



On the Henstock-Kurzweil integral for Riesz-space-valued functions on time scales

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Abstract

We introduce and investigate the Henstock-Kurzweil (HK) integral for Riesz-space-valued functions on time scales. Some basic properties of the HK delta integral for Riesz-space-valued functions are proved. Further, we prove uniform and monotone convergence theorems. ©2017 All rights reserved.

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1. Introduction

It is well-known that the Henstock-Kurzweil integral integrates highly oscillating functions and encompasses Newton, Riemann and Lebesgue integrals. This integral was introduced by Kurzweil and Henstock independently in 1957/58 [34, 39]. It has been shown that the Henstock-Kurzweil integral is equivalent to the Denjoy-Perron integral. For fundamental results and some applications in the theory of Henstock-Kurzweil integration, we refer the reader to the papers [23–25, 27, 32–34, 60, 61, 64, 66] and monographs [4, 30, 35, 36, 38, 40–45, 48, 55, 56].

A time scale \mathbb{T} is an arbitrary nonempty closed subset of real numbers \mathbb{R} with the subspace topology inherited from the standard topology of \mathbb{R} . The theory of time scales was born in 1988 with the Ph.D. thesis of Hilger [37]. The aim of this theory is to unify various definitions and results from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems. It has been extensively studied on various aspects by several authors; see, e.g., [5, 6, 19–22, 31, 46]. In [47], Peterson and Thompson introduced a more general concept of integral on time scales, i.e., the Henstock-Kurzweil delta integral, which contains the Riemann delta and the Lebesgue delta integrals as special cases. The theory of Henstock-Kurzweil integration for real-valued

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and vector-valued functions on time scales has developed rather intensively in the past few years; see, for instance, the papers [1–3, 26, 28, 54, 57–59, 62, 65] and the references cited therein.

One of the interesting points of integration theories is the problem when functions with values in general spaces have to be integrated. The Henstock-Kurzweil integral for Riesz-space-valued functions was investigated in [7–18, 49–53, 63]. Surprisingly enough, the Henstock-Kurzweil integral for Riesz-space-valued functions has not received attention in the literature of time scales. The main goal of this paper is to generalize the results above by constructing the Henstock-Kurzweil integral for Riesz-space-valued functions on time scales.

The paper is organized as follows. Section 2 contains basic concepts of Riesz space, time scales and Henstock-Kurzweil integral. In Section 3, the definition of Henstock-Kurzweil delta integral for Riesz-space-valued functions is introduced, and the basic properties of the Henstock-Kurzweil delta integral for Riesz-space-valued functions are investigated. In Section 4, we prove a uniformly convergence theorem and a monotone convergence theorem for the Henstock-Kurzweil delta integral for Riesz-space-valued functions.

2. Preliminaries

The following conventions and notations will be used, unless stated otherwise. Let \mathbb{N} , \mathbb{R} , and \mathbb{R}^+ be the sets of all natural, real, and positive real numbers, respectively, and let X be a Riesz space. A decreasing sequence $(b_n)_n$ in X , such that $\bigwedge_n b_n = 0$, is called an (o) -sequence. A bounded double sequence $(a_{i,j})_{i,j}$ in X is a (D) -sequence or a regulator if $(a_{i,j})_{i,j}$ is an (o) -sequence for all $i \in \mathbb{N}$. A Riesz space X is said to be Dedekind complete if every nonempty subset X_1 of X , bounded from above, has a lattice supremum in X denoted by $\bigvee X_1$. A Dedekind complete Riesz space X is said to be weakly σ -distributive if for every (D) -sequence $(a_{i,j})_{i,j}$ in X one has:

$$\bigwedge_{\varphi \in \mathbb{N}^\mathbb{N}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0.$$

Let \mathbb{T} be a time scale, i.e., a nonempty closed subset of \mathbb{R} . For $a, b \in \mathbb{T}$, we define the closed interval $[a, b]_\mathbb{T}$ by $[a, b]_\mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}$. The open and half-open intervals are defined in a similar way. For $t \in \mathbb{T}$, we define the forward jump operator σ by $\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$, where $\inf \emptyset = \sup\{\mathbb{T}\}$, while the backward jump operator ρ is defined by $\rho(t) = \sup\{s < t : s \in \mathbb{T}\}$, where $\sup \emptyset = \inf\{\mathbb{T}\}$.

If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$, then we say that t is left-scattered. If $\sigma(t) = t$, then we say that t is right-dense, while if $\rho(t) = t$, then we say that t is left-dense. A point $t \in \mathbb{T}$ is dense if it is right and left dense; isolated, if it is right and left scattered. The forward graininess function $\mu(t)$ and the backward graininess function $\eta(t)$ are defined by $\mu(t) = \sigma(t) - t$ and $\eta(t) = t - \rho(t)$ for all $t \in \mathbb{T}$, respectively. If $\sup \mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^k := \mathbb{T} \setminus \sup \mathbb{T}$, otherwise $\mathbb{T}^k := \mathbb{T}$; if $\inf \mathbb{T}$ is finite and right-scattered, then $\mathbb{T}_k := \mathbb{T} \setminus \inf \mathbb{T}$, otherwise $\mathbb{T}_k := \mathbb{T}$. We set $\mathbb{T}_k^k := \mathbb{T}^k \cap \mathbb{T}_k$.

Throughout this paper, all considered intervals will be intervals in \mathbb{T} . A partition \mathcal{D} of $[a, b]_\mathbb{T}$ is a finite collection of interval-point pairs $\{([t_{i-1}, t_i]_\mathbb{T}, \xi_i)\}_{i=1}^n$, where

$$\{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$$

and $\xi_i \in [a, b]_\mathbb{T}$ for $i = 1, 2, \dots, n$. By $\Delta t_i = t_i - t_{i-1}$ we denote the length of the i th subinterval in the partition \mathcal{D} . We say that $\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$ is Δ -gauge for $[a, b]_\mathbb{T}$ provided $\delta_L(\xi) > 0$ on $(a, b]_\mathbb{T}$, $\delta_R(\xi) > 0$ on $[a, b)_\mathbb{T}$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$ and $\delta_R(\xi) \geq \mu(\xi)$ for all $\xi \in [a, b]_\mathbb{T}$. Let $\delta^1(\xi), \delta^2(\xi)$ be Δ -gauges for $[a, b]_\mathbb{T}$ such that $0 < \delta_L^1(\xi) < \delta_L^2(\xi)$ for all $\xi \in (a, b]_\mathbb{T}$ and $0 < \delta_R^1(\xi) < \delta_R^2(\xi)$ for all $\xi \in [a, b)_\mathbb{T}$. We say $\delta^1(\xi)$ is finer than $\delta^2(\xi)$ and write $\delta^1(\xi) < \delta^2(\xi)$. We say that $\mathcal{D} = \{([t_{i-1}, t_i]_\mathbb{T}, \xi_i)\}_{i=1}^n$ is

- (1) a partial partition of $[a, b]_\mathbb{T}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_\mathbb{T} \subset [a, b]_\mathbb{T}$;
- (2) a partition of $[a, b]_\mathbb{T}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_\mathbb{T} = [a, b]_\mathbb{T}$;

- (3) a δ -fine Henstock-Kurzweil (HK) partition of $[a, b]_{\mathbb{T}}$ if $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}} \subset (\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i))_{\mathbb{T}}$ for all $i = 1, 2, \dots, n$.

Given a δ -fine HK partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ of $[a, b]_{\mathbb{T}}$, we write

$$S(f, \mathcal{D}, \delta) = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1})$$

for integral sums over D , whenever $f : [a, b]_{\mathbb{T}} \rightarrow X$. In what follows, we shall always assume that X is a Dedekind complete weakly σ -distributive Riesz space.

3. The Henstock-Kurzweil delta integral for Riesz-space-valued functions

Before formulating and giving the proof of our first result, we need the following definition.

Definition 3.1. A function $f : [a, b]_{\mathbb{T}} \rightarrow X$ is called Henstock-Kurzweil delta integrable (HK Δ -integrable) on $[a, b]_{\mathbb{T}}$, if there exists $x \in X$ and a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a Δ -gauge, δ , for $[a, b]_{\mathbb{T}}$, such that

$$|S(f, \mathcal{D}, \delta) - x| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for each δ -fine HK partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ of $[a, b]_{\mathbb{T}}$. In this case, x is called the HK Δ -integral of f on $[a, b]_{\mathbb{T}}$ and is denoted by $x = \int_a^b f(t) \Delta t$.

Theorem 3.2. If $f : [a, b]_{\mathbb{T}} \rightarrow X$ is HK Δ -integrable on $[a, b]_{\mathbb{T}}$, then the integral of f is determined uniquely.

Proof. Suppose there exist $x_1, x_2 \in X$ and (D)-sequences $(a_{i,j})_{i,j}, (b_{i,j})_{i,j}$ of elements of X and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist two Δ -gauges, δ_1, δ_2 , for $[a, b]_{\mathbb{T}}$, such that

$$|S(f, \mathcal{D}_1, \delta_1) - x_1| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}, \quad |S(f, \mathcal{D}_2, \delta_2) - x_2| < \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

for each δ_1 -fine HK partition \mathcal{D}_1 and δ_2 -fine HK partition \mathcal{D}_2 , respectively. Let $\delta = \min\{\delta_1, \delta_2\}$ and $(c_{i,j})_{i,j}$ be a (D)-sequence of elements of X such that

$$\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$$

for every $\varphi \in \mathbb{N}^{\mathbb{N}}$. Then,

$$|x_1 - x_2| \leq |S(f, \mathcal{D}, \delta) - x_1| + |S(f, \mathcal{D}, \delta) - x_2| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$$

for each δ -fine HK partition \mathcal{D} . Since X is weakly σ -distributive, we obtain that

$$|x_1 - x_2| \leq \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \right) = 0.$$

The proof is complete. \square

Theorem 3.3. If $f, g : [a, b]_{\mathbb{T}} \rightarrow X$ are HK Δ -integrable on $[a, b]_{\mathbb{T}}$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is HK Δ -integrable on $[a, b]_{\mathbb{T}}$ and

$$\int_a^b (\alpha f(t) + \beta g(t)) \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t.$$

Proof. We shall prove that if f, g are HK Δ -integrable on $[a, b]_{\mathbb{T}}$ and $c \in \mathbb{R}$, then $f + g$ and cf are HK Δ -integrable too and

$$\int_a^b (f(t) + g(t)) \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t, \quad \int_a^b cf(t) \Delta t = c \int_a^b f(t) \Delta t.$$

If f is HK Δ -integrable on $[a, b]_{\mathbb{T}}$, then there exists a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a Δ -gauge, δ_1 , for $[a, b]_{\mathbb{T}}$, such that

$$\left| S(f, \mathcal{D}_1, \delta_1) - \int_a^b f(t) \Delta t \right| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for each δ_1 -fine HK partition \mathcal{D}_1 . Similarly, there exists a (D)-sequence $(b_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a Δ -gauge, δ_2 , for $[a, b]_{\mathbb{T}}$, such that

$$\left| S(g, \mathcal{D}_2, \delta_2) - \int_a^b g(t) \Delta t \right| < \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

for each δ_2 -fine HK partition \mathcal{D}_2 . Let $\delta = \min\{\delta_1, \delta_2\}$, and consider a (D)-sequence $(c_{i,j})_{i,j}$ of elements of X such that

$$\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$$

for every $\varphi \in \mathbb{N}^{\mathbb{N}}$. Then,

$$\begin{aligned} \left| S(f+g, \mathcal{D}, \delta) - \int_a^b f(t) \Delta t - \int_a^b g(t) \Delta t \right| &= \left| S(f, \mathcal{D}, \delta) - \int_a^b f(t) \Delta t + S(g, \mathcal{D}, \delta) - \int_a^b g(t) \Delta t \right| \\ &\leq \left| S(f, \mathcal{D}, \delta) - \int_a^b f(t) \Delta t \right| + \left| S(g, \mathcal{D}, \delta) - \int_a^b g(t) \Delta t \right| \\ &< \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \end{aligned}$$

for each δ -fine HK partition \mathcal{D} . Hence, $f + g$ is HK Δ -integrable and

$$\int_a^b (f(t) + g(t)) \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t.$$

For $c \in \mathbb{R}$, $(|c|a_{i,j})_{i,j}$ is a (D)-sequence. Then,

$$\left| S(cf, \mathcal{D}, \delta) - c \int_a^b f(t) \Delta t \right| \leq |c| \left| S(f, \mathcal{D}, \delta) - \int_a^b f(t) \Delta t \right| < |c| \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} = \bigvee_{i=1}^{\infty} |c| a_{i,\varphi(i)}$$

for each δ -fine HK partition \mathcal{D} . This implies that cf is HK Δ -integrable and

$$\int_a^b cf(t) \Delta t = c \int_a^b f(t) \Delta t.$$

The proof is complete. \square

Theorem 3.4 (Cauchy-Bolzano condition). *A function $f : [a, b]_{\mathbb{T}} \rightarrow X$ is HK Δ -integrable on $[a, b]_{\mathbb{T}}$ if and only if there exists a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a Δ -gauge, δ , for $[a, b]_{\mathbb{T}}$, such that*

$$|S(f, \mathcal{D}_1, \delta) - S(f, \mathcal{D}_2, \delta)| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for each δ -fine HK partition $\mathcal{D}_1, \mathcal{D}_2$ of $[a, b]_{\mathbb{T}}$.

Proof.

(Necessity). This follows from the inequality

$$|S(f, \mathcal{D}_1, \delta) - S(f, \mathcal{D}_2, \delta)| \leq \left| S(f, \mathcal{D}_1, \delta) - \int_a^b f(t) \Delta t \right| + \left| S(f, \mathcal{D}_2, \delta) - \int_a^b f(t) \Delta t \right|$$

and some routine arguments.

(Sufficiency). To every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a Δ -gauge $\delta_{\varphi}(\xi)$ with the following property. Let

$$\delta_{[a,b]_T} = \{ \delta(\xi) : \exists \varphi \in \mathbb{N}^{\mathbb{N}}, \delta(\xi) = \delta_{\varphi}(\xi), \xi \in [a, b]_T \}.$$

Then, for $\delta(\xi) \in \delta_{[a,b]_T}$ and a δ -fine HK partition \mathcal{D} , the set $\{S(f, \mathcal{D}, \delta)\}$ is bounded. Indeed, for X boundedly complete, there exist

$$a_{\delta} = \bigwedge_{\mathcal{D}} S(f, \mathcal{D}, \delta), \quad b_{\delta} = \bigvee_{\mathcal{D}} S(f, \mathcal{D}, \delta).$$

For $\delta_1(\xi), \delta_2(\xi) \in \delta_{[a,b]_T}$, let $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$. Then,

$$a_{\delta_1} = \bigwedge_{\mathcal{D}} S(f, \mathcal{D}, \delta_1) \leq \bigwedge_{\mathcal{D}} S(f, \mathcal{D}, \delta) \leq \bigvee_{\mathcal{D}} S(f, \mathcal{D}, \delta) \leq \bigvee_{\mathcal{D}} S(f, \mathcal{D}, \delta_2) = b_{\delta_2}.$$

Therefore,

$$\bigvee_{\delta(\xi) \in \delta_{[a,b]_T}} a_{\delta} \leq \bigwedge_{\delta(\xi) \in \delta_{[a,b]_T}} b_{\delta}.$$

Hence, there exists $x \in X$ such that $a_{\delta} \leq x \leq b_{\delta}$ for all $\delta(\xi) \in \delta_{[a,b]_T}$. Now, let $\varphi \in \mathbb{N}^{\mathbb{N}}$. Then there exists a Δ -gauge $\delta_{\varphi}(\xi)$ for $[a, b]_T$ such that

$$S(f, \mathcal{D}_1, \delta_{\varphi}) \leq S(f, \mathcal{D}_2, \delta_{\varphi}) + \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}$$

for each δ_{φ} -fine HK partition $\mathcal{D}_1, \mathcal{D}_2$. Fix \mathcal{D}_2 . Then

$$b_{\delta_{\varphi}} \leq S(f, \mathcal{D}_2, \delta_{\varphi}) + \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}.$$

Since the inequality holds for every δ_{φ} -fine HK partition \mathcal{D}_2 , we have

$$b_{\delta_{\varphi}} \leq a_{\delta_{\varphi}} + \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}.$$

By the weak σ -distributivity of X , we obtain that

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i, \varphi(i)} \right) = 0$$

and so

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} b_{\delta_{\varphi}} - \bigvee_{\varphi \in \mathbb{N}^{\mathbb{N}}} a_{\delta_{\varphi}} \leq \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} (b_{\delta_{\varphi}} - a_{\delta_{\varphi}}) = 0.$$

Consequently,

$$x = \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} b_{\delta_{\varphi}} = \bigvee_{\varphi \in \mathbb{N}^{\mathbb{N}}} a_{\delta_{\varphi}}.$$

Then, for every δ_φ -fine HK partition \mathcal{D} , we have

$$S(f, \mathcal{D}, \delta_\varphi) - x \leq b_{\delta_\varphi} - a_{\delta_\varphi} \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}, \quad x - S(f, \mathcal{D}, \delta_\varphi) \leq b_{\delta_\varphi} - a_{\delta_\varphi} \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}.$$

It follows that

$$|S(f, \mathcal{D}, \delta_\varphi) - x| \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}$$

and the proof is complete. \square

Theorem 3.5. If $f : [a, b]_{\mathbb{T}} \rightarrow X$, then f is HK- Δ integrable on $[a, b]_{\mathbb{T}}$ if and only if f is HK- Δ integrable on $[a, c]_{\mathbb{T}}$ and $[c, b]_{\mathbb{T}}$. Moreover, in this case

$$\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t.$$

Proof.

(Necessity). By Theorem 3.4, there exists a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a Δ -gauge, δ , for $[a, b]_{\mathbb{T}}$, such that

$$|S(f, \mathcal{D}_1, \delta) - S(f, \mathcal{D}_2, \delta)| < \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}$$

for each δ -fine HK partition $\mathcal{D}_1, \mathcal{D}_2$ of $[a, b]_{\mathbb{T}}$. Take any two δ -fine HK partitions of $[a, c]_{\mathbb{T}}$, say, \mathcal{D}_3 and \mathcal{D}_4 . Similarly, take another δ -fine HK partition \mathcal{D}_5 of $[c, b]_{\mathbb{T}}$. Then, we have

$$|S(f, \mathcal{D}_3, \delta) - S(f, \mathcal{D}_4, \delta)| = |S(f, \mathcal{D}_3 + \mathcal{D}_5, \delta) - S(f, \mathcal{D}_5, \delta) + S(f, \mathcal{D}_5, \delta) - S(f, \mathcal{D}_4 + \mathcal{D}_5, \delta)| < \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}.$$

Hence, f is HK- Δ integrable on $[a, c]_{\mathbb{T}}$. Similarly, f is HK- Δ integrable on $[c, b]_{\mathbb{T}}$. Consequently, there exist (D)-sequences $(a_{i,j})_{i,j}$, $(b_{i,j})_{i,j}$ and $(c_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a Δ -gauge, δ , for $[a, b]_{\mathbb{T}}$, such that

$$\begin{aligned} \left| S(f, \mathcal{D}, \delta) - \int_a^b f(t) \Delta t \right| &< \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}, \quad \left| S(f, \mathcal{D}', \delta) - \int_a^c f(t) \Delta t \right| < \bigvee_{i=1}^{\infty} b_{i, \varphi(i)}, \\ \left| S(f, \mathcal{D} - \mathcal{D}', \delta) - \int_c^b f(t) \Delta t \right| &< \bigvee_{i=1}^{\infty} c_{i, \varphi(i)} \end{aligned}$$

for each δ -fine HK partition \mathcal{D} of $[a, b]_{\mathbb{T}}$, \mathcal{D}' , of $[a, c]_{\mathbb{T}}$ and $\mathcal{D} - \mathcal{D}'$ of $[c, b]_{\mathbb{T}}$. Then, there exists a (D)-sequence $(d_{i,j})_{i,j}$ of elements of X such that

$$\begin{aligned} &\left| \int_a^b f(t) \Delta t - \int_a^c f(t) \Delta t - \int_c^b f(t) \Delta t \right| \\ &\leq \left| S(f, \mathcal{D}, \delta) - \int_a^b f(t) \Delta t \right| + \left| S(f, \mathcal{D}', \delta) - \int_a^c f(t) \Delta t \right| + \left| S(f, \mathcal{D} - \mathcal{D}', \delta) - \int_c^b f(t) \Delta t \right| \\ &\leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)} + \bigvee_{i=1}^{\infty} b_{i, \varphi(i)} + \bigvee_{i=1}^{\infty} c_{i, \varphi(i)} \leq \bigvee_{i=1}^{\infty} d_{i, \varphi(i)} \end{aligned}$$

and the result follows.

(Sufficiency). Let f be HK- Δ integrable on $[a, c]_{\mathbb{T}}$ and $[c, b]_{\mathbb{T}}$. Then there exist (D)-sequences $(a_{i,j})_{i,j}$, $(b_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist Δ -gauges,

$$\delta^1(\xi) = (\delta_L^1(\xi), \delta_R^1(\xi)), \quad \delta^2(\xi) = (\delta_L^2(\xi), \delta_R^2(\xi)),$$

for $[a, b]_{\mathbb{T}}$ such that

$$\left| S(f, \mathcal{D}_1, \delta^1) - \int_a^c f(t) \Delta t \right| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}, \quad \left| S(f, \mathcal{D}_2, \delta^2) - \int_c^b f(t) \Delta t \right| < \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

for each δ^1 -fine HK partition $\mathcal{D}_1 = \{([t_{k-1}^1, t_k^1]_{\mathbb{T}}, \xi_k^1)\}_{k=1}^n$ of $[a, c]_{\mathbb{T}}$ and for each δ^2 -fine HK partition $\mathcal{D}_2 = \{([t_{k-1}^2, t_k^2]_{\mathbb{T}}, \xi_k^2)\}_{k=1}^m$ of $[c, b]_{\mathbb{T}}$, respectively. We define a Δ -gauge $\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$, on $[a, b]_{\mathbb{T}}$, by first defining $\delta_L(\xi)$ as

$$\delta_L(\xi) = \begin{cases} \delta_L^1(\xi), & \text{if } \xi \in [a, c]_{\mathbb{T}}, \\ \delta_L^1(\xi), & \text{if } \xi = c = \rho(c), \\ \min \left\{ \delta_L^1(\xi), \frac{\eta(c)}{2} \right\}, & \text{if } \xi = c > \rho(c), \\ \min \left\{ \delta_L^2(\xi), \frac{\xi - c}{2} \right\}, & \text{if } \xi \in (c, b]_{\mathbb{T}}, \end{cases}$$

and then defining $\delta_R(\xi)$ as

$$\delta_R(\xi) = \begin{cases} \min \left\{ \delta_R^1(\xi), \max \left\{ \mu(\xi), \frac{c-\xi}{2} \right\} \right\}, & \text{if } \xi \in [a, c]_{\mathbb{T}}, \\ \min \{ \delta_R^2(\xi) \}, & \text{if } \xi \in [c, b]_{\mathbb{T}}. \end{cases}$$

Now, let $\mathcal{D} = \{([t_{k-1}, t_k]_{\mathbb{T}}, \xi_k)\}_{k=1}^p$ be a δ -fine HK partition of $[a, b]_{\mathbb{T}}$. Then, either c is a tag point for \mathcal{D} , say $c = \xi_q$, and $t_q > c$; or $\rho(c) < c$, and $\rho(c)$ is a tag point for \mathcal{D} , say $\rho(c) = \xi_q$, and $t_q = c$. In the first case, there exists a (D)-sequence $(c_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ we have

$$\begin{aligned} \left| S(f, \mathcal{D}, \delta) - \int_a^c f(t) \Delta t - \int_c^b f(t) \Delta t \right| &= \left| \sum_{k=1}^p f(\xi_k)(t_k - t_{k-1}) - \int_a^c f(t) \Delta t - \int_c^b f(t) \Delta t \right| \\ &\leq \left| \sum_{k=1}^{q-1} f(\xi_k)(t_k - t_{k-1}) + f(c)(c - t_{q-1}) - \int_a^c f(t) \Delta t \right| \\ &\quad + \left| \sum_{k=q+1}^p f(\xi_k)(t_k - t_{k-1}) + f(c)(t_q - c) - \int_c^b f(t) \Delta t \right| \\ &< \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} < \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \end{aligned}$$

Using the weak σ -distributivity, we get the corresponding results. The other case is easy and is omitted. Hence, f is HK- Δ -integrable on $[a, b]_{\mathbb{T}}$ and

$$\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t.$$

This concludes the proof. \square

Lemma 3.6 (The Saks-Henstock lemma). *Let $f : [a, b]_{\mathbb{T}} \rightarrow X$ be HK- Δ integrable on $[a, b]_{\mathbb{T}}$. Then there exists a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a Δ -gauge, δ , for $[a, b]_{\mathbb{T}}$, such that*

$$\left| S(f, \mathcal{D}, \delta) - \int_a^b f(t) \Delta t \right| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for each δ -fine HK partition \mathcal{D} of $[a, b]_{\mathbb{T}}$. In particular, if $\mathcal{D}' = \{([t_{k-1}, t_k]_{\mathbb{T}}, \xi_k)\}_{k=1}^m$ is an arbitrary δ -fine partial HK partition of $[a, b]_{\mathbb{T}}$, then

$$\left| S(f, \mathcal{D}', \delta) - \sum_{k=1}^m \int_{t_{k-1}}^{t_k} f(t) \Delta t \right| \leq \bigvee_{i=1}^{\infty} a_{i, \varphi(i)}.$$

Proof. Assume $\mathcal{D}' = \{([t_{k-1}, t_k]_{\mathbb{T}}, \xi_k)\}_{k=1}^m$ is an arbitrary δ' -fine partial HK partition of $[a, b]_{\mathbb{T}}$. Then the complement $[a, b]_{\mathbb{T}} \setminus \bigcup_{k=1}^m [t_{k-1}, t_k]_{\mathbb{T}}$ can be expressed as a fine collection of closed subintervals and we denote

$$[a, b]_{\mathbb{T}} \setminus \bigcup_{k=1}^m [t_{k-1}, t_k]_{\mathbb{T}} = \sum_{k=1}^n [t'_{k-1}, t'_k]_{\mathbb{T}}.$$

From Theorem 3.5, we know that $\int_{t'_{k-1}}^{t'_k} f(t) \Delta t$ exists. Then, there exist (D)-sequences $(b_{k,i,j})_{k,i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist Δ -gauges $\delta_1, \delta_2, \dots, \delta_n$ for $[a, b]_{\mathbb{T}}$, such that

$$\left| S(f, \mathcal{D}_k, \delta_k) - \int_{t'_{k-1}}^{t'_k} f(t) \Delta t \right| < \bigvee_{i=1}^{\infty} b_{k,i,\varphi(i)}$$

for each δ_k -fine HK partition \mathcal{D}_k of $[t_{k-1}, t_k]_{\mathbb{T}}$. Assume that $\delta \leq \delta', \delta_1, \delta_2, \dots, \delta_n$. Let

$$\mathcal{D}_0 = \mathcal{D}' + \mathcal{D}_1 + \mathcal{D}_2 + \dots + \mathcal{D}_n.$$

Obviously, \mathcal{D}_0 is a δ -fine HK partition of $[a, b]_{\mathbb{T}}$. Then, there exists a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that

$$\left| S(f, \mathcal{D}_0, \delta) - \int_a^b f(t) \Delta t \right| = \left| S(f, \mathcal{D}', \delta) + \sum_{k=1}^n S(f, \mathcal{D}_k, \delta) - \int_a^b f(t) \Delta t \right| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for every $\varphi \in \mathbb{N}^{\mathbb{N}}$. Consequently, we obtain

$$\begin{aligned} \left| S(f, \mathcal{D}', \delta) - \sum_{k=1}^m \int_{t_{k-1}}^{t_k} f(t) \Delta t \right| &= \left| S(f, \mathcal{D}_0, \delta) - \sum_{k=1}^n S(f, \mathcal{D}_k, \delta) - \left(\int_a^b f(t) \Delta t - \sum_{k=1}^n \int_{t'_{k-1}}^{t'_k} f(t) \Delta t \right) \right| \\ &\leq \left| S(f, \mathcal{D}_0, \delta) - \int_a^b f(t) \Delta t \right| + \sum_{k=1}^n \left| S(f, \mathcal{D}_k, \delta) - \int_{t'_{k-1}}^{t'_k} f(t) \Delta t \right| \\ &< \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \sum_{k=1}^n \bigvee_{i=1}^{\infty} b_{k,i,\varphi(i)} \\ &\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \sum_{k=1}^n \bigvee_{i=1}^{\infty} \sum_{k=1}^n b_{k,i,\varphi(i)} \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + n \bigvee_{i=1}^{\infty} \sum_{k=1}^n b_{k,i,\varphi(i)}. \end{aligned}$$

Let $c_{i,j} = n \sum_{k=1}^n b_{k,i,\varphi(j)}$. Then, $(c_{i,j})_{i,j}$ is a (D)-sequence and, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, we have

$$\left| S(f, \mathcal{D}', \delta) - \sum_{k=1}^m \int_{t_{k-1}}^{t_k} f(t) \Delta t \right| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

The proof is complete. \square

4. Convergence theorems

In this section we prove two convergence theorems. We begin with the following two definitions.

Definition 4.1. We say that $f_n \rightarrow f$ converges with a common regulating sequence (w.c.r.s.) if there exists

a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $t \in [a, b]_{\mathbb{T}}$ there exists $p = p(t)$ such that

$$|f_n(t) - f(t)| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for any $n \geq p$.

Definition 4.2. We say that $\{f_n\}_{n=1}^{\infty}$ is uniformly HK Δ -integrable on $[a, b]_{\mathbb{T}}$ if each f_n is HK Δ -integrable on $[a, b]_{\mathbb{T}}$ and there exists a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a Δ -gauge δ for $[a, b]_{\mathbb{T}}$, such that

$$\left| S(f_n, \mathcal{D}, \delta) - \int_a^b f_n(t) \Delta t \right| < \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

for each δ -fine HK partition \mathcal{D} of $[a, b]_{\mathbb{T}}$ and $n \in \mathbb{N}$.

For uniformly HK Δ -integrable sequences of integrable functions, we have the following convergence theorem.

Theorem 4.3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of uniformly HK Δ -integrable functions on $[a, b]_{\mathbb{T}}$ and assume that $f_n \rightarrow f$ converges with a common regulating sequence. Then f is HK Δ -integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(t) \Delta t = \int_a^b f(t) \Delta t.$$

Proof. We will prove the theorem in two steps.

Step 1. By assumption, there exists a (D)-sequence $(b_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exist two Δ -gauges δ_1 and δ_2 for $[a, b]_{\mathbb{T}}$, such that

$$\left| S(f_n, \mathcal{D}_1, \delta_1) - \int_a^b f_n(t) \Delta t \right| < \bigvee_{i=1}^{\infty} b_{i,\varphi(i+1)}, \quad \left| S(f_n, \mathcal{D}_2, \delta_2) - \int_a^b f_n(t) \Delta t \right| < \bigvee_{i=1}^{\infty} b_{i,\varphi(i+2)}$$

for each δ_1 -fine (δ_2 -fine) HK partition $\mathcal{D}_1 = \{[t_{i-1}, t_i], \xi_i\}_i$ ($\mathcal{D}_2 = \{[t'_{i-1}, t'_i], \xi'_i\}_i$) of $[a, b]_{\mathbb{T}}$ and $n \in \mathbb{N}$. Let $\delta = \min\{\delta_1, \delta_2\}$. By the w.c.r.s. convergence,

$$|S(f, \mathcal{D}_1, \delta) - S(f_n, \mathcal{D}_1, \delta)| \leq \sum_i |f(\xi_i) - f_n(\xi_i)|(t_i - t_{i-1}) < (b - a) \bigvee_{i=1}^{\infty} a_{i,\varphi(i+1)}$$

for each δ -fine HK partition \mathcal{D}_1 and $n \geq p_1$. Similarly, we have

$$|S(f, \mathcal{D}_2, \delta) - S(f_n, \mathcal{D}_2, \delta)| \leq \sum_i |f(\xi'_i) - f_n(\xi'_i)|(t'_i - t'_{i-1}) < (b - a) \bigvee_{i=1}^{\infty} a_{i,\varphi(i+2)}$$

for each δ -fine HK partition \mathcal{D}_2 and $n \geq p_2$. We can choose a (D)-sequence $(c_{i,j})_{i,j}$ of elements of X such that

$$\bigvee_{i=1}^{\infty} b_{i,\varphi(i+1)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i+2)} + (b - a) \bigvee_{i=1}^{\infty} a_{i,\varphi(i+1)} + (b - a) \bigvee_{i=1}^{\infty} a_{i,\varphi(i+2)} \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$

Let $n > \max\{p_1, p_2\}$. Then,

$$\begin{aligned} |S(f, \mathcal{D}_1, \delta) - S(f, \mathcal{D}_2, \delta)| &= |S(f, \mathcal{D}_1, \delta) - S(f_n, \mathcal{D}_1, \delta)| + \left| S(f_n, \mathcal{D}_1, \delta) - \int_a^b f_n(t) \Delta t \right| \\ &\quad + \left| \int_a^b f_n(t) \Delta t - S(f_n, \mathcal{D}_2, \delta) \right| + |S(f_n, \mathcal{D}_2, \delta) - S(f, \mathcal{D}_2, \delta)| < \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \end{aligned}$$

for each δ -fine HK partition \mathcal{D}_1 and \mathcal{D}_2 of $[a, b]_{\mathbb{T}}$. Therefore, by Theorem 3.4, f is HK Δ -integrable.

Step 2. Since f is HK Δ -integrable, there exists a (D)-sequence $(c_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a Δ -gauge δ' for $[a, b]_{\mathbb{T}}$, such that

$$\left| S(f, \mathcal{D}', \delta') - \int_a^b f(t) \Delta t \right| < \bigvee_{i=1}^{\infty} c_{i,\varphi(i+1)}$$

for each δ' -fine HK partition \mathcal{D}' of $[a, b]_{\mathbb{T}}$. By the uniform HK Δ -integrability, there exists a (D)-sequence $(e_{i,j})_{i,j}$ of elements of X for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that

$$\left| S(f_n, \mathcal{D}', \delta') - \int_a^b f_n(t) \Delta t \right| < \bigvee_{i=1}^{\infty} e_{i,\varphi(i+3)}$$

for each δ' -fine HK partition \mathcal{D}' of $[a, b]_{\mathbb{T}}$ and $n \in \mathbb{N}$. By the w.c.r.s. convergence,

$$\left| S(f, \mathcal{D}', \delta') - S(f_n, \mathcal{D}', \delta') \right| < (b - a) \bigvee_{i=1}^{\infty} a_{i,\varphi(i+3)}$$

for each δ' -fine HK partition \mathcal{D}' and $n \geq p$. Choose a (D)-sequence $(d_{i,j})_{i,j}$ of elements of X such that

$$\bigvee_{i=1}^{\infty} c_{i,\varphi(i+1)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i+3)} + (b - a) \bigvee_{i=1}^{\infty} a_{i,\varphi(i+3)} \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}.$$

Then,

$$\begin{aligned} \left| \int_a^b f(t) \Delta t - \int_a^b f_n(t) \Delta t \right| &= \left| \int_a^b f(t) \Delta t - S(f, \mathcal{D}', \delta') \right| + \left| S(f, \mathcal{D}', \delta') - S(f_n, \mathcal{D}', \delta') \right| \\ &\quad + \left| S(f_n, \mathcal{D}', \delta') - \int_a^b f_n(t) \Delta t \right| < \bigvee_{i=1}^{\infty} d_{i,\varphi(i)} \end{aligned}$$

for each δ' -fine HK partition \mathcal{D}' of $[a, b]_{\mathbb{T}}$. It follows that

$$\lim_{n \rightarrow \infty} \int_a^b f_n(t) \Delta t = \int_a^b f(t) \Delta t$$

and the proof is complete. \square

We now recall the well-known Fremlin lemma.

Lemma 4.4 ([29]). *Let $\{(a_{i,j}^n)_{i,j} : n \in \mathbb{N}\}$ be any countable family of regulators. Then, for each fixed element $x \in X$, $x \geq 0$, there exists a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that*

$$x \bigwedge \sum_{i=1}^{\infty} \left(\bigvee_{j=1}^{\infty} a_{i,\varphi(i)+j}^n \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for every $\varphi \in \mathbb{N}^{\mathbb{N}}$.

Theorem 4.5 (Monotone convergence theorem). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of HK Δ -integrable functions on $[a, b]_{\mathbb{T}}$, and f_1 be bounded from below. Let $f : [a, b]_{\mathbb{T}} \rightarrow X$ be a bounded function such that $f_n \leq f_{n+1}$ and $f_n \rightarrow f$ converges with a common regulating sequence. Then, f is HK Δ -integrable and*

$$\lim_{n \rightarrow \infty} \int_a^b f_n(t) \Delta t = \int_a^b f(t) \Delta t.$$

Proof. Since f_n are HK Δ -integrable functions on $[a, b]_{\mathbb{T}}$, there exists a (D)-sequence $(a_{n,i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a Δ -gauge δ_n for $[a, b]_{\mathbb{T}}$, such that

$$\left| S(f_n, \mathcal{D}_n, \delta_n) - \int_a^b f_n(t) \Delta t \right| < \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n+1)}$$

for each δ_n -fine HK partition \mathcal{D}_n of $[a, b]_{\mathbb{T}}$. By the w.c.r.s. convergence, there exists a (D)-sequence $(a_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and every $t \in [a, b]_{\mathbb{T}}$, there exists $p = p(t)$ such that

$$|f_n(t) - f(t)| < \bigvee_{i=1}^{\infty} a_{i,\varphi(i+1)}$$

for any $n \geq p$. Let $b_{1,i,j} = 2(b-a)a_{i,j}$, $b_{m,i,j} = a_{m-1,i,j}$, $m = 2, 3, \dots$, and $x = (b-a)(L-l)$, where $L, l \in X$ are such that $l \leq f_1(\xi) \leq f(\xi) \leq L$ for any $\xi \in [a, b]_{\mathbb{T}}$. By Fremlin's Lemma 4.4, there exists a (D)-sequence $(b_{i,j})_{i,j}$ of elements of X such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$

$$x \wedge \left(\sum_{m=1}^{\infty} \bigvee_{i=1}^{\infty} b_{m,i,\varphi(i+m)} \right) \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}.$$

Let $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi), \dots, \delta_{p(t)}(\xi)\}$, $\mathcal{D}^0 = \mathcal{D}^1 \cup \mathcal{D}^2$, where

$$\mathcal{D}^1 = \{([t_{k-1}, t_k]_{\mathbb{T}}, \xi_k) \in \mathcal{D} \mid p(\xi_k) \geq n\}, \quad \mathcal{D}^2 = \bigcup_{p(\xi_k) < n} \mathcal{D}_k,$$

with \mathcal{D}_k a sufficiently fine partition of $[t_{k-1}, t_k]_{\mathbb{T}}$ such that \mathcal{D}^0 is δ_n -fine. Thanks to Henstock's Lemma 3.6, we have

$$\left| \sum_{p(\xi_k) \geq n} f_n(\xi_k)(t_k - t_{k-1}) - \sum_{p(\xi_k) \geq n} \int_{t_{k-1}}^{t_k} f_n(t) \Delta t \right| \leq \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n+1)}.$$

Consequently, we obtain

$$\begin{aligned} & \left| S(f_n, \mathcal{D}_n, \delta) - \int_a^b f_n(t) \Delta t \right| \\ & \leq \left| \sum_{p(\xi_k) \geq n} f_n(\xi_k)(t_k - t_{k-1}) - \sum_{p(\xi_k) \geq n} \int_{t_{k-1}}^{t_k} f_n(t) \Delta t \right| \\ & \quad + \left| \sum_{p(\xi_k) < n} f_n(\xi_k)(t_k - t_{k-1}) - \sum_{p(\xi_k) < n} \int_{t_{k-1}}^{t_k} f_n(t) \Delta t \right| \\ & \leq \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n+1)} + \left| \sum_{m=1}^{n-1} \sum_{p(\xi_k)=m} f_n(\xi_k)(t_k - t_{k-1}) - \sum_{m=1}^{n-1} \sum_{p(\xi_k)=m} \int_{t_{k-1}}^{t_k} f_n(t) \Delta t \right| \\ & \leq \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n+1)} + \left| \sum_{m=1}^{n-1} \sum_{p(\xi_k)=m} f_n(\xi_k)(t_k - t_{k-1}) - \sum_{m=1}^{n-1} \sum_{p(\xi_k)=m} \int_{t_{k-1}}^{t_k} f_{\xi_k}(t) \Delta t \right| \\ & \leq \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n+1)} + \left| \sum_{m=1}^{n-1} \sum_{p(\xi_k)=m} f_n(\xi_k)(t_k - t_{k-1}) - \sum_{m=1}^{n-1} \sum_{p(\xi_k)=m} f_{p(\xi_k)}(\xi_k)(t_k - t_{k-1}) \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \sum_{m=1}^{n-1} \sum_{p(\xi_k)=m} f_{p(\xi_k)}(\xi_k)(t_k - t_{k-1}) - \sum_{m=1}^{n-1} \sum_{p(\xi_k)=m} \int_{t_{k-1}}^{t_k} f_{\xi_k}(t) \Delta t \right| \\
& \leq \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n+1)} + \sum_{m=1}^{n-1} \sum_{p(\xi_k)=m} |f_n(\xi_k) - f_{p(\xi_k)}(\xi_k)| (t_k - t_{k-1}) \\
& \quad + \sum_{m=1}^{n-1} \left| \sum_{p(\xi_k)=m} f_{p(\xi_k)}(\xi_k)(t_k - t_{k-1}) - \sum_{p(\xi_k)=m} \int_{t_{k-1}}^{t_k} f_{\xi_k}(t) \Delta t \right| \\
& < \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n+1)} + 2(b-a) \bigvee_{i=1}^{\infty} a_{i,\varphi(i+1)} + \sum_{m=1}^{n-1} \bigvee_{i=1}^{\infty} a_{m,i,\varphi(i+m+1)} \\
& < \bigvee_{i=1}^{\infty} b_{1,i,\varphi(i+n+1)} + \sum_{m=1}^{n-1} \bigvee_{i=1}^{\infty} a_{m,i,\varphi(i+m+1)} \\
& < \bigvee_{i=1}^{\infty} b_{1,i,\varphi(i+n+1)} + \sum_{m=2}^{n-1} \bigvee_{i=1}^{\infty} b_{m,i,\varphi(i+m+1)} \\
& < \sum_{m=1}^{\infty} \bigvee_{i=1}^{\infty} b_{m,i,\varphi(i+m)}.
\end{aligned}$$

On the other hand, we have

$$\left| S(f_n, \mathcal{D}, \delta) - \int_a^b f_n(t) \Delta t \right| \leq (L-l)(b-a) = x.$$

Then,

$$\left| S(f_n, \mathcal{D}, \delta) - \int_a^b f_n(t) \Delta t \right| \leq x \wedge \left(\sum_{m=1}^{\infty} \bigvee_{i=1}^{\infty} b_{m,i,\varphi(i+m)} \right) \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}.$$

Now, we prove that $\{f_n\}_{n=1}^{\infty}$ is a sequence of uniformly HK Δ -integrable functions on $[a, b]_{\mathbb{T}}$. By Theorem 4.3, f is HK Δ -integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(t) \Delta t = \int_a^b f(t) \Delta t.$$

The proof is complete. \square

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