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Fixed point results for generalized Θ-contractions

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Abstract

The aim of this paper is to extend the result of [M. Jleli, B. Samet, J. Inequal. Appl., **2014** (2014), 8 pages] by applying a simple condition on the function Θ. With this condition, we also prove some fixed point theorems for Suzuki-Berinde type Θ-contractions which generalize various results of literature. Finally, we give one example to illustrate the main results in this paper.

Keywords: Complete metric space, Θ-contraction, Suzuki-Berinde type Θ-contraction, fixed point.
2010 MSC: 46S40, 47H10, 54H25.

1. Introduction and preliminaries

Banach's contraction principle [4] is one of the pivotal results of nonlinear analysis and its applications, which establishes that, if F is a mapping from a complete metric space (X, d) into itself and there exists a constant $k \in [0, 1)$ such that

$$d(Fx, Fy) \leqslant kd(x, y),$$

for all $x, y \in X$, then F has a unique fixed point in X.

Due to its importance and simplicity, many authors have obtained a lot of interesting extensions and generalizations of Banach's contraction principle (see [1–3, 6, 10, 12] and references therein).

Especially, in 1962, Edelstein [7] established the following version of Banach's contraction principle for a compact metric space.

Theorem 1.1. *Let (X, d) be a compact metric space and $F : X \to X$ be a self-mapping. Assume that*

$$d(Fx, Fy) < d(x, y)$$

holds for all $x, y \in X$ with $x \neq y$. Then F has a unique fixed point in X.

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In 2008, Suzuki [11] proved generalized versions of Edelstein's results in a compact metric space as follows:

Theorem 1.2. *Let* (X, d) *be a compact metric space and* $F : X \to X$ *be a self-mapping. Assume that*

$$\frac{1}{2}d(x, Fx) < d(x, y) \implies d(Fx, Fy) < d(x, y)$$

holds for all $x, y \in X$ *with* $x \neq y$*. Then* F *has a unique fixed point in* X*.*

On the other hand, Berinde [5] gave the following well-known result as a generalization of Banach's contraction principle:

Theorem 1.3. *Let* (X, d) *be a complete metric space and* $F : X \to X$ *be a self-mapping. If there exist a constant* $k \in [0, 1)$ *and a constant* $L \geqslant 0$ *such that*

$$d(Fx, Fy) \leqslant kd(x, y) + L\min\{d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)\},$$

for all $x, y \in X$*, then* F *has a unique fixed point in* X*.*

Recently, Jleli and Samet [9] introduced a new type of contraction which is called the Θ-contraction and established some new fixed point theorems for such a contraction in the context of generalized metric spaces.

Definition 1.4.

(1) Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying the following conditions:
  (Θ_1) Θ is nondecreasing;
  (Θ_2) for each sequence $\{\alpha_n\} \subseteq \mathbb{R}^+$,

$$\lim_{n\to\infty} \Theta(\alpha_n) = 1 \iff \lim_{n\to\infty} \alpha_n = 0;$$

  (Θ_3) there exist $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$.

(2) A mapping $F : X \to X$ is called the Θ-*contraction* if there exists the function Θ satisfying (Θ_1)-(Θ_3) and a constant $k \in (0, 1)$ such that, for all $x, y \in X$,

$$d(Fx, Fy) \neq 0 \implies \Theta(d(Fx, Fy)) \leqslant [\Theta(d(x, y))]^k.$$

Theorem 1.5 ([9])**.** *Let* (X, d) *be a complete metric space and* $F : X \to X$ *be a* Θ*-contraction. Then* F *has a unique fixed point.*

Also, they showed that any Banach contraction is a particular case of Θ-contraction while there exist Θ-contractions which are not Banach contractions.

To be consistent with Jleli and Samet [9], we denote by Ψ the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions (Θ_1)-(Θ_3).

In 2015, Hussain et al. [8] modified and extended the above result and proved the following fixed point theorem for a generalized Θ-contractive condition in the setting of complete metric spaces:

Theorem 1.6. *Let* (X, d) *be a complete metric space and* $F : X \to X$ *be a self-mapping. If there exists a function* $\Theta \in \Omega$ *and positive real numbers* $\alpha, \beta, \gamma, \delta$ *with* $0 \leqslant \alpha + \beta + \gamma + 2\delta < 1$ *such that*

$$\Theta(d(Fx, Fy)) \leqslant [\Theta(d(x, y))]^\alpha \cdot [\Theta(d(x, Fx))]^\beta \cdot [\Theta(d(y, Fy))]^\gamma \cdot [\Theta((d(x, Fy) + d(y, Fx))]^\delta,$$

for all $x, y \in X$*, then* F *has a unique fixed point.*

In this paper, we use the following condition instead of the condition (Θ_3) in Definition 1.4.

(Θ_3') Θ is continuous on $(0, \infty)$.

We denote by Ω the set of all functions satisfying the conditions (Θ_1), (Θ_2), and (Θ_3').

Example 1.7. Define some functions as follows: for all $t > 0$,

(1) $\Theta_1(t) = e^{\sqrt{t}}$;
(2) $\Theta_2(t) = e^{\sqrt{te^t}}$;
(3) $\Theta_3(t) = e^t$;
(4) $\Theta_4(t) = \cosh t$;
(5) $\Theta_5(t) = 1 + \ln(1 + t)$;
(6) $\Theta_6(t) = e^{te^t}$.

Then Θ_1, Θ_2, Θ_3, Θ_4, Θ_5, $\Theta_6 \in \Omega$.

Example 1.8. Note that the conditions Θ_3 and Θ_3' are independent of each other. Indeed, for $p \geqslant 1$, $\Theta(t) = e^{t^p}$ satisfies the conditions (Θ_1) and (Θ_2), but it does not satisfy (Θ_3), while it satisfies the condition (Θ_3'). Therefore, $\Omega \not\subseteq \Psi$. Again, for any $p > 1$ and $m \in (0, \frac{1}{p})$, a function $\Theta(t) = 1 + t^m(1 + [t])$, where $[t]$ denotes the integer part of t, satisfies the conditions (Θ_1) and (Θ_2), but it does not satisfy (Θ_3'), while it satisfies the condition (Θ_3) for any $k \in (\frac{1}{p}, 1)$. Therefore, $\Psi \not\subseteq \Omega$. Also, if we define $\Theta(t) = e^{\sqrt{t}}$, then $\Theta \in \Psi$ and $\Theta \in \Omega$. Therefore, $\Psi \cap \Omega \neq \emptyset$.

2. Main results

In this section, we define the Θ-contraction for a new family of functions Ω and establish some fixed point theorems in the context of complete metric spaces.

Definition 2.1. Let (X, d) be a metric space and F be a self-mapping on X. We say that F is the Θ-*contraction* if there exist $\Theta \in \Omega$ and a constant $k \in (0, 1)$ such that

$$\Theta(d(Fx, Fy)) \leqslant [\Theta(d(x, y))]^k,$$

for all $x, y \in X$ with $Fx \neq Fy$.

In view of Example 1.8, it is meaningful to consider the result of Jleli and Samet [9] with the function $\Theta \in \Omega$ instead of $\Theta \in \Psi$.

Theorem 2.2. *Let (X, d) be a complete metric space and $F : X \to X$ be the Θ-contraction. Then F has a unique fixed point $z \in X$ and, for any $x_0 \in X$, the sequence $\{F^n x_0\}$ converges to the point z.*

Proof. Let $x_0 \in X$, we define a sequence $\{x_n\}$ by $x_{n+1} = F^n x_0 = Fx_n$ for each $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of F and we have nothing to prove. So, without loss of generality, we assume that $x_n \neq x_{n+1}$, i.e., $Fx_{n-1} \neq Fx_n$ for all $n \in \mathbb{N}$. It follows from the assumption that

$$\begin{aligned} 1 < \Theta(d(x_n, x_{n+1})) = \Theta(d(Fx_{n-1}, Fx_n)) &\leqslant [\Theta(d(x_{n-1}, x_n))]^k = [\Theta(d(Fx_{n-2}, Fx_{n-1}))]^k \\ &\leqslant [\Theta(d(x_{n-2}, x_{n-1}))]^{k^2} \\ &\;\vdots \\ &\leqslant [\Theta(d(x_0, x_1))]^{k^n}, \end{aligned} \tag{2.1}$$

for all $n \in \mathbb{N}$. Since $\Theta \in \Omega$, by taking the limit as $n \to \infty$ in (2.1), we have

$$\lim_{n\to\infty} \Theta(d(x_n, x_{n+1})) = 1 \iff \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.2}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and the sequences $\{p(n)\}$ and $\{q(n)\}$ of natural numbers such that, for any $p(n) > q(n) > n$,

$$d(x_{p(n)}, x_{q(n)}) \geqslant \varepsilon, \quad d(x_{p(n)-1}, x_{q(n)}) < \varepsilon, \tag{2.3}$$

for each $n \in \mathbb{N}$. So, by the triangle inequality and (2.3), we have

$$\varepsilon \leqslant d(x_{p(n)}, x_{q(n)}) \leqslant d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < d(x_{p(n)-1}, x_{p(n)}) + \varepsilon. \tag{2.4}$$

By taking the limit as $n \to \infty$ in (2.4), we have

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \varepsilon. \tag{2.5}$$

From (2.2), we can choose a natural number $n_0 \in \mathbb{N}$ such that

$$d(x_{p(n)}, x_{p(n)+1}) < \frac{\varepsilon}{4}, \quad d(x_{q(n)}, x_{q(n)+1}) < \frac{\varepsilon}{4}, \tag{2.6}$$

for each $n \geqslant n_0$.

Next, we claim that $Fx_{p(n)} \neq Fx_{q(n)}$ for all $n \geqslant n_0$, that is,

$$d(x_{p(n)+1}, x_{q(n)+1}) = d(Fx_{p(n)}, Fx_{q(n)}) > 0. \tag{2.7}$$

Suppose that there exists $n \geqslant n_0$ such that $d(x_{p(n)+1}, x_{q(n)+1}) = 0$. It follows from (2.2), (2.5), and (2.6) that

$$\begin{aligned}\varepsilon \leqslant d(x_{p(n)}, x_{q(n)}) &\leqslant d(x_{p(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)})\\ &< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4}\\ &= \frac{\varepsilon}{2},\end{aligned}$$

which is a contradiction. Thus the relation (2.7) holds. Then, by the assumption, we have

$$\Theta(d(Fx_{p(n)}, Fx_{q(n)})) \leqslant [\Theta(d(x_{p(n)}, x_{q(n)}))]^k.$$

By taking the limit as $n \to \infty$ and using (Θ_3') and (2.5), it follows that

$$\Theta(\varepsilon) \leqslant [\Theta(\varepsilon)]^k,$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. The completeness of X ensures that there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Finally, the continuity of F yields

$$d(z, Fz) = \lim_{n\to\infty} d(x_n, Fx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = d(z, z) = 0.$$

Hence z is a fixed point of F.

Now, we show the uniqueness of the fixed point z. Suppose that there exists another fixed point u of F distinct from z, that is,

$$Fz = z \neq u = Fu.$$

Then it follows from the assumption that

$$\Theta(d(z, u)) = \Theta(d(Fz, Fu)) \leqslant [\Theta(d(z, u))]^k,$$

which is a contradiction since $k \in (0, 1)$. Thus z is the unique fixed point of F. This completes the proof. □

Note that the family Ω consists of a large class of functions. For example, if we take

$$\Theta(t) = 2 - \frac{2}{\pi}\arctan\left(\frac{1}{t^a}\right),$$

where $0 < a < 1$ and $t > 0$, then $\Theta \in \Omega$ and we can obtain the following result from Theorem 2.2.

Theorem 2.3. *Let (X, d) be a complete metric space and F be a self-mapping on X. If there exist constants $a, k \in [0, 1)$ such that*

$$2-\frac{2}{\pi}\arctan\left(\frac{1}{d(Fx,Fy)^{a}}\right) \leqslant \left[2-\frac{2}{\pi}\arctan\left(\frac{1}{d(x,y)^{a}}\right)\right]^{k},$$

for all $x, y \in X$ with $Fx \neq Fy$, then F has a unique fixed point $z \in X$ and, for all $x_0 \in X$, the sequence $\{F^n x_0\}$ converges to the point z.

3. Fixed point results for the Suzuki-Berinde type Θ-contraction

In the present section, we define the Suzuki-Berinde type Θ-contraction to prove some fixed point theorems in the context of complete metric spaces.

Definition 3.1. Let (X, d) be a metric space and F be a self-mapping on X. We say that F is the *Suzuki-Berinde type Θ-contraction* if there exist $\Theta \in \Omega$, $k \in (0, 1)$ and $L \geqslant 0$ such that, for all $x, y \in X$ with $Fx \neq Fy$,

$$\frac{1}{2}d(x,Fx) < d(x,y) \implies \Theta(d(Fx,Fy)) \leqslant [\Theta(d(x,y))]^{k} + L\min\{d(x,Fx), d(x,Fy), d(y,Fx)\}.$$

Theorem 3.2. *Let (X, d) be a complete metric space and $F : X \to X$ be a self-mapping satisfying the Suzuki-Berinde type Θ-contraction. Then F has a unique fixed point $z \in X$ and, for any $x_0 \in X$, the sequence $\{F^n x_0\}$ converges to the point z.*

Proof. For any $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_{n+1} = F^n x_0 = Fx_n$ for each $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of F and we have nothing to prove. So we assume that $x_{n-1} \neq x_n$ or

$$0 < d(x_{n-1}, Fx_{n-1}),$$

for each $n \in \mathbb{N}$. Therefore, we have

$$\frac{1}{2}d(x_{n-1},Fx_{n-1}) < d(x_{n-1},Fx_{n-1}) = d(x_{n-1},x_n), \tag{3.1}$$

for each $n \in \mathbb{N}$. It follows from the assumption that

$$\Theta(d(Fx_{n-1},Fx_n)) \leqslant [\Theta(d(x_{n-1},x_n))]^{k} + L\min\{d(x_{n-1},Fx_{n-1}), d(x_{n-1},Fx_n), d(x_n,Fx_{n-1})\},$$

which implies that

$$\Theta(d(Fx_{n-1},Fx_n)) \leqslant [\Theta(d(x_{n-1},x_n))]^{k} + L\min\{d(x_{n-1},x_n), d(x_{n-1},x_{n+1}), d(x_n,x_n)\} = [\Theta(d(x_{n-1},x_n))]^{k}.$$

Therefore, we have

$$1 < \Theta(d(x_n,x_{n+1})) = \Theta(d(Fx_{n-1},Fx_n)) \leqslant [\Theta(d(x_{n-1},x_n))]^{k} \leqslant \cdots \leqslant [\Theta(d(x_0,x_1))]^{k^n}, \tag{3.2}$$

for each $n \in \mathbb{N}$. Since $\Theta \in \Omega$, by taking the limit as $n \to \infty$ in (3.2), we have

$$\lim_{n\to\infty}\Theta(d(x_n,x_{n+1})) = 1 \iff \lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3.3}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and the sequences $\{p(n)\}$ and $\{q(n)\}$ of natural numbers such that, for any $p(n) > q(n) > n$,

$$d(x_{p(n)},x_{q(n)}) \geqslant \varepsilon, \quad d(x_{p(n)-1},x_{q(n)}) < \varepsilon, \tag{3.4}$$

for each $n \in \mathbb{N}$. So, by the triangle inequality and (3.4), we have

$$\varepsilon \leqslant d(x_{p(n)}, x_{q(n)}) \leqslant d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < d(x_{p(n)-1}, Fx_{p(n)-1}) + \varepsilon. \tag{3.5}$$

By taking the limit as $n \to \infty$ in (3.5) and using the inequality (3.3), we have

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \varepsilon. \tag{3.6}$$

From (3.1) and (3.4), we can choose a natural number $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_{p(n)}, Fx_{p(n)}) < \frac{\varepsilon}{2} < d(x_{p(n)}, x_{q(n)}),$$

for all $n \geqslant n_0$. On the other hand, by the assumption, we have

$$\begin{aligned}\Theta(d(Fx_{p(n)}, Fx_{q(n)})) &\leqslant [\Theta(d(x_{p(n)}, x_{q(n)}))]^k \\ &\quad + L\min\{d(x_{p(n)}, Fx_{p(n)}), d(x_{p(n)}, Fx_{q(n)}), d(x_{q(n)}, Fx_{p(n)})\} \\ &= [\Theta(d(x_{p(n)}, x_{q(n)}))]^k \\ &\quad + L\min\{d(x_{p(n)}, x_{p(n)+1}), d(x_{p(n)}, x_{q(n)+1}), d(x_{q(n)}, x_{p(n)+1})\}.\end{aligned} \tag{3.7}$$

By taking the limit as $n \to \infty$ in (3.7) and using (Θ_3') and (3.6), we have

$$\Theta(\varepsilon) \leqslant [\Theta(\varepsilon)]^k,$$

which is a contradiction since $k \in (0, 1)$. Thus $\{x_n\}$ is a Cauchy sequence. Thus the completeness of X ensures that there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Next, we claim that

$$\frac{1}{2}d(x_n, Fx_n) < d(x_n, z) \ \text{ or } \ \frac{1}{2}d(Fx_n, F^2x_n) < d(Fx_n, z), \tag{3.8}$$

for each $n \in \mathbb{N}$. Suppose that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_m, Fx_m) \geqslant d(x_m, z) \ \text{ and } \ \frac{1}{2}d(Fx_m, F^2x_m) \geqslant d(Fx_m, z). \tag{3.9}$$

Then we have

$$2d(x_m, z) \leqslant d(x_m, Fx_m) \leqslant d(x_m, z) + d(z, Fx_m),$$

which implies that

$$d(x_m, z) \leqslant d(z, Fx_m). \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$d(x_m, z) \leqslant d(z, Fx_m) \leqslant \frac{1}{2}d(Fx_m, F^2x_m).$$

Since $\frac{1}{2}d(x_m, Fx_m) < d(x_m, Fx_m)$, by the assumption, we have

$$\Theta(d(Fx_m, F^2x_m)) \leqslant [\Theta(d(x_m, Fx_m))]^k + L\min\{d(x_m, Fx_m), d(x_m, F^2x_m), d(Fx_m, Fx_m)\},$$

which implies that

$$\Theta(d(Fx_m, F^2x_m)) \leqslant [\Theta(d(x_m, Fx_m))]^k.$$

Since Θ is strictly increasing, we have

$$d(Fx_m, F^2x_m) < d(x_m, Fx_m). \tag{3.11}$$

It follows from (3.9), (3.10), and (3.11) that

$$d(Fx_m, F^2x_m) < d(x_m, Fx_m) \leqslant d(x_m, z) + d(z, Fx_m) \leqslant \frac{1}{2}d(Fx_m, F^2x_m) + \frac{1}{2}d(Fx_m, F^2x_m) = d(Fx_m, F^2x_m),$$

which is a contradiction. Hence (3.8) holds and so, for each $n \in \mathbb{N}$,

$$1 < \Theta(d(Fx_n, Fz)) \leqslant [\Theta(d(x_n, z))]^k + L\min\{d(x_n, Fx_n), d(x_n, Fz), d(z, Fx_n)\},$$

which implies that

$$1 < \Theta(d(Fx_n, Fz)) \leqslant [\Theta(d(x_n, z))]^k + L\min\{d(x_n, x_{n+1}), d(x_n, Fz), d(z, x_{n+1})\}. \tag{3.12}$$

Using (3.12) and (Θ_2), we have

$$\lim_{n\to\infty} \Theta(d(Fx_n, Fz)) = 1$$

and so, from (Θ_2),

$$\lim_{n\to\infty} d(Fx_n, Fz) = 0.$$

Therefore, we have

$$d(z, Fz) = \lim_{n\to\infty} d(x_{n+1}, Fz) = \lim_{n\to\infty} d(Fx_n, Fz) = 0.$$

Hence z is a fixed point of F.

Now, we show the uniqueness of the fixed point z. Suppose that there exists another fixed point u of F distinct from z, that is,

$$Fz = z \neq u = Fu.$$

Thus we have $\frac{1}{2}d(z, Fz) < d(z, u)$ and so, from the assumption,

$$\Theta(d(z, u)) = \Theta(d(Fz, Fu)) \leqslant [\Theta(d(z, u))]^k + L\min\{d(z, Fz), d(z, Fu), d(u, Fz)\},$$

which implies that

$$\Theta(d(z, u)) \leqslant [\Theta(d(z, u))]^k,$$

which is a contradiction since $k \in (0, 1)$. Thus z is the unique fixed point of F. This completes the proof. □

Theorem 3.3. *Let (X, d) be a complete metric space and $F : X \to X$ be a self-mapping. If there exists $\Theta \in \Omega$ such that, for all $x, y \in X$ with $Fx \neq Fy$,*

$$\frac{1}{2}d(x, Fx) < d(x, y) \implies \Theta(d(Fx, Fy)) \leqslant [\Theta(d(x, y))]^k,$$

then F has a unique fixed point $z \in X$ and, for any $x_0 \in X$, the sequence $\{F^n x_0\}$ is convergent to the point z.

Theorem 3.4. *Let (X, d) be a complete metric space and F be a self-mapping on X. If there exist constants $a, k \in (0, 1)$ and $L \geqslant 0$ such that*

$$\frac{1}{2}d(x, Fx) < d(x, y) \Longrightarrow 2 - \frac{2}{\pi}\arctan\left(\frac{1}{d(Fx, Fy)^a}\right) \leqslant \left[2 - \frac{2}{\pi}\arctan\left(\frac{1}{d(x, y)^a}\right)\right]^k + L\min\{d(x, Fx), d(x, Fy), d(y, Fx)\},$$

for all $x, y \in X$ with $Fx \neq Fy$, then F has a unique fixed point $z \in X$ and, for any $x_0 \in X$, the sequence $\{F^n x_0\}$ converges to the point z.

Proof. Taking $\Theta(t) = 2 - \frac{2}{\pi}\arctan(\frac{1}{t^a})$ in Theorem 3.2, we have the conclusion. □

Corollary 3.5. *Let (X, d) be a complete metric space and F be a self-mapping on X. If there exist constants $a, k \in (0, 1)$ such that*

$$\frac{1}{2}d(x, Fx) < d(x, y) \Longrightarrow 2 - \frac{2}{\pi}\arctan\left(\frac{1}{d(Fx, Fy)^a}\right) \leqslant \left[2 - \frac{2}{\pi}\arctan\left(\frac{1}{d(x, y)^a}\right)\right]^k,$$

for all $x, y \in X$ with $Fx \neq Fy$, then F has a unique fixed point $z \in X$ and, for any $x_0 \in X$, the sequence $\{F^n x_0\}$ converges to the point z.

Proof. Taking $\Theta(t) = 2 - \frac{2}{\pi}\arctan(\frac{1}{t^a})$ in Theorem 3.3, we have the conclusion. □

Example 3.6. Consider the sequence $\{S_n\}$ defined as follows:

$$S_1 = 1, \quad S_2 = 1 + 5, \cdots,$$

and

$$S_n = 1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1), \cdots.$$

Let $X = \{S_n : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$ be the usual metric. Then (X, d) is a complete metric space. Define a mapping $F : X \to X$ by

$$F(S_1) = S_1, \quad F(S_n) = S_{n-1},$$

for each $n > 1$. Clearly, F does not satisfy Banach's contraction. In fact, we can easily check the following:

$$\lim_{n\to\infty} \frac{d(F(S_n), F(S_1))}{d(S_n, S_1)} = \lim_{n\to\infty} \frac{S_{n-1} - 1}{S_n - 1} = \lim_{n\to\infty} \frac{(n-1)(2n-3) - 1}{n(2n-1) - 1} = 1.$$

Also, F does not satisfy the Suzuki-Berinde contraction. On the other hand, by considering the mapping $\Theta : (0, \infty) \to (1, \infty)$ defined by

$$\Theta(t) = e^{te^t},$$

we can easily show that $\Theta \in \Omega$ and F is the Suzuki-Berinde type Θ-contraction, that is, there exist $k \in (0, 1)$ and $L \geqslant 0$ such that

$$\frac{1}{2}d(S_n, F(S_n)) < d(S_n, S_m) \Longrightarrow e^{d(F(S_n), F(S_m))e^{d(F(S_n), F(S_m))}}$$
$$\leqslant e^{kd(S_n, S_m)e^{d(S_n, S_m)}} + L\min\{d(S_n, F(S_n)), d(S_n, F(S_m)), d(S_m, F(S_n))\}.$$

Now, we consider the following two cases:

Case 1. For $1 = n$ and $m > 2$, we have

$$e^{(2m^2-5m+3)e^{2m^2-5m+2}} < e^{k(2m^2-m-1)e^{2m^2-m-1}},$$

for $k = e^{-1} \in (0, 1)$ and so

$$e^{d(F(S_1), F(S_m))e^{d(F(S_1), F(S_m))}} \leqslant e^{kd(S_1, S_m)e^{d(S_1, S_m)}} + L\min\{d(S_1, F(S_1)), d(S_1, F(S_m)), d(S_m, F(S_1))\},$$

for some $L \geqslant 0$.

Case 2. For $m > n > 1$, we have

$$e^{(2m^2-5m-2n^2+5n)e^{2m^2-5m-2n^2+5n}} < e^{k(2m^2-m-2n^2+n)e^{2m^2-m-2n^2+n}},$$

for $k = e^{-1} \in (0, 1)$ and so

$$e^{d(F(S_n), F(S_m))e^{d(F(S_n), F(S_m))}} \leqslant e^{kd(S_n, S_m)e^{d(S_n, S_m)}} + L\min\{d(S_n, F(S_n)), d(S_n, F(S_m)), d(S_m, F(S_n))\},$$

for some $L \geqslant 0$. Hence all of the conditions of Theorem 3.2 are satisfied and S_1 is a unique fixed point of the mapping F. But F does not satisfy the condition (Θ_3) and so the result [Theorem 5] of Jleli and Samet [9] and the result of Hussain et al. [8] can not be applied to this example.

References

[1] A. Ahmad, A. S. Al-Rawashdeh, A. Azam, *Fixed point results for* $\{\alpha, \xi\}$*-expansive locally contractive mappings*, J. Inequal. Appl., **2014** (2014), 10 pages. 1

[2] J. Ahmad, A. Al-Rawashdeh, A. Azam, *New fixed point theorems for generalized* F*-contractions in complete metric spaces*, Fixed Point Theory Appl., **2015** (2015), 18 pages.

[3] A. Al-Rawashdeh, J. Ahmad, *Common fixed point theorems for JS-contractions*, Bull. Math. Anal. Appl., **8** (2016), 12–22. 1

[4] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181. 1

[5] V. Berinde, *General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces*, Carpathian J. Math., **24** (2008), 10–19. 1

[6] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273. 1

[7] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc., **37** (1962), 74–79. 1

[8] N. Hussain, V. Parvaneh, B. Samet, C. Vetro, *Some fixed point theorems for generalized contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2015** (2015), 17 pages. 1, 3.6

[9] M. Jleli, B. Samet, *A new generalization of the Banach contraction principle*, J. Inequal. Appl., **2014** (2014), 8 pages. 1, 1.5, 1, 2, 3.6

[10] Z.-L. Li, S.-J. Jiang, *Fixed point theorems of JS-quasi-contractions*, Fixed Point Theory Appl., **2016** (2016), 11 pages. 1

[11] T. Suzuki, *A new type of fixed point theorem in metric spaces*, Nonlinear Anal., **71** (2009), 5313–5317. 1

[12] F. Vetro, *A generalization of Nadler fixed point theorem*, Carpathian J. Math., **31** (2015), 403–410. 1