Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 2440–2448
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Variational approach to non-instantaneous impulsive nonlinear differential equations

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Communicated by X.-J. Yang

Abstract

In this paper, a class of nonlinear differential equations with non-instantaneous impulses are considered. By using variational methods and critical point theory, a criterion is obtained to guarantee that the non-instantaneous impulsive problem has at least two distinct nonzero bounded weak solutions.

Keywords: Non-instantaneous impulse, mountain pass theorem, bounded solution.
2010 MSC: 34K45, 34B37, 47J30.

1. Introduction

Non-instantaneous impulsive differential equations were introduced by Hernádez and O'Regan in [7], motivated by a problem related to the hemodynamical equilibrium of a person: in the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the above situation as an impulsive action which starts abruptly and stays active on a finite time interval.

Impulsive effects arise from the real world and are used to describe sudden, discontinuous jumps. Differential equation with not instantaneous impulses is a generalization of the classical theory of impulsive differential equations. For some general and recent works on the theory of impulsive differential equations we refer the readers to [1, 2, 4, 6, 9, 10, 13–16].

The existence of solutions of non-instantaneous impulsive problem has been studied via some approaches, such as fixed point theory and theory of analytic semigroup, see, for example, [5, 7, 11, 12]. Recently, the variational structure of non-instantaneous impulsive linear problem has been developed in [3].

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doi:10.22436/jnsa.010.05.14

Received 2017-03-10

Inspired by the above facts, in this paper a class of non-instantaneous impulsive nonlinear problems which has variational structure will be studied by critical point theory.

Consider the following non-instantaneous impulsive problem

$$\begin{cases} -u''(t) = D_x F_i(t, u(t) - u(t_{i+1})), & t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N, \\ u'(t) = \alpha_i, & t \in (t_i, s_i],\ i = 1, 2, \cdots, N, \\ u'(s_i^+) = u'(s_i^-), & i = 1, 2, \cdots, N, \\ u(0) = u(T) = 0, & u'(0) = \alpha_0, \end{cases} \tag{1.1}$$

where $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_N < s_N < t_{N+1} = T$, the impulses start abruptly at the points t_i and keep the derivative constant on a finite time interval $(t_i, s_i]$. Here $u'(s_i^{\pm}) = \lim_{s \to s_i^{\pm}} u'(s)$ and α_i are given constants. For each $i = 0, 1, \cdots, N$, the nonlinear functions $D_x F_i(t, x)$ are the derivatives of $F_i(t, x)$ with respect to x and F_i satisfies the following assumption:

(A) $F_i(t, x)$ is measurable in t for every $x \in \mathbb{R}$ and continuously differentiable in x for a.e. $t \in (s_i, t_{i+1}]$, and there exist $a \in C(\mathbb{R}^+; \mathbb{R}^+)$ and $b \in L^1(s_i, t_{i+1}; \mathbb{R}^+)$ such that

$$|F_i(t, x)| \leqslant a(|x|) b(t), \quad |D_x F_i(t, x)| \leqslant a(|x|) b(t)$$

for all $x \in \mathbb{R}$ and a.e. $t \in (s_i, t_{i+1}]$.

Moreover, without loss of generality, it is supposed that $F_i(t, 0) = 0$ for a.e. $t \in (s_i, t_{i+1}]$ and each $i = 0, 1, \cdots, N$.

Our results are presented as follows.

Theorem 1.1. *Assume condition* (A) *holds and*

(H) *for each* $i = 0, 1, \cdots, N$, *there exist constants* $\sigma_i > 2$ *such that* $0 < \sigma_i F_i(t, x) \leqslant x D_x F_i(t, x)$ *for a.e.* $t \in (s_i, t_{i+1}]$ *and* $x \in \mathbb{R} \setminus \{0\}$,

then problem (1.1) *has at least one nonzero bounded weak solution in* $H_0^1(0, T)$ *provided*

$$\frac{\pi^2}{4T(1+\pi)^2} > \sum_{i=1}^{N} |\alpha_{i-1} - \alpha_i| + 2 \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} \max_{|x|=1} F_i(t, x) dt. \tag{1.2}$$

In addition, if $\alpha_{j-1} \neq \alpha_j$ *for some* $j = 1, 2, \cdots, N$, *then problem* (1.1) *has at least two distinct nonzero bounded weak solutions in* $H_0^1(0, T)$.

Example 1.2. It follows from Theorem 1.1 that the following non-instantaneous impulsive problem has at least two distinct nonzero bounded weak solutions.

$$\begin{cases} -u''(t) = t(u(t) - u(t_{i+1}))^3, \quad t \in (s_i, t_{i+1}],\ i = 0, 1, \\ u'(t) = 0.1, \quad t \in (t_1, s_1], \\ u'(s_1^+) = u'(s_1^-), \\ u(0) = u(1) = 0, \quad u'(0) = 0, \end{cases}$$

where $0 = s_0 < t_1 = \frac{1}{16} < s_1 = \frac{15}{16} < t_2 = 1$.

2. Preliminaries

We recall some facts which will be used in the proof of our main result. It is a consequence of Poincaré's inequality that

$$\int_0^T |u(t)|^2 dt \leqslant \frac{1}{\lambda_1} \int_0^T |u'(t)|^2 dt, \tag{2.1}$$

where $\lambda_1 = \pi^2/T^2$ is the first eigenvalue of the Dirichlet problem

$$-u''(t) = \lambda u(t),\ t \in [0,T]; \quad u(0) = u(T) = 0.$$

Let $H_0^1(0,T)$ be the Sobolev space endowed with the norm

$$\|u\|_{H_0^1} := \left(\int_0^T |u'(t)|^2 + |u(t)|^2 dt\right)^{1/2}.$$

Obviously, $H_0^1(0,T)$ is a reflexive Banach space and by the Poincaré's inequality (2.1), we know that

$$\|u\| := \left(\int_0^T |u'(t)|^2 dt\right)^{1/2}$$

is equivalent to the norm $\|u\|_{H_0^1}$ in $H_0^1(0,T)$. Let

$$\|u\|_{L^2} := \left(\int_0^T |u(t)|^2 dt\right)^{1/2} \quad \text{and} \quad \|u\|_\infty := \max_{t\in[0,T]} |u(t)|.$$

Then for $u \in H_0^1(0,T)$, we have

$$\|u\|_\infty \leqslant \beta\|u\|, \tag{2.2}$$

where $\beta = (T\lambda_1)^{-1/2} + T^{1/2}$. In fact, it follows from the mean value theorem that $\frac{1}{T}\int_0^T u(s)ds = u(\tau)$ for some $\tau \in (0,T)$. Hence, for $t \in [0,T]$, using Hölder inequality,

$$|u(t)| = \left|u(\tau) + \int_\tau^t u'(s)ds\right| \leqslant \frac{1}{T}\left|\int_0^T u(s)ds\right| + \int_0^T |u'(t)|dt \leqslant T^{-1/2}\|u\|_{L^2} + T^{1/2}\|u'\|_{L^2}.$$

Lemma 2.1 ([17, Theorem 38.A])**.** *For the functional* $F : M \subseteq X \to [-\infty,+\infty]$ *with* $M \neq \emptyset$, $\min_{u\in M} F(u) = \alpha$ *has a solution in case the following hold:*

(i) X *is a real reflexive Banach space;*
(ii) M *is bounded and weak sequentially closed;*
(iii) F *is sequentially weakly lower semi-continuous on* M.

Lemma 2.2 ([8, Theorem 4.10])**.** *Let* E *be a Banach space and* $\varphi \in C^1(E,\mathbb{R})$. *Assume that there exist* $u_0 \in E$, $u_1 \in E$ *and a bounded open neighborhood* Ω *of* u_0 *such that* $u_1 \in E \setminus \Omega$ *and* $\inf_{\partial\Omega} \varphi > \max\{\varphi(u_0), \varphi(u_1)\}$. *Let*

$$\Gamma = \{g \in C([0,1],E) : g(0) = u_0, g(1) = u_1\} \quad \textit{and} \quad c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \varphi(g(s)).$$

If φ *satisfies the* $(PS)_c$*-condition, then* c *is a critical value of* φ *and* $c > \max\{\varphi(u_0), \varphi(u_1)\}$.

To follow the ideas of the variational approach for impulsive differential equations of [10, 16], for each $v \in H_0^1(0,T)$, we have

$$\begin{aligned}\int_0^T u''(t)v(t)dt &= \int_0^{t_1} u''(t)v(t)dt + \sum_{i=1}^N \int_{t_i}^{s_i} u''(t)v(t)dt + \sum_{i=1}^{N-1} \int_{s_i}^{t_{i+1}} u''(t)v(t)dt + \int_{s_N}^T u''(t)v(t)dt \\ &= -\int_0^T u'(t)v'(t)dt + \sum_{i=1}^N \left[u'(t_i^-) - u'(t_i^+)\right] v(t_i) + \sum_{i=1}^N \left[u'(s_i^-) - u'(s_i^+)\right] v(s_i),\end{aligned}$$

which combined with (1.1) yields that

$$\int_0^T u''(t)v(t)dt = -\int_0^T u'(t)v'(t)dt + \sum_{i=1}^{N} [\alpha_{i-1} - \alpha_i]\, v(t_i) \\ - \sum_{i=0}^{N-1} \Big(\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))dt \Big) v(t_{i+1}). \tag{2.3}$$

On the other hand,

$$\int_0^T u''(t)v(t)dt = -\sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))v(t)dt + \sum_{i=1}^{N} \int_{t_i}^{s_i} \frac{d}{dt}[\alpha_i]v(t)dt \\ = -\sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))v(t)dt. \tag{2.4}$$

Thus, in view of $v(t_{N+1}) = v(T) = 0$, (2.3), and (2.4), we find that

$$-\int_0^T u'(t)v'(t)dt + \sum_{i=1}^{N} [\alpha_{i-1} - \alpha_i]\, v(t_i) = -\sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))(v(t) - v(t_{i+1}))dt. \tag{2.5}$$

Considering the aforementioned equality, we introduce the following concept of weak solution for (1.1).

Definition 2.3. A function $u \in H_0^1(0, T)$ is a weak solution of (1.1) if (2.5) holds for any $v \in H_0^1$.

Consider the functional $\Phi : H_0^1 \to \mathbb{R}$ defined by

$$\Phi(u) := \frac{1}{2} \int_0^T |u'(t)|^2 dt - \sum_{i=1}^{N} (\alpha_{i-1} - \alpha_i)\, u(t_i) - \sum_{i=0}^{N} \varphi_i(u),$$

where

$$\varphi_i(u) := \int_{s_i}^{t_{i+1}} F_i(t, u(t) - u(t_{i+1}))dt.$$

For u and v fixed in $H_0^1(0, T)$ and $\lambda \in [-1, 1]$, it follows from (2.2) that

$$|u(t) - u(t_{i+1})| \leqslant 2\|u\|_\infty \leqslant 2\beta\|u\|, \tag{2.6}$$

which combined with assumption (A) yields that $\varphi_i(u)$ is well-defined on $H_0^1(0, T)$. Using assumption (A) again, we see that

$$\lim_{\lambda \to 0} \frac{1}{\lambda} [F_i(t, u(t) - u(t_{i+1}) + \lambda v(t) - \lambda v(t_{i+1})) - F_i(t, u(t) - u(t_{i+1}))] \\ = D_x F_i(t, u(t) - u(t_{i+1}))(v(t) - v(t_{i+1})), \quad \text{for a.e. } t \in (s_i, t_{i+1}].$$

It follows from (2.6) that

$$|u(t) - u(t_{i+1}) + \lambda\theta(v(t) - v(t_{i+1}))| \leqslant 2\beta(\|u\| + \|v\|), \quad \text{for } \theta \in (0, 1),$$

which combined with mean value theorem and assumption (A) yields to

$$\left| \frac{1}{\lambda} [F_i(t, u(t) - u(t_{i+1}) + \lambda(v(t) - v(t_{i+1}))) - F_i(t, u(t) - u(t_{i+1}))] \right|$$

$$
\begin{aligned}
&= |D_x F_i(t, u(t) - u(t_{i+1}) + \lambda\theta(v(t) - v(t_{i+1})))(v(t) - v(t_{i+1}))| \\
&\leqslant \max_{z\in[0,2\beta(\|u\|+\|v\|)]} a(z) 2\beta\|v\| b(t) \in L^1(s_i, t_{i+1}; \mathbb{R}^+),
\end{aligned}
$$

for some $\theta \in (0,1)$. Lebesgue's dominated convergence theorem shows that φ_i has at every point u a directional derivative

$$
\langle \varphi_i'(u), v\rangle = \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))(v(t) - v(t_{i+1}))dt. \tag{2.7}
$$

Moreover, in view of assumption (A) and (2.6), we have

$$
|\langle \varphi_i'(u), v\rangle| \leqslant \int_{s_i}^{t_{i+1}} |D_x F_i(t, u(t) - u(t_{i+1}))||(v(t) - v(t_{i+1}))|dt \leqslant 2\beta \int_{s_i}^{t_{i+1}} b(t)dt \max_{z\in[0,2\beta\|u\|]} a(z)\|v\|.
$$

Thus $\varphi_i'(u) \in (H_0^1(0,T))^*$. Suppose $u_n \to u$ in $H_0^1(0,T)$, then $\{u_n\}$ converges uniformly to u on [0,T] by (2.2). Furthermore, it follows from (2.7) that

$$
\|\varphi_i'(u_n) - \varphi_i'(u)\| \leqslant 2\beta \int_{s_i}^{t_{i+1}} |D_x F_i(t, u_n(t) - u_n(t_{i+1})) - D_x F_i(t, u(t) - u(t_{i+1}))|dt.
$$

Thus φ_i' is continuous from $H_0^1(0,T)$ into $(H_0^1(0,T))^*$. So $\Phi \in C^1(H_0^1(0,T), \mathbb{R})$ and

$$
\langle \Phi'(u), v\rangle = \int_0^T u'(t)v'(t)dt - \sum_{i=1}^{N} (\alpha_{i-1} - \alpha_i)\, v(t_i) - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))(v(t) - v(t_{i+1}))dt.
$$

Then critical points of Φ correspond to weak solutions of the problem (1.1).

Lemma 2.4. *If assumption* (H) *holds, then for each* $i = 0, 1, \cdots, N$ *there exist* $M_i, m_i, b_i \in L^1(s_i, t_{i+1})$ *which are almost everywhere positive such that*

$$
F_i(t, x) \leqslant M_i(t)|x|^{\sigma_i}, \quad \textit{for a.e. } t \in (s_i, t_{i+1}] \textit{ and } |x| \leqslant 1 \tag{2.8}
$$

and

$$
F_i(t, x) \geqslant m_i(t)|x|^{\sigma_i} - b_i(t), \quad \textit{for a.e. } t \in (s_i, t_{i+1}] \textit{ and } x \in \mathbb{R}, \tag{2.9}
$$

where

$$
m_i(t) := \min_{|x|=1} F_i(t, x) \quad \textit{and} \quad M_i(t) := \max_{|x|=1} F_i(t, x), \quad \textit{for a.e. } t \in (s_i, t_{i+1}].
$$

Proof. It follows from assumption (H) that

$$
0 < m_i(t) \leqslant F_i\left(t, \frac{x}{|x|}\right) \leqslant M_i(t), \quad \text{for a.e. } t \in (s_i, t_{i+1}]
$$

and

$$
M_i(t) \leqslant \max_{|x|\leqslant 1} F_i(t, x) \leqslant \max_{|x|\leqslant 1} a(|x|)b(t) \in L^1(s_i, t_{i+1}).
$$

For each $i = 0, 1, \cdots, N$, define $T_i : (0, +\infty) \to \mathbb{R}$ by

$$
T_i(z) = F_i\left(t, \frac{x}{z}\right) z^{\sigma_i}, \quad \text{for a.e. } t \in (s_i, t_{i+1}] \text{ and } x \neq 0.
$$

Condition (H) implies that T_i is nonincreasing. Thus, for a.e. $t \in (s_i, t_{i+1}]$, we have

$$
F_i(t, x) \leqslant F_i\left(t, \frac{x}{|x|}\right) |x|^{\sigma_i}, \quad \text{if } 0 < |x| \leqslant 1
$$

and

$$F_i(t,x) \geqslant F_i\left(t, \frac{x}{|x|}\right)|x|^{\sigma_i}, \quad \text{if } |x| \geqslant 1. \tag{2.10}$$

So (2.8) holds. Moreover, for a.e. $t \in (s_i, t_{i+1}]$ and $|x| \leqslant 1$, we have

$$|F_i(t,x) - m_i(t)|x|^{\sigma_i}| \leqslant \max_{|x|\leqslant 1} a(|x|)b(t) + m_i(t) := b_i(t),$$

which combined with (2.10) yields (2.9). □

3. Proof of Theorem 1.1

Proof. We complete the proof in four steps.

Step 1. $\Phi(u)$ satisfies PS condition on $H_0^1(0,T)$, that is, every sequence $\{u_k\}$ in $H_0^1(0,T)$ such that $\Phi(u_k)$ is bounded and $\Phi'(u_k) \to 0$ as $k \to +\infty$ contains a convergent subsequence. It is clear that

$$\left|\sum_{i=1}^{N}(\alpha_{i-1}-\alpha_i)\,u(t_i)\right| \leqslant \sum_{i=1}^{N}|\alpha_{i-1}-\alpha_i|\,\|u\|_\infty \leqslant \sum_{i=1}^{N}|\alpha_{i-1}-\alpha_i|\,\beta\|u\|. \tag{3.1}$$

Let $\sigma := \min\{\sigma_i : i = 0, 1, \cdots, N\}$, then $\sigma > 2$. By (H) and (3.1), we have

$$\begin{aligned}
\sigma\Phi(u_k) - \langle\Phi'(u_k), u_k\rangle &= \left(\frac{\sigma}{2}-1\right)\int_0^T |u_k'(t)|^2 dt - (\sigma-1)\sum_{i=1}^{N}(\alpha_{i-1}-\alpha_i)\,u_k(t_i) \\
&\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \sigma F_i(t, u_k(t)-u_k(t_{i+1})) - D_xF_i(t, u_k(t)-u_k(t_{i+1})) \\
&\quad \times (u_k(t)-u_k(t_{i+1}))dt \\
&\geqslant \left(\frac{\sigma}{2}-1\right)\|u_k\|^2 - (\sigma-1)\beta\sum_{i=1}^{N}|\alpha_{i-1}-\alpha_i|\,\|u_k\|,
\end{aligned} \tag{3.2}$$

which implies that $\{u_k\}$ is bounded in $H_0^1(0,T)$. Since $H_0^1(0,T)$ is a reflexive Banach space, passing to a subsequence if necessary, we may assume that there is a $u_0 \in H_0^1(0,T)$ such that $u_k \rightharpoonup u_0$ in $H_0^1(0,T)$. Then $\{u_k\}$ converges uniformly to u_0 on $[0,T]$ and $u_k \to u_0$ in $L^2(0,T)$. Notice that

$$\begin{aligned}
&\langle\Phi'(u_m) - \Phi'(u_n), u_m - u_n\rangle \\
&= \int_0^T |u_m'(t) - u_n'(t)|^2 dt - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} [D_xF_i(t, u_m(t)-u_m(t_{i+1})) \\
&\quad - D_xF_i(t, u_n(t)-u_n(t_{i+1}))]\,[u_m(t)-u_m(t_{i+1})-u_n(t)+u_n(t_{i+1})]\,dt.
\end{aligned} \tag{3.3}$$

In view of assumption (A) and

$$\begin{aligned}
|u_m(t)-u_m(t_{i+1})-u_n(t)+u_n(t_{i+1})| &\leqslant |u_m(t)-u_n(t)| + |u_m(t_{i+1})-u_n(t_{i+1})| \\
&\leqslant 2\|u_m-u_n\|_\infty \to 0, \quad \text{as } m, n \to \infty,
\end{aligned}$$

we have that the second term on the right hand of (3.3) converges to 0 as $m, n \to \infty$. What is more, the fact that $\Phi'(u_k) \to 0$ as $k \to +\infty$ implies

$$\left|\langle\Phi'(u_m) - \Phi'(u_n), u_m - u_n\rangle\right| \leqslant \|\Phi'(u_m) - \Phi'(u_n)\|\|u_m - u_n\| \to 0,$$

as $m, n \to \infty$. Consequently, $\|u_m - u_n\| \to 0$ as $m, n \to \infty$. By the completeness of $H_0^1(0,T)$, we know that $\{u_k\}$ possesses a convergent subsequence in $H_0^1(0,T)$.

Step 2. $\Phi(u)$ has mountain pass geometric structure.

Let B_r be the open ball in $H_0^1(0,T)$ with radius r and centered at 0 and let ∂B_r and $\overline{B_r}$ denote the boundary and closure of B_r, respectively. For each $u \in \partial B_{(2\beta)^{-1}}$, where β is listed in (2.2), we have

$$|u(t) - u(t_{i+1})| \leqslant 2\|u\|_\infty \leqslant 2\beta\|u\| = 1.$$

Thus, using (2.8), we find that, for each $u \in \partial B_{(2\beta)^{-1}}$,

$$\int_{s_i}^{t_{i+1}} F_i(t, u(t) - u(t_{i+1}))dt \leqslant \int_{s_i}^{t_{i+1}} M_i(t)|u(t) - u(t_{i+1})|^{\sigma_i} dt \leqslant \int_{s_i}^{t_{i+1}} M_i(t)dt,$$

which combined with (3.1) yields that, for any $u \in \partial B_{(2\beta)^{-1}}$,

$$\Phi(u) \geqslant \frac{1}{2}\left[\frac{1}{4\beta^2} - \sum_{i=1}^{N}|\alpha_{i-1} - \alpha_i| - 2\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} M_i(t)dt\right] := \gamma.$$

In view of (1.2), we have $\Phi(u) \geqslant \gamma > 0 = \Phi(0)$ for any $u \in \partial B_{(2\beta)^{-1}}$. Thus

$$\inf_{u \in \partial B_{(2\beta)^{-1}}} \Phi(u) > \Phi(0).$$

Let $\xi > 0$ and $w \in H_0^1(0,T)$ with $\|w\| = 1$ and $w(t)$ is not a constant for a.e. $[0, t_1]$. It follows from (2.9) that

$$\int_{s_i}^{t_{i+1}} F_i(t, (w(t) - w(t_{i+1}))\xi)dt \geqslant \Big(\int_{s_i}^{t_{i+1}} m_i(t)|w(t) - w(t_{i+1})|^{\sigma_i} dt\Big)\xi^{\sigma_i} - \int_{s_i}^{t_{i+1}} b_i(t)dt. \tag{3.4}$$

Let $Q_i := \int_{s_i}^{t_{i+1}} m_i(t)|w(t) - w(t_{i+1})|^{\sigma_i} dt$. Then we have that

$$0 \leqslant Q_i \leqslant (2\beta)^{\sigma_i}\int_{s_i}^{t_{i+1}} m_i(t)dt \quad \text{and} \quad Q_0 > 0.$$

In fact, suppose that $\int_0^{t_1} m_0(t)|w(t) - w(t_1)|^{\sigma_0} dt = 0$, since $m_0(t)$ is almost everywhere positive, we have $w(t) = w(t_1)$ for a.e. $[0, t_1]$, which is a contradiction to the assumption on w. In view of (3.4), we have

$$\begin{aligned}\Phi(\xi w) &= \frac{1}{2}\xi^2 - \sum_{i=1}^{N}(\alpha_{i-1} - \alpha_i)\, w(t_i)\xi - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t, (w(t) - w(t_{i+1}))\xi)dt \\ &\leqslant \frac{1}{2}\xi^2 + \sum_{i=1}^{N}|\alpha_{i-1} - \alpha_i|\,\beta\xi - \sum_{i=0}^{N} Q_i\xi^{\sigma_i} + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b_i(t)dt.\end{aligned} \tag{3.5}$$

Since $\sigma_i > 2$, we have $\Phi(\xi w) \to -\infty$ as $\xi \to +\infty$. Thus, there exists ξ_0 with $\|\xi_0 w\| > (2\beta)^{-1}$ such that

$$\inf_{u \in \partial B_{(2\beta)^{-1}}} \Phi(u) > \Phi(\xi_0 w).$$

Therefore it follows from Steps 1-2 and Lemma 2.2 that there exists $u_1^* \in H_0^1(0,T)$ such that $\Phi'(u_1^*) = 0$ with

$$\Phi(u_1^*) > \max\{\Phi(0), \Phi(\xi_0 w)\} \geqslant \Phi(0) = 0, \tag{3.6}$$

so u_1^* is a nonzero weak solution of problem (1.1).

Step 3. If $\alpha_{j-1} \neq \alpha_j$ for some $j = 1, 2, \cdots, N$, then $\Phi(u)$ has a nonzero local minimum u_2^* in $\overline{B_{(2\beta)^{-1}}}$, where β is listed in (2.2).

Since $\overline{B_{(2\beta)^{-1}}}$ is a closed convex set, $\overline{B_{(2\beta)^{-1}}}$ is weak sequentially closed. Furthermore, $\Phi(u)$ is sequentially weakly lower semi-continuous on $H_0^1(0,T)$ as the sum of a convex continuous function and of a weakly continuous one. Thus it follows from Lemma 2.1 that there exists a $u_2^* \in \overline{B_{(2\beta)^{-1}}}$ such that

$$\Phi(u_2^*) = \min_{\overline{B_{(2\beta)^{-1}}}} \Phi(u).$$

What is more, $u_2^* \neq 0$. In fact, since $F_i(t,0) = 0$ for a.e. $t \in (s_i, t_{i+1}]$, we find that $F_i(t,x) \geqslant 0$ for a.e. $t \in (s_i, t_{i+1}]$ and $x \in \mathbb{R}$ by (H). Thus $\varphi_i \geqslant 0$. Let

$$u^\sharp(t) := \begin{cases} \alpha_{j-1} - \alpha_j, & \text{if } t = t_j, \\ 0, & \text{if } t \in [0,T] \text{ and } t \neq t_j, \end{cases}$$

then $u^\sharp \in \overline{B_{(2\beta)^{-1}}}$ and $\Phi(u^\sharp) \leqslant -(\alpha_{j-1} - \alpha_j)^2 < 0$. So

$$\Phi(u_2^*) \leqslant \Phi(u^\sharp) < 0 \tag{3.7}$$

and the assertion follows.

Step 4. u_1^* and u_2^* are different and both bounded.

In view of (3.6) and (3.7), we have

$$\Phi(u_1^*) > 0 > \Phi(u_2^*), \tag{3.8}$$

so u_1^* and u_2^* are different. From the inf max characterization of u_1^* in Lemma 2.2 and (3.5), we find that

$$\Phi(u_1^*) = \inf_{g \in \Gamma} \max_{s \in [0,1]} \Phi(g(s)) \leqslant \max_{s \in [0,1]} \Phi(\xi_0 w s) \leqslant \max_{s \in [0,1]} h(s),$$

where

$$h(s) := \frac{\xi_0^2 s^2}{2} + \sum_{i=1}^{N} |\alpha_{i-1} - \alpha_i|\, \beta \xi_0 s - \sum_{i=0}^{N} Q_i \xi_0^{\sigma_i} s^{\sigma_i} + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} b_i(t)dt.$$

Since $h(s)$ is continuous on $[0,1]$, we have $\Phi(u_1^*)$ is bounded above and so is $\Phi(u_2^*)$ by (3.8). Similar as (3.2), for $u^* \in H_0^1(0,T)$, we get

$$\left(\frac{\sigma}{2} - 1\right) \|u^*\|^2 - (\sigma - 1)\beta \sum_{i=1}^{N} |\alpha_{i-1} - \alpha_i| \, \|u^*\| \leqslant \sigma\Phi(u^*) - \langle \Phi'(u^*), u^* \rangle. \tag{3.9}$$

Since u_1^* and u_2^* are both critical points of Φ, furthermore $\Phi(u_1^*)$ and $\Phi(u_2^*)$ are both bounded above, (3.9) implies that u_1^* and u_2^* are both bounded in $H_0^1(0,T)$. This completes the proof. □

Acknowledgment

This work has been partially supported by National Natural Science Foundation of China (No.11401420, No.11326117, No.11626182), SXNSF (No.2013021001-2, No.201601D102002), YFTUT (No.2013T062), TFTUT (No.tyut-rc201212a), the Ministerio de Economia y Competitividad of Spain under Grant MTM2016-75140-P of the AEI and co-financed by European Community fund FEDER and XUNTA de Galicia under grants GRC2015-004 and R2016/022.

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