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Fixed point results for generalized contractive multivalued maps

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Abstract

In this paper, we prove some results on the existence of fixed points for multivalued maps with respect to general distance. Our results improve and generalize a number of known fixed point results including the fixed point results.

Keywords: Metric space, fixed point, w-distance, multivalued contractive map, Banach limit.
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1. Introduction and preliminaries

The well-known Banach contraction principle, which asserts that each single-valued contraction self-map on a complete metric space has a unique fixed point, plays a central role in nonlinear analysis and has many generalizations in a number of different directions. Susuki [16] proved a simple but new type of generalization of the Banach contraction principle for single-valued maps on metric spaces. Recently, using the concept of a mean on the Banach space ℓ^∞, Hasegawa et al. [3] proved a useful result on the existence of unique fixed point for single-valued self-maps on metric spaces. Using the concept of the Hausdorff metric, Nadler [13] introduced a notion of multivalued contraction map and proved a multivalued version of the Banach contraction principle, which states that each closed bounded valued contraction map on a complete metric space has a fixed point. Since then various fixed point results concerning multivalued contractions have appeared. In [6], Kikawa and Susuki proved a beautiful fixed point result which generalizes the above mentioned result of Suzuki and Nadler's fixed point result. Also see, [1, 7, 15] and references therein. In fact, without using the Hausdorff metric, the existence part can be proved under much less stringent conditions. Among others Husain and Latif [4], Feng and Liu [2] have generalized Nadler's fixed point result without using the Hausdorff metric. Fixed point theory of multivalued maps has lots of applications in convex optimization, see [11, 12, 14, 21].

Introducing a notion of w-distance on a metric space, Kada et al. [5] improved several classical results in metric fixed point theory. While, Suzuki and Takahashi [17] have introduced notions of single-valued

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and multivalued weakly contractive maps with respect to w-distance and proved fixed point results for such maps. Recent fixed point results concerning w-distance can be found in [8, 9, 10, 18, 19]. Recently, using the mean and w-distance, Takahashi et al. [20] proved fixed point result for single-valued maps, which contains the fixed point result of Hasegawa et al. [3] as a special case.

In this paper, we prove some results on the existence of fixed points for multivalued maps with respect to w-distance. Our results improve and generalize a number of known fixed point results including the fixed point results of [3, 6, 13, 20].

Let us recall some useful notions and facts.

Let (X, d) be a metric space. We use 2^X to denote a collection of nonempty subsets of X and $Cl(X)$ a collection of all nonempty closed subsets of X. An element $x \in X$ is called a *fixed point* of a multivalued map $T : X \to 2^X$ if $x \in T(x)$. We denote $\mathrm{Fix}(T) = \{x \in X : x \in T(x)\}$.

A sequence $\{x_n\}$ in X is called an orbit of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \geqslant 1$.

A map $f : X \to \mathbb{R}$ is called *lower semicontinuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ implies that $f(x) \leqslant \liminf\limits_{n\to\infty} f(x_n)$.

A multivalued map $T : X \to 2^X$ is said to be contractive [4] if there exists $r \in [0, 1)$ such that for any $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ with $d(u, v) \leqslant r d(x, y)$. In [4], it has been proved that each closed valued contractive map has fixed point, which is an improved version of the Nadler's result [13].

Kada et al. [5] introduced the w-distance on a metric space as follows:

Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is said to be a w-distance on X if the following are satisfied for all $x, y, z \in X$:

(i) $p(x, z) \leqslant p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(ii) for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is lower semicontinuous;

(iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) < \delta$ and $p(z, y) < \delta$ imply $d(x, y) \leqslant \epsilon$.

Note that, in general for $x, y \in X$, $p(x, y) \neq p(y, x)$ and not either of the implications $p(x, y) = 0 \Leftrightarrow x = y$ necessarily hold. Clearly, the metric d is a w-distance on X. Let $(Y, \|.\|)$ be a normed space. Then the functions $p_1, p_2 : Y \times Y \to [0, \infty)$ defined by $p_1(x, y) = \|y\|$ and $p_2(x, y) = \|x\| + \|y\|$ for all $x, y \in Y$ are w-distances [5]. Many other examples and properties of the w-distance can be found in [5, 18]. We denote by $W(X)$ the set of all w-distances on X. A w-distance p on X is called symmetric if $p(x, y) = p(y, x)$ for all $x, y \in X$. We denote by $W_0(X)$ the set of all symmetric w-distances on X.

The following lemma concerning w-distance is fundamental.

Lemma 1.1 ([5]). *Let X be a metric space with metric d and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$. Then, the following hold:*

(a) *if $p(x_n, y) \leqslant \alpha_n$ and $p(x_n, z) \leqslant \beta_n$ for any $n \in \mathbb{N}$, then $y = z$; in particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;*
(b) *if $p(x_n, y_n) \leqslant \alpha_n$ and $p(x_n, z) \leqslant \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*
(c) *if $p(x_n, x_m) \leqslant \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(d) *if $p(y, x_n) \leqslant \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Let (X, d) be a metric space. A multivalued mapping $T : X \to 2^X$ is said to be p-contractive [17] if there exist $p \in W(X)$ and $r \in [0, 1)$ such that for any $x, y \in X$ and $u \in Tx$ there is $v \in Ty$ with

$$p(u, v) \leqslant r p(x, y).$$

In particular, a single-valued mapping $f : X \to X$ is said to be p-contractive if there exist $p \in W(X)$ and $r \in [0, 1)$ such that $p(f(x), f(y)) \leqslant r p(x, y)$ for all $x, y \in X$. Using w-distance, Suzuki and Takahashi [17] proved that each p-contractive multivalued map $T : X \to Cl(X)$ has a fixed point, which is an improved version of the corresponding results in [4, 13]. Further, they deduced that each single-valued p-contractive self-map on X has a unique fixed point, which is an improved version of the Banach contraction principle.

Now, we recall the notion of the mean.

Let ℓ^∞ be the Banach space of bounded sequence with supremum norm and let $(\ell^\infty)^*$ be the dual space of ℓ^∞. We denote by $\mu(x)$ the value of μ at $x = \{x_n\} \in \ell^\infty$. The value of $\mu(x)$ is also denoted by $\mu_n(x_n)$. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, ...)$. A mean μ is called a Banach limit on ℓ^∞ if $\mu_n(x_n) = \mu_n(x_{n+1})$ for all $\{x_n\} \in \ell^\infty$. If μ is a Banach limit on ℓ^∞, then for $x = \{x_n\} \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leqslant \mu_n(x_n) \leqslant \limsup_{n\to\infty} x_n.$$

In particular, if $x = \{x_n\} \in \ell^\infty$, and $x_n \to a \in \mathbb{R}$, then we have $\mu(x) = \mu(x_n) = a$. For the proof of the existence of a Banach limit and its other elementary properties, see [19].

2. Fixed point results

We consider a decreasing function $\beta : [0, 1) \to (\frac{1}{2}, 1]$ defined by $\beta(\lambda) = \frac{1}{1+\lambda}$.

First we prove the following useful lemma for multivalued maps with respect to w-distance.

Lemma 2.1. *Let (X, d) be a metric space, $p \in W(X)$ and Let $T : X \to Cl(X)$. Assume that there exists $r \in [0, 1)$ such that for any $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that*

$$\beta(r)p(x, Tx) \leqslant p(x, y) \quad \textit{implies} \quad p(u, v) \leqslant rp(x, y).$$

Then there exists an orbit of T in X which is a Cauchy sequence.

Proof. Let z_0 be an arbitrary but fixed element of X and $z_1 \in Tz_0$. Then, clearly we have

$$\beta(r)p(z_0, Tz_0) \leqslant \beta(r)p(z_0, z_1) \leqslant p(z_0, z_1).$$

Thus, by hypothesis, there is $z_2 \in Tz_1$ such that

$$p(z_1, z_2) \leqslant rp(z_0, z_1).$$

Continuing this process, we get a sequence $\{z_n\}$ in X such that for each $n \in \mathbb{N}$, $z_n \in Tz_{n-1}$ satisfying

$$p(z_n, z_{n+1}) \leqslant rp(z_{n-1}, z_n).$$

Note that for each $n \in \mathbb{N}$, we have

$$p(z_n, z_{n+1}) \leqslant r^n p(z_0, z_1).$$

Now, for all $n, m \in \mathbb{N}$, $m > n$ we get

$$\begin{aligned} p(z_n, z_m) &\leqslant p(z_n, z_{n+1}) + p(z_{n+1}, z_{n+2}) + \ldots + p(z_{m-1}, z_m) \\ &\leqslant r^n p(z_0, z_1)[1 + r + r^2 + \cdots + r^{m-n-1}] \\ &\leqslant \frac{r^n}{1-r} p(z_0, z_1). \end{aligned}$$

Take $\alpha_n = \frac{r^n}{1-r}[p(z_0, z_1)]$, then clearly $\alpha_n \to 0$. Thus, by Lemma 1.1, $\{z_n\}$ is a Cauchy sequence in X. □

Now, using Lemma 2.1 we prove some fixed point results for multivalued maps which improve the fixed point result of Kikawa and Suzuki [6, Theorem 2] for w-distance.

Theorem 2.2. *Let (X, d) be a complete metric space. Assume that all the hypotheses of Lemma 2.1 hold. Further assume that for any orbit $\{u_n\}$ of T in X with limit u, we have*

$$\beta(r)p(u_n, Tu_n) \leqslant p(u_n, u). \tag{2.1}$$

Then the map T has a fixed point.

Proof. It follows from Lemma 2.1 that there exists an orbit $\{z_n\}$ of T at $z_0 \in X$ which is a Cauchy sequence. Due to the completeness of X, there exists some $z \in X$ such that $z_n \to z$. Now, using the lower semicontinuity of $p(z_n, .)$ and the proof of Lemma 2.1, we have

$$p(z_n, z) \leqslant \liminf_{m\to\infty}[p(z_n, z_m)] \leqslant \frac{r^n}{1-r}[p(z_0, z_1)].$$

Since $z_n \in Tz_{n-1}$ and $z \in X$, using the hypothesis and the definition of T, there exists $w_n \in Tz$ such that

$$p(z_n, w_n) \leqslant rp(z_{n-1}, z).$$

Then

$$p(z_n, w_n) \leqslant \frac{r^n}{1-r}p(u_0, u_1)$$

and thus, by Lemma 1.1, we have $w_n \to z$. Since Tz is closed, we get $z \in Tz$. □

Now, we prove a fixed point result, where the condition (2.1) of Theorem 2.2 is dispensable with somewhat natural condition.

Theorem 2.3. *Assume that all the hypotheses of Theorem 2.2 except the inequality (2.1) hold. Further assume that*

$$\inf\{p(x, u) + p(x, Tx) : x \in X\} > 0,$$

for every $u \in X$ with $u \notin Tu$. Then, the map T has a fixed point.

Proof. Due to Lemma 2.1 there exists an orbit sequence $\{z_n\}$ of T at $z_0 \in X$ which is a Cauchy sequence and for each $n \in \mathbb{N}$ we have

$$p(z_n, z_{n+1}) \leqslant r^n p(z_0, z_1),$$

where $0 < r < 1$. Further, due to the completeness of X, and the lower semicontinuity of $p(z_n, .)$, we have $z_n \to z \in X$ and

$$p(z_n, z) \leqslant \frac{r^n}{1-r}p(z_0, z_1).$$

Assume that $z \notin Tz$. Then, using the hypothesis, we have

$$\begin{aligned}
0 &< \inf\{p(u, z) + p(u, Tu) : u \in X\} \\
&\leqslant \inf\{p(z_n, z) + p(z_n, Tz_n) :: n \in \mathbb{N}\} \\
&\leqslant \inf\left\{\frac{r^n}{1-r}p(z_0, z_1) + r^n p(z_0, z_1) : n \in \mathbb{N}\right\} \\
&\leqslant \left\{\frac{2-r}{1-r}p(z_0, z_1)\right\}\inf\{r^n : n \in \mathbb{N}\} = 0,
\end{aligned}$$

which is impossible and hence z is a fixed point of T. □

Now, using notions of mean and w-distance we prove a general result on the existence of fixed points for multivalued maps, which improve and generalize a number of known results.

Theorem 2.4. *Let (X, d) be a metric space, $p \in W_0(X)$, $T : X \to 2^X$, $\{x_n\}$ be a sequence such that $\{p(x_n, z)\}$ is bounded for some $z \in X$, and $T : X \to 2^X$ be a multivalued mapping. Suppose there exist $r \in [0, 1)$ and a mean μ on ℓ^∞ such that for any $y \in X$ there is $v \in Ty$ with*

$$\mu_n p(x_n, v) \leqslant r\mu_n p(x_n, y). \tag{2.2}$$

Then there exists $u \in X$ such that $u \in Tu$ and $p(u, u) = 0$.

Proof. First we note that for any $y \in X$, the sequence $\{p(x_n, y)\}$ is bounded. Indeed, by the hypothesis, the sequence $\{p(x_n, z)\}$ is bounded for some $z \in X$, so for any $y \in X$ and $n \in \mathbb{N}$ we have

$$p(x_n, y) \leqslant p(x_n, z) + p(z, y) \leqslant \sup_{j \in \mathbb{N}} p\left(x_j, z\right) + p(z, y) < \infty.$$

Using a mean μ on ℓ^∞, we define a function

$$g(y) = \mu_n p(x_n, y), \quad y \in X.$$

Then, clearly g is a real-valued continuous function on X. Let z_0 be an arbitrary but fixed element of X, then by definition there exists $z_1 \in Tz_0$ such that

$$\mu_n p(x_n, z_1) \leqslant r \mu_n p(x_n, z_0).$$

Continuing this process, we get a sequence $z_m \in Tz_{m-1}$ such that

$$\mu_n p(x_n, z_m) \leqslant r^m \mu_n p(x_n, z_0). \tag{2.3}$$

Now, for $m, n \in \mathbb{N}$, we have

$$p(z_m, z_{m+1}) \leqslant p(z_m, x_n) + p(x_n, z_{m+1}).$$

Applying μ_n, we have

$$\begin{aligned} p(z_m, z_{m+1}) &\leqslant \mu_n p(z_m, x_n) + \mu_n p(x_n, z_{m+1}) \\ &= \mu_n p(x_n, z_m) + \mu_n p(x_n, z_{m+1}) \\ &\leqslant r^m \mu_n p(x_n, z_0) + r^{m+1} \mu_n p(x_n, z_0) \\ &= r^m (1 + r) g(z_0). \end{aligned}$$

Thus, for any $k, m \in \mathbb{N}$ with $m > k$, we have

$$\begin{aligned} p(z_k, z_m) &\leqslant p(z_k, z_{k+1}) + p(z_{k+1}, z_{k+2}) + \cdots + p(z_{m-1}, z_m) \\ &\leqslant r^k (1+r) g(z_0) + r^{k+1} (1+r) g(z_0) + \cdots + r^{m-1} (1+r) g(z_0) + \cdots \\ &= r^k (1+r) g(z_0) \left[1 + r + r^2 + r^3 + \cdots + r^{m-1}\right] \\ &\leqslant r^k (1+r) g(z_0) \left[\frac{1}{1-r}\right]. \end{aligned}$$

Since $r \in [0, 1)$, we have $r^k (1+r) g(z_0) \left[\frac{1}{1-r}\right] \to 0$ as $l \to \infty$. Thus by Lemma 1.1 (c), $\{z_m\}$ is a Cauchy sequence in X. Since X is complete, then, there exists some $q \in X$ such that $\{z_m\}$ converges to q. Now, since p is lower-semicontinuous, for any $n \in \mathbb{N}$, we have

$$p(x_n, q) \leqslant \liminf_{m \to \infty} p(x_n, z_m),$$

and thus, we have

$$g(q) = \mu_n p(x_n, q) \leqslant \mu_n \liminf_{m \to \infty} p(x_n, z_m).$$

Also note that for any k, m, n, $m > k$

$$p(x_n, z_m) \leqslant p(x_n, z_k) + p(z_k, z_m) \leqslant p(x_n, z_k) + r^k (1+r) g(z_0) \frac{1}{1-r}.$$

Thus, we get

$$\mu_n \limsup_{m \to \infty} p(x_n, z_m) \leqslant \mu_n p(x_n, z_k) + r^k (1+r) g(z_0) \frac{1}{1-r}.$$

Now, passing the limit as $k \to \infty$, we have

$$\mu_n \limsup_{m\to\infty} p(x_n, z_m) \leqslant \liminf_{k\to\infty} \mu_n p(x_n, z_k) + \lim_{k\to\infty} r^k (1+r)\, g(z_0) \frac{1}{1-r} \leqslant \liminf_{k\to\infty} \mu_n p(x_n, z_k).$$

We get

$$\begin{aligned} g(q) \leqslant \mu_n \liminf_{m\to\infty} p(x_n, z_m) &\leqslant \mu_n \limsup_{m\to\infty} p(x_n, z_m) \\ &\leqslant \liminf_{m\to\infty} \mu_n p(x_n, z_m) \\ &= \liminf_{m\to\infty} g(z_m) \\ &\leqslant \limsup_{m\to\infty} g(z_m). \end{aligned}$$

Further, note that for each $m \in \mathbb{N}$ and $z_m \in Tz_{m-1}$, we have

$$g(z_m) = \mu_n p(x_n, z_m) \leqslant r\mu_n p(x_n, z_{m-1}) \leqslant \cdots \leqslant r^m \mu_n p(x_n, z_0) = r^m g(z_0),$$

and letting $m \to \infty$, we get

$$\limsup_{m\to\infty} g(z_m) \leqslant 0,$$

and thus, using (2.2) and (2.3), we get $g(q) \leqslant 0$. Hence

$$g(q) = \mu_n p(x_n, q) = 0.$$

Now, we show that $q \in Tq$ and $p(q, q) = 0$. Since $q \in X$, so there exists $w \in Tq$ such that

$$\mu_n p(x_n, w) \leqslant r\mu_n p(x_n, q) = rg(q).$$

Also, for each $n \in \mathbb{N}$,

$$p(w, q) \leqslant p(w, x_n) + p(x_n, q).$$

Thus,

$$p(w, q) \leqslant \mu_n p(w, x_n) + \mu_n p(x_n, q) = \mu_n p(x_n, w) + \mu_n p(x_n, q) \leqslant (1+r)g(q) = 0,$$

that is, $p(w, q) = 0$. Now, it is enough to show that $q = w$. Note that

$$p(w, w) \leqslant \mu_n p(w, x_n) + \mu_n p(x_n, w) \leqslant r\mu_n p(x_n, q) + rp(x_n, q) = 0$$

and thus, $p(w, w) = 0$. By Lemma 1.1 we get $q = w \in Tq$ and $p(q, q) = 0$. □

Remark 2.5. Theorems 2.2 and 2.3 are improved versions of [6, Theorem 2] for w-distance. Theorem 2.4 improves and generalizes a number of existence results including the corresponding results of [3, 20].

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