\hat{u})\|+2\xi\|\mathcal{B}\psi(\hat{u})\|)\\&=[1-(1-\mu L/\delta)(1-\xi_n)\eta_n]\|\psi(x_n)-\psi(\hat{u})\|\\&\quad+(1-\mu L/\delta)(1-\xi_n)\eta_n\frac{\|\mu\vartheta(\hat{u})-\psi(\hat{u})\|+2\xi\|\mathcal{B}\psi(\hat{u})\|}{1-\mu L/\delta}.\end{aligned}$$

By induction

$$\|\psi(x_n)-\psi(\hat{u})\|\leqslant \max\left\{\|\psi(x_0)-\psi(\hat{u})\|,\frac{\|\mu\vartheta(\hat{u})-\psi(\hat{u})\|+2\xi\|\mathcal{B}\psi(\hat{u})\|}{1-\mu L/\delta}\right\}.$$

It follows that

$$\|x_n-\hat{u}\|\leqslant\frac{1}{\delta}\|\psi(x_n)-\psi(\hat{u})\|\leqslant\frac{1}{\delta}\max\left\{\|\psi(x_0)-\psi(\hat{u})\|,\frac{\|\mu\vartheta(\hat{u})-\psi(\hat{u})\|+2\xi\|\mathcal{B}\psi(\hat{u})\|}{1-\mu L/\delta}\right\}.$$

Hence, $\{\psi(x_n)\}$, $\{x_n\}$, $\{u_n\}$, and $\{v_n\}$ are all bounded.

From (3.3), we have

$$\psi(x_{n+1})-\psi(x_n)=(1-\xi_n)(v_n-\psi(x_n)). \tag{3.10}$$

Thus,

$$\langle\psi(x_{n+1})-\psi(x_n),\psi(x_n)-\psi(\hat{u})\rangle=(1-\xi_n)\langle v_n-\psi(x_n),\psi(x_n)-\psi(\hat{u})\rangle. \tag{3.11}$$

Observe that

$$\begin{aligned}2\langle\psi(x_{n+1})-\psi(x_n),\psi(x_n)-\psi(\hat{u})\rangle=\|\psi(x_{n+1})-\psi(\hat{u})\|^2-\|\psi(x_n)-\psi(\hat{u})\|^2\\-\|\psi(x_{n+1})-\psi(x_n)\|^2\end{aligned} \tag{3.12}$$

and

$$2\langle v_n-\psi(x_n),\psi(x_n)-\psi(\hat{u})\rangle=\|v_n-\psi(\hat{u})\|^2-\|\psi(x_n)-\psi(\hat{u})\|^2-\|v_n-\psi(x_n)\|^2. \tag{3.13}$$

By virtue of (3.11), (3.12), and (3.13), we deduce

$$\begin{aligned}&\|\psi(x_{n+1})-\psi(\hat{u})\|^2-\|\psi(x_n)-\psi(\hat{u})\|^2-\|\psi(x_{n+1})-\psi(x_n)\|^2\\&\quad=(1-\xi_n)[\|v_n-\psi(\hat{u})\|^2-\|\psi(x_n)-\psi(\hat{u})\|^2-\|v_n-\psi(x_n)\|^2].\end{aligned} \tag{3.14}$$

Combining (3.9), (3.10) with (3.14), we have

$$\begin{aligned}&\|\psi(x_{n+1})-\psi(\hat{u})\|^2-\|\psi(x_n)-\psi(\hat{u})\|^2\\&=(1-\xi_n)[\|v_n-\psi(\hat{u})\|^2-\|\psi(x_n)-\psi(\hat{u})\|^2-\|v_n-\psi(x_n)\|^2]+(1-\xi_n)^2\|v_n-\psi(x_n)\|^2\\&=(1-\xi_n)[\|v_n-\psi(\hat{u})\|^2-\|\psi(x_n)-\psi(\hat{u})\|^2]-\xi_n(1-\xi_n)\|v_n-\psi(x_n)\|^2\\&\leqslant(1-\xi_n)[\eta_n\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2+(1-\eta_n)\|u_n-\psi(\hat{u})\|^2\\&\quad-\|\psi(x_n)-\psi(\hat{u})\|^2]-\xi_n(1-\xi_n)\|v_n-\psi(x_n)\|^2.\end{aligned} \tag{3.15}$$

Returning to (3.8), we get

$$\begin{aligned}\|v_n-\psi(\hat{u})\|^2\leqslant[1-(1-\mu L/\delta)\eta_n]\|\psi(x_n)-\psi(\hat{u})\|^2\\+(1-\mu L/\delta)\eta_n\left(\frac{\|\mu\vartheta(\hat{u})-\psi(\hat{u})\|+2\xi\|\mathcal{B}\psi(\hat{u})\|}{(1-\mu L/\delta)}\right)^2.\end{aligned} \tag{3.16}$$

Next, we consider two possible cases.

Case 1. Assume there exists some integer $m>0$ such that $\{\|\psi(x_n)-\psi(\hat{u})\|\}$ is decreasing for all $n\geqslant m$.

In this case, we know that $\lim_{n\to\infty}\|\psi(x_n)-\psi(\hat{u})\|$ exists. From (3.15) and (3.16), we have

$$\begin{aligned}
&\xi_n(1-\xi_n)\|\nu_n-\psi(x_n)\|^2\\
&\quad\leqslant \|\psi(x_n)-\psi(\hat{u})\|^2-\|\psi(x_{n+1})-\psi(\hat{u})\|^2+(1-\xi_n)[\|\nu_n-\psi(\hat{u})\|^2-\|\psi(x_n)-\psi(\hat{u})\|^2]\\
&\quad\leqslant \|\psi(x_n)-\psi(\hat{u})\|^2-\|\psi(x_{n+1})-\psi(\hat{u})\|^2\\
&\qquad+(1-\mu L/\delta)\eta_n\left(\frac{\|\mu\vartheta(\hat{u})-\psi(\hat{u})\|+2\xi\|\mathcal{B}\psi(\hat{u})\|}{(1-\mu L/\delta)}\right)^2\\
&\quad\to 0.
\end{aligned}$$

This together with (C2) implies that

$$\lim_{n\to\infty}\|\nu_n-\psi(x_n)\|=0.$$

Furthermore, it follows from (3.10) that

$$\lim_{n\to\infty}\|\psi(x_{n+1})-\psi(x_n)\|=0. \tag{3.17}$$

By the convexity of the norm and (3.9), we have

$$\begin{aligned}
\|\psi(x_{n+1})-\psi(\hat{u})\|^2&=\|\xi_n(\psi(x_n)-\psi(\hat{u}))+(1-\xi_n)(\nu_n-\psi(\hat{u}))\|^2\\
&\leqslant \xi_n\|\psi(x_n)-\psi(\hat{u})\|^2+(1-\xi_n)\|\nu_n-\psi(\hat{u})\|^2\\
&\leqslant \xi_n\|\psi(x_n)-\psi(\hat{u})\|^2+\eta_n(1-\xi_n)\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2\\
&\quad+(1-\eta_n)(1-\xi_n)\|\psi(x_n)-\psi(\hat{u})\|^2\\
&\quad+(1-\eta_n)(1-\xi_n)\delta_n(\delta_n-2\xi)\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|^2\\
&\quad+(1-\eta_n)(1-\xi_n)\zeta_n(\zeta_n-2\eta)\|\mathcal{A}x_n-\mathcal{A}\hat{u}\|^2\\
&\leqslant \|\psi(x_n)-\psi(\hat{u})\|^2+\eta_n(1-\xi_n)\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2\\
&\quad+(1-\eta_n)(1-\xi_n)\delta_n(\delta_n-2\xi)\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|^2\\
&\quad+(1-\eta_n)(1-\xi_n)\zeta_n(\zeta_n-2\eta)\|\mathcal{A}x_n-\mathcal{A}\hat{u}\|^2.
\end{aligned} \tag{3.18}$$

Thus,

$$\begin{aligned}
&(1-\eta_n)(1-\xi_n)\delta_n(2\xi-\delta_n)\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|^2+(1-\eta_n)(1-\xi_n)\zeta_n(2\eta-\zeta_n)\|\mathcal{A}x_n-\mathcal{A}\hat{u}\|^2\\
&\quad\leqslant \|\psi(x_n)-\psi(\hat{u})\|^2-\|\psi(x_{n+1})-\psi(\hat{u})\|^2+\eta_n(1-\xi_n)\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2\\
&\quad\leqslant (\|\psi(x_n)-\psi(\hat{u})\|+\|\psi(x_{n+1})-\psi(\hat{u})\|)\|\psi(x_{n+1})-\psi(x_n)\|\\
&\qquad+\eta_n(1-\xi_n)\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2\\
&\quad\to 0\quad \text{(by (C1) and (3.17))}.
\end{aligned}$$

Since $\liminf_{n\to\infty}(1-\xi_n)(1-\eta_n)\zeta_n(2\eta-\zeta_n)>0$ and $\liminf_{n\to\infty}(1-\eta_n)(1-\xi_n)\delta_n(2\xi-\delta_n)>0$, we obtain

$$\lim_{n\to\infty}\|\mathcal{A}x_n-\mathcal{A}\hat{u}\|=0\quad\text{and}\quad \lim_{n\to\infty}\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|=0. \tag{3.19}$$

Set $z_n=\psi(x_n)-\zeta_n\mathcal{A}x_n-(\psi(\hat{u})-\zeta_n\mathcal{A}\hat{u})$ for all n. Using the property (2.1) of projection, we get

$$\begin{aligned}
\|u_n-\psi(\hat{u})\|^2&=\|\mathrm{proj}_{\mathcal{C}}[\psi(x_n)-\zeta_n\mathcal{A}x_n]-\mathrm{proj}_{\mathcal{C}}[\psi(\hat{u})-\zeta_n\mathcal{A}\hat{u}]\|^2\\
&\leqslant\langle z_n,u_n-\psi(\hat{u})\rangle\\
&=\frac{1}{2}\{\|z_n\|^2+\|u_n-\psi(\hat{u})\|^2-\|z_n-u_n+\psi(\hat{u})\|^2\}\\
&\leqslant\frac{1}{2}\{\|\psi(x_n)-\psi(\hat{u})\|^2+\|u_n-\psi(\hat{u})\|^2-\|\psi(x_n)-u_n-\zeta_n(\mathcal{A}x_n-\mathcal{A}\hat{u})\|^2\}
\end{aligned}$$

$$= \frac{1}{2}\{\|\psi(x_n)-\psi(\hat{u})\|^2 + \|u_n-\psi(\hat{u})\|^2 - \|\psi(x_n)-u_n\|^2 \\ - \zeta_n^2\|\mathcal{A}x_n-\mathcal{A}\hat{u}\|^2 + 2\zeta_n\langle\psi(x_n)-u_n, \mathcal{A}x_n-\mathcal{A}\hat{u}\rangle\}.$$

It follows that

$$\|u_n-\psi(\hat{u})\|^2 \leqslant \|\psi(x_n)-\psi(\hat{u})\|^2 - \|\psi(x_n)-u_n\|^2 + 2\zeta_n\|\psi(x_n)-u_n\|\|\mathcal{A}x_n-\mathcal{A}\hat{u}\|. \tag{3.20}$$

Set $z'_n = u_n - \delta_n\mathcal{B}u_n - (\psi(\hat{u})-\delta_n\mathcal{B}\psi(\hat{u}))$ for all n. We also have

$$\begin{aligned}\|v_n-\psi(\hat{u})\|^2 &= \|\mathrm{proj}_{\mathcal{C}}[\eta_n\mu\vartheta(x_n)+(1-\eta_n)(u_n-\delta_n\mathcal{B}u_n)] - \mathrm{proj}_{\mathcal{C}}[\psi(\hat{u})-\delta_n\mathcal{B}\psi(\hat{u})]\|^2\\ &\leqslant \langle\eta_n(\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u}))+(1-\eta_n)z'_n, v_n-\psi(\hat{u})\rangle\\ &= \frac{1}{2}\{\|\eta_n(\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u}))+(1-\eta_n)z'_n\|^2 + \|v_n-\psi(\hat{u})\|^2\\ &\quad - \|\eta_n(\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u}))+(1-\eta_n)z'_n - v_n+\psi(\hat{u})\|^2\}\\ &\leqslant \frac{1}{2}\{\eta_n\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2 + (1-\eta_n)\|u_n-\psi(\hat{u})\|^2\\ &\quad + \|v_n-\psi(\hat{u})\|^2 - \|\eta_n(\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z'_n)\\ &\quad + u_n-v_n-\delta_n(\mathcal{B}u_n-\mathcal{B}\psi(\hat{u}))\|^2\}\\ &= \frac{1}{2}\{\eta_n\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2 + (1-\eta_n)\|u_n-\psi(\hat{u})\|^2\\ &\quad + \|v_n-\psi(\hat{u})\|^2 - \|u_n-v_n\|^2 - \eta_n^2\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z'_n\|^2\\ &\quad + 2\delta_n\eta_n\langle\mathcal{B}u_n-\mathcal{B}\psi(\hat{u}), \mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z'_n\rangle\\ &\quad - \delta_n^2\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|^2 + 2\delta_n\langle u_n-v_n, \mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\rangle\\ &\quad - 2\eta_n\langle u_n-v_n, \mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z'_n\rangle\}.\end{aligned}$$

It follows that

$$\begin{aligned}\|v_n-\psi(\hat{u})\|^2 \leqslant \eta_n\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2 &+ (1-\eta_n)\|u_n-\psi(\hat{u})\|^2\\ &+ 2\delta_n\eta_n\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z'_n\|\\ &- \|u_n-v_n\|^2 + 2\delta_n\|u_n-v_n\|\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|\\ &+ 2\eta_n\|u_n-v_n\|\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z'_n\|.\end{aligned} \tag{3.21}$$

In the light of (3.18), (3.20), and (3.21), we derive

$$\begin{aligned}&\|\psi(x_{n+1})-\psi(\hat{u})\|^2\\ &\quad\leqslant \xi_n\|\psi(x_n)-\psi(\hat{u})\|^2 + (1-\xi_n)\|v_n-\psi(\hat{u})\|^2\\ &\quad\leqslant \|\psi(x_n)-\psi(\hat{u})\|^2 + \eta_n\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2 - (1-\eta_n)\|\psi(x_n)-u_n\|^2\\ &\qquad - (1-\xi_n)\|u_n-v_n\|^2 + 2\zeta_n\|\psi(x_n)-u_n\|\|\mathcal{A}x_n-\mathcal{A}\hat{u}\| + 2\delta_n\|u_n-v_n\|\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|\\ &\qquad + 2\delta_n\eta_n\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z'_n\|\\ &\qquad + 2\eta_n\|u_n-v_n\|\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z'_n\|.\end{aligned}$$

Then,

$$\begin{aligned}&(1-\eta_n)\|\psi(x_n)-u_n\|^2 + (1-\xi_n)\|u_n-v_n\|^2\\ &\quad\leqslant (\|\psi(x_n)-\psi(\hat{u})\| + \|\psi(x_{n+1})-\psi(\hat{u})\|)\|\psi(x_{n+1})-\psi(x_n)\|\\ &\qquad + 2\zeta_n\|\psi(x_n)-u_n\|\|\mathcal{A}x_n-\mathcal{A}\hat{u}\| + 2\delta_n\|u_n-v_n\|\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|\\ &\qquad + 2\delta_n\eta_n\|\mathcal{B}u_n-\mathcal{B}\psi(\hat{u})\|\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z'_n\|\end{aligned}$$

$$\begin{aligned}&+2\eta_n\|u_n-v_n\|\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})-z_n'\|\\&+\eta_n\|\mu\vartheta(x_n)-\psi(\hat{u})+\delta_n\mathcal{B}\psi(\hat{u})\|^2\\ \to\ & 0 \quad (\text{by (C1), (3.17), and (3.19)}).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\|\psi(x_n)-u_n\|=0 \quad \text{and} \quad \lim_{n\to\infty}\|u_n-v_n\|=0. \tag{3.22}$$

Now, we prove $\overline{\lim}_{n\to\infty}\langle\mu\vartheta(\hat{u})-\psi(\hat{u}),v_n-\psi(\hat{u})\rangle\leqslant 0$. Choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\overline{\lim}_{n\to\infty}\langle\mu\vartheta(\hat{u})-\psi(\hat{u}),v_n-\psi(\hat{u})\rangle=\lim_{i\to\infty}\langle\mu\vartheta(\hat{u})-\psi(\hat{u}),v_{n_i}-\psi(\hat{u})\rangle. \tag{3.23}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some point $z\in\mathcal{C}$. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup z$. This implies that $\psi(x_{n_i})\rightharpoonup\psi(z)$ due to the weak continuity of ψ. Thus, $v_{n_i}\rightharpoonup\psi(z)$ by (3.22). Next, we show $z\in VI(\mathcal{A},\psi,\mathcal{C})$. Set

$$Ru=\begin{cases}\mathcal{A}u+N_{\mathcal{C}}(u), & u\in\mathcal{C},\\ \emptyset, & u\notin\mathcal{C}.\end{cases}$$

By [35], we know that R is maximal ψ-monotone. Let $(u,w)\in G(R)$. Since $w-Bu\in N_{\mathcal{C}}(u)$ and $x_n\in\mathcal{C}$, we have $\langle\psi(u)-\psi(x_n),w-\mathcal{A}u\rangle\geqslant 0$. Noting that $u_n=\mathrm{proj}_{\mathcal{C}}[\psi(x_n)-\zeta_n\mathcal{A}x_n]$, we get

$$\langle\psi(u)-u_n,u_n-(\psi(x_n)-\zeta_n\mathcal{A}x_n)\rangle\geqslant 0.$$

It follows that

$$\langle\psi(u)-u_n,\frac{u_n-\psi(x_n)}{\zeta_n}+\mathcal{A}x_n\rangle\geqslant 0.$$

Then,

$$\begin{aligned}\langle\psi(u)-\psi(x_{n_i}),w\rangle&\geqslant\langle\psi(u)-\psi(x_{n_i}),\mathcal{A}v\rangle\\&\geqslant\langle\psi(u)-\psi(x_{n_i}),\mathcal{A}u\rangle-\left\langle\psi(u)-u_{n_i},\frac{u_{n_i}-\psi(x_{n_i})}{\zeta_{n_i}}\right\rangle-\langle\psi(u)-u_{n_i},\mathcal{A}x_{n_i}\rangle\\&=\langle\psi(u)-\psi(x_{n_i}),\mathcal{A}u-\mathcal{A}x_{n_i}\rangle+\langle\psi(u)-\psi(x_{n_i}),\mathcal{A}x_{n_i}\rangle\\&\quad-\left\langle\psi(u)-u_{n_i},\frac{u_{n_i}-\psi(x_{n_i})}{\zeta_{n_i}}\right\rangle-\langle\psi(u)-u_{n_i},\mathcal{A}x_{n_i}\rangle\\&\geqslant-\left\langle\psi(u)-u_{n_i},\frac{u_{n_i}-\psi(x_{n_i})}{\zeta_{n_i}}\right\rangle-\langle\psi(x_{n_i})-u_{n_i},\mathcal{A}x_{n_i}\rangle.\end{aligned} \tag{3.24}$$

Since $\|\psi(x_{n_i})-u_{n_i}\|\to 0$ and $\psi(x_{n_i})\rightharpoonup\psi(z)$, we deduce that $\langle\psi(u)-\psi(z),w\rangle\geqslant 0$ by taking $i\to\infty$ in (3.24). Thus, $z\in R^{-1}0$ by the maximal ψ-monotonicity of R. Hence, $z\in VI(\mathcal{A},\psi,\mathcal{C})$.

Next, we need to prove $\psi(z)\in VI(\mathcal{B},\mathcal{C})$. Set

$$R'v=\begin{cases}\mathcal{B}v+N_{\mathcal{C}}(v), & v\in\mathcal{C},\\ \emptyset, & v\notin\mathcal{C}.\end{cases}$$

By [21], we know that R' is maximal monotone. Let $(v,w)\in G(R')$. Since $w-\mathcal{B}v\in N_{\mathcal{C}}(v)$ and $v_n\in\mathcal{C}$, we have $\langle v-v_n,w-\mathcal{B}v\rangle\geqslant 0$. Noting that $v_n=\mathrm{proj}_{\mathcal{C}}[\eta_n\mu\vartheta(x_n)+(1-\eta_n)(u_n-\delta_n\mathcal{B}u_n)]$, we get

$$\langle v-v_n,v_n-[\eta_n\mu\vartheta(x_n)+(1-\eta_n)(u_n-\delta_n\mathcal{B}u_n)]\rangle\geqslant 0.$$

It follows that

$$\left\langle v-v_n,\frac{v_n-u_n}{\delta_n}+\mathcal{B}u_n-\frac{\eta_n}{\delta_n}(\mu\vartheta(x_n)-u_n+\delta_n\mathcal{B}u_n)\right\rangle\geqslant 0.$$

Then,

$$\begin{aligned}
\langle v - v_{n_i}, w\rangle &\geqslant \langle v - v_{n_i}, \mathcal{B}v\rangle \\
&\geqslant \langle v - v_{n_i}, \mathcal{B}v\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\delta_{n_i}}\right\rangle - \langle v - v_{n_i}, \mathcal{B}u_{n_i}\rangle \\
&\quad + \frac{\eta_{n_i}}{\delta_{n_i}}\langle v - v_{n_i}, \mu\vartheta(x_{n_i}) - u_{n_i} + \delta_{n_i}\mathcal{B}u_{n_i}\rangle \\
&= \langle v - v_{n_i}, \mathcal{B}v - \mathcal{B}v_{n_i}\rangle + \langle v - v_{n_i}, \mathcal{B}v_{n_i}\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\delta_{n_i}}\right\rangle \\
&\quad - \langle v - v_{n_i}, \mathcal{B}u_{n_i}\rangle + \frac{\eta_{n_i}}{\delta_{n_i}}\langle v - v_{n_i}, \mu\vartheta(x_{n_i}) - u_{n_i} + \delta_{n_i}\mathcal{B}u_{n_i}\rangle \\
&\geqslant -\left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\delta_{n_i}}\right\rangle - \langle v - v_{n_i}, \mathcal{B}u_{n_i} - \mathcal{B}v_{n_i}\rangle \\
&\quad + \frac{\eta_{n_i}}{\delta_{n_i}}\langle v - v_{n_i}, \mu\vartheta(x_{n_i}) - u_{n_i} + \delta_{n_i}\mathcal{B}u_{n_i}\rangle.
\end{aligned} \tag{3.25}$$

Since $\|v_{n_i} - u_{n_i}\| \to 0$ and $v_{n_i} \rightharpoonup \psi(z)$, we deduce that $\langle v - \psi(z), w\rangle \geqslant 0$ by taking $i \to \infty$ in (3.25). Thus, $\psi(z) \in R'^{-1}0$ by the maximal monotonicity of R'. Hence, $\psi(z) \in VI(\mathcal{B}, \mathcal{C})$. Therefore, $z \in \Lambda$.

From (3.23), we obtain

$$\begin{aligned}
\overline{\lim}_{n\to\infty}\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), v_n - \psi(\hat{u})\rangle &= \lim_{i\to\infty}\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), \psi(x_{n_i}) - \psi(\hat{u})\rangle \\
&= \langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), \psi(z) - \psi(\hat{u})\rangle \leqslant 0.
\end{aligned} \tag{3.26}$$

Note that

$$\begin{aligned}
\|v_n - \psi(\hat{u})\|^2 &= \|\mathrm{proj}_{\mathcal{C}}[\eta_n\mu\vartheta(x_n) + (1-\eta_n)(u_n - \delta_n\mathcal{B}u_n)] - \mathrm{proj}_{\mathcal{C}}[\psi(\hat{u}) - (1-\eta_n)\delta_n\mathcal{B}\psi(\hat{u})]\|^2 \\
&\leqslant \langle \eta_n(\mu\vartheta(x_n) - \psi(\hat{u})) + (1-\eta_n)z'_n, v_n - \psi(\hat{u})\rangle \\
&\leqslant \eta_n\mu\langle \vartheta(x_n) - \vartheta(\hat{u}), v_n - \psi(\hat{u})\rangle + \eta_n\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), v_n - \psi(\hat{u})\rangle \\
&\quad + (1-\eta_n)\|u_n - \delta_n\mathcal{B}u_n - (\psi(\hat{u}) - \delta_n\mathcal{B}\psi(\hat{u}))\|\|v_n - \psi(\hat{u})\| \\
&\leqslant \eta_n L\mu\|x_n - \hat{u}\|\|v_n - \psi(\hat{u})\| + \eta_n\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), v_n - \psi(\hat{u})\rangle \\
&\quad + (1-\eta_n)\|u_n - \psi(\hat{u})\|\|v_n - \psi(\hat{u})\| \\
&\leqslant \eta_n(\mu L/\delta)\|\psi(x_n) - \psi(\hat{u})\|\|v_n - \psi(\hat{u})\| + \eta_n\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), v_n - \psi(\hat{u})\rangle \\
&\quad + (1-\eta_n)\|\psi(x_n) - \psi(\hat{u})\|\|v_n - \psi(\hat{u})\| \\
&= [1 - (1 - L\mu/\delta)\eta_n]\|\psi(x_n) - \psi(\hat{u})\|\|v_n - \psi(\hat{u})\| + \eta_n\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), v_n - \psi(\hat{u})\rangle \\
&\leqslant \frac{1 - (1 - L\mu/\delta)\eta_n}{2}\|\psi(x_n) - \psi(\hat{u})\|^2 + \frac{1}{2}\|v_n - \psi(\hat{u})\|^2 + \eta_n\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), v_n - \psi(\hat{u})\rangle.
\end{aligned}$$

It follows that

$$\|v_n - \psi(\hat{u})\|^2 \leqslant [1 - (1 - L\mu/\delta)\eta_n]\|\psi(x_n) - \psi(\hat{u})\|^2 + 2\eta_n\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), v_n - \psi(\hat{u})\rangle.$$

Therefore,

$$\begin{aligned}
\|\psi(x_{n+1}) - \psi(\hat{u})\|^2 &\leqslant \xi_n\|\psi(x_n) - \psi(\hat{u})\|^2 + (1-\xi_n)\|v_n - \psi(\hat{u})\|^2 \\
&\leqslant \xi_n\|\psi(x_n) - \psi(\hat{u})\|^2 + (1-\xi_n)[1 - (1 - \mu L/\delta)\eta_n]\|\psi(x_n) - \psi(\hat{u})\|^2 \\
&\quad + 2(1-\xi_n)\eta_n\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), v_n - \psi(\hat{u})\rangle \\
&= [1 - (1 - \mu L/\delta)(1-\xi_n)\eta_n]\|\psi(x_n) - \psi(\hat{u})\|^2 \\
&\quad + 2(1-\xi_n)\eta_n\langle \mu\vartheta(\hat{u}) - \psi(\hat{u}), v_n - \psi(\hat{u})\rangle \\
&= [1 - (1 - \mu L/\delta)(1-\xi_n)\eta_n]\|\psi(x_n) - \psi(\hat{u})\|^2
\end{aligned} \tag{3.27}$$

$$+ (1-\mu L/\delta)(1-\xi_n)\eta_n\left(\frac{2}{1-\mu L/\delta}\langle \mu\vartheta(\hat{u})-\psi(\hat{u}), \nu_n-\psi(\hat{u})\rangle\right).$$

We can apply Lemma 2.6 to (3.27) to conclude that $\psi(x_n) \to \psi(\hat{u})$ and $x_n \to \hat{u}$.

Case 2. Assume there exists an integer n_0 such that $\|\psi(x_{n_0})-\psi(\hat{u})\| \leqslant \|\psi(x_{n_0+1})-\psi(\hat{u})\|$. At this case, we set $\omega_n = \{\|\psi(x_n)-\psi(\hat{u})\|\}$. Then, we have $\omega_{n_0} \leqslant \omega_{n_0+1}$. Define an integer sequence $\{\tau_n\}$ for all $n \geqslant n_0$ as follows:

$$\tau(n) = \max\{l \in \mathbb{N} | n_0 \leqslant l \leqslant n, \omega_l \leqslant \omega_{l+1}\}.$$

It is clear that $\tau(n)$ is a non-decreasing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty$$

and

$$\omega_{\tau(n)} \leqslant \omega_{\tau(n)+1},$$

for all $n \geqslant n_0$.

By the similar argument as that of (3.26) and (3.27), we can prove that

$$\overline{\lim}_{n\to\infty}\langle \mu\vartheta(\hat{u})-\psi(\hat{u}), \nu_{\tau(n)}-\psi(\hat{u})\rangle \leqslant 0 \tag{3.28}$$

and

$$\begin{aligned}\omega^2_{\tau(n)+1} \leqslant{}& [1-(1-\mu L/\delta)(1-\xi_{\tau(n)})\eta_{\tau(n)}]\omega^2_{\tau(n)} \\ &+ (1-\mu L/\delta)(1-\xi_{\tau(n)})\eta_{\tau(n)}\left(\frac{2}{1-\mu L/\delta}\langle \mu\vartheta(\hat{u})-\psi(\hat{u}), \nu_{\tau(n)}-\psi(\hat{u})\rangle\right).\end{aligned} \tag{3.29}$$

Since $\omega_{\tau(n)} \leqslant \omega_{\tau(n)+1}$, we have from (3.29) that

$$\omega^2_{\tau(n)} \leqslant \frac{2}{1-\mu L/\delta}\langle \mu\vartheta(\hat{u})-\psi(\hat{u}), \nu_{\tau(n)}-\psi(\hat{u})\rangle. \tag{3.30}$$

Combining (3.28) with (3.30), we have

$$\overline{\lim}_{n\to\infty}\omega_{\tau(n)} \leqslant 0,$$

and hence

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0. \tag{3.31}$$

From (3.28) and (3.29), we also obtain

$$\overline{\lim}_{n\to\infty}\omega_{\tau(n)+1} \leqslant \overline{\lim}_{n\to\infty}\omega_{\tau(n)}.$$

This together with (3.31) implies that

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

Applying Lemma 2.7 we get

$$0 \leqslant \omega_n \leqslant \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$. That is, $x_n \to \hat{u}$. This completes the proof. □

Algorithm 3.7. For given initial value $x_0 \in \mathcal{C}$, define the sequence $\{x_n\}$ by the following form

$$\begin{cases} u_n = \mathrm{proj}_{\mathcal{C}}[x_n - \zeta_n \mathcal{A} x_n], \\ \nu_n = \mathrm{proj}_{\mathcal{C}}[\eta_n \mu\vartheta(x_n) + (1-\eta_n)(u_n - \delta_n \mathcal{B} u_n)], \\ x_{n+1} = \xi_n x_n + (1-\xi_n)\nu_n, n \geqslant 0, \end{cases} \tag{3.32}$$

where $\vartheta : \mathcal{C} \to \mathcal{H}$ is an L-Lipschitz continuous mapping, $\{\eta_n\}$, $\{\xi_n\}$, and $\{\delta_n\}$ are three real number sequences in $[0, 1]$, $\{\zeta_n\}$ is a real number sequence in $(0, \infty)$, and $0 < \mu < 1/L$ is a constant.

Corollary 3.8. *Assume the following conditions are satisfied:*

(C1): $\lim_{n\to\infty} \eta_n = 0$ *and* $\sum_n \eta_n = \infty$;
(C2): $0 < \underline{\lim}_{n\to\infty} \xi_n \leqslant \overline{\lim}_{n\to\infty} \xi_n < 1$;
(C3): $0 < \underline{\lim}_{n\to\infty} \zeta_n \leqslant \overline{\lim}_{n\to\infty} \zeta_n < 2\eta$ *and* $0 < \underline{\lim}_{n\to\infty} \delta_n \leqslant \overline{\lim}_{n\to\infty} \delta_n < 2\xi$.

Then the sequence $\{x_n\}$ *generated by* (3.32) *converges strongly to* $\hat{u} \in VI(A, C) \cap VI(B, C)$ *which solves the following variational inequality*

$$\langle \mu\vartheta(\hat{u}) - \hat{u}, \hat{v} - \hat{u} \rangle \leqslant 0, \quad \forall \hat{v} \in VI(A, C) \cap VI(B, C).$$

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