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Ulam-Hyers stability, well-posedness and limit shadowing property of the fixed point problems for some contractive mappings in M_s-metric spaces

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Abstract

In this paper, first, we introduce several types of the Ulam-Hyers stability, the well-posedness and the limit shadowing property of fixed point problems in M_s-metric spaces. Second, we give such results for fixed point problems of Banach and Kannan contractive mappings in M_s-metric spaces. Finally, we give some examples to illustrate the validity of our main results. ©2017 All rights reserved.

1. Introduction and preliminaries

In 1922, Banach [1] proved a famous theorem, which is well-known as "Banach's contraction principle" or "Banach's fixed point theorem", to establish the existence of solutions for integral equations.

Theorem 1.1 ([1])**.** *Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ *be a contractive mapping, that is, there exists* $k \in [0, 1)$ *such that*

$$d(Tx, Ty) \leqslant kd(x, y),$$

for all $x, y \in X$*. Then* T *has a unique fixed point* $z \in X$*, that is,* $Tz = z$*. Furthermore, for each* $x_0 \in X$*, the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = Tx_n,$$

for all $n \geqslant 0$ *converges to the fixed point* z *of* T*.*

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Since then, because of its simplicity, usefulness and applications, it has become a very popular and useful tool in solving the existence problems in many branches of mathematical analysis.

From Banach's contraction principle, it follows that the contractive mapping is continuous. It is inspired to consider that *do there exist some contractive conditions which need the mapping* T *has not to be necessarily continuous?*

The answer for this inquiry was provided by Kannan [5] in 1969 who proved "Kannan's fixed point theorem" for the following contractive mapping, which is called *Kannan's contraction*.

Theorem 1.2 ([5]). *Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ *be a contractive mapping, that is, there exists* $k \in [0, \frac{1}{2})$ *such that*

$$d(Tx, Ty) \leqslant k[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$. *Then,* T *has a unique fixed point* $z \in X$, *that is,* $Tz = z$.

In 1975, Subrahmanyam [12] proved that Kannan's contraction characterizes the metric completeness, that is, if every Kannan's contractive mapping has a fixed point, then a metric space (X, d) is complete.

Recently, some authors have introduced some extensions of metric spaces in several ways and studied fixed point theory and applications in these metric spaces.

Especially, in 2012, Sedghi et al. [11] introduced the concept of S-metric spaces which is a space with three dimensions and they studied the topological properties of S-metric spaces and showed the existence and uniqueness of a fixed point for certain generalized contractive mappings in the setting of S-metric spaces. In 2014, Mlaiki et al. [7] established an extension of S-metric spaces to partial S-metric spaces and, also, they pointed out that every S-metric space is a partial S-metric space, but the converse is not true. Recently, the concept of a partial S-metric space was extended to the concept of M_s-space by Mlaiki et al. [8]. Moreover, they showed the existence and uniqueness of a fixed point for a self-mapping in M_s-metric spaces under Banach's and Kannan's contractive mappings, which are the generalizations of Banach's and Kannan's fixed point theorems in the framework of partial S-metric spaces.

Theorem 1.3 ([8]). *Let* (X, m_s) *be a complete* M_s-*metric space and* $T : X \mapsto X$ *be a self-mapping on* X *satisfying the following condition:*

$$m_s(Tx, Ty, Tz) \leqslant km_s(x, y, z),$$

for all $x, y, z \in X$, *where* $k \in [0, 1)$. *Then* T *has a unique fixed point* $u \in X$, *such that* $m_s(u, u, u) = 0$.

Theorem 1.4 ([8]). *Let* (X, m_s) *be a complete* M_s-*metric space and* $T : X \mapsto X$ *be a self-mapping on* X *satisfying the following condition:*

$$m_s(Tx, Tx, Ty) \leqslant \lambda[m_s(x, x, Tx) + m_s(y, y, Ty)],$$

for all $x, y \in X$, *where* $\lambda \in [0, \frac{1}{2})$. *Then* T *has a unique fixed point* $u \in X$ *such that* $m_s(u, u, u) = 0$.

On the other hand, recently, some authors studied and extended the Ulam-Hyers stability, the well-posedness and the limit shadowing property of for fixed point problems in various spaces (see [2–4, 6, 9, 10]).

In this paper, we first present various types of the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem in an M_s-metric space which is an extension of a metric space. Second, we study the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem for Banach's and Kannan's contractive mappings in M_s-metric spaces. Finally, we give some examples to illustrate the validity of our main results.

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}^+$, and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers, and real numbers, respectively. The following definitions, notations and lemmas are needed in the following discussion.

Definition 1.5 ([7]). Let X be a nonempty set. A partial S-metric on X is a function $S_p : X^3 \mapsto [0, \infty)$ satisfying the following conditions: for all $x, y, z, t \in X$,

(s_p1) $S_p(x, y, z) \geqslant 0$;
(s_p2) $x = y$ if and only if $s_p(x, x, x) = s_p(y, y, y) = s_p(x, x, y)$;
(s_p3) $s_p(x, y, z) \leqslant s_p(x, x, t) + s_p(y, y, t) + s_p(z, z, t) - s_p(t, t, t)$;
(s_p4) $s_p(x, x, x) \leqslant s_p(x, y, z)$;
(s_p5) $s_p(x, x, y) = s_p(y, y, x)$.

The pair (X, s_p) is called a *partial S-metric space*.

Note that every S-metric space is a partial S-metric space, but not every partial S-metric space is an S-metric space.

Now, we give an example to show that a partial S-metric space needs not to be necessarily an S-metric space.

Example 1.6. Let $X = [0, 1]$ and define a function $s_p : X^3 \mapsto \mathbb{R}^+$ by $s_p(x, y, z) = \max\{x, y, z\}$ for all $x, y, z \in X$. Then s_p is a partial S-metric, but it is not an S-metric on X. Indeed, for any $0 < x < 1$, we have $s_p(x, x, x) = x \neq 0$.

Next, we present the definition of an M_s-metric space, but first we introduce the following notations which are useful in the sequel:

(1) $m_{s_{x,y,z}} = \min\{m_s(x, x, x), m_s(y, y, y), m_s(z, z, z)\}$;
(2) $M_{s_{x,y,z}} = \max\{m_s(x, x, x), m_s(y, y, y), m_s(z, z, z)\}$.

Definition 1.7 ([8]). An M_s-metric on a nonempty set X is a function $m_s : X^3 \mapsto \mathbb{R}^+$ that satisfies the following conditions: for all $x, y, z, t \in X$,

(m_s1) $m_s(x, x, x) = m_s(y, y, y) = m_s(z, z, z) = m_s(x, y, z)$ if and only if $x = y = z$;
(m_s2) $m_{s_{x,y,z}} \leqslant m_s(x, y, z)$;
(m_s3) $m_s(x, x, y) = m_s(y, y, x)$;
(m_s4) $(m_s(x, y, z) - m_{s_{x,y,z}}) \leqslant (m_s(x, x, t) - m_{s_{x,x,t}}) + (m_s(y, y, t) - m_{s_{y,y,t}}) + (m_s(z, z, t) - m_{s_{z,z,t}})$.

Then the pair (X, m_s) is called an M_s-*metric space*.

Example 1.8. Let $X = [0, \infty)$ and $m_s : X^3 \mapsto \mathbb{R}^+$ be a mapping defined by

$$m_s(x, y, z) = \max\{x, y, z\} - \min\{x, y, z\},$$

for all $x, y, z \in X$. Then m_s is an M_s-metric on X.

Example 1.9. Let X be a nonempty set and d be the ordinary metric on X. Define the mapping $m_s : X^3 \mapsto [0, \infty)$ by

$$m_s(x, y, z) = d(x, y) + d(x, z) + d(y, z),$$

for all $x, y, z \in X$. Then m_s is an M_s-metric on X.

Example 1.10. Let $X = \{1, 2, 3\}$ and define a mapping m_s on X by

$m_s(1, 2, 3) = 6$, $m_s(1, 1, 2) = m_s(2, 2, 1) = 10$, $m_s(1, 1, 3) = m_s(3, 3, 1) = m_s(2, 2, 3) = m_s(3, 3, 2) = 7$,
$m_s(2, 2, 2) = 9$, $m_s(3, 3, 3) = 5$, $m_s(1, 1, 1) = 8$.

Then m_s is an M_s-metric on X, but m_s is not a partial S-metric since $m_s(1, 1, 1) \nleqslant m_s(1, 2, 3)$.

Example 1.11. Let (X, s) be an S-metric space with the S-metric s. Let $\phi : [0, \infty) \mapsto [0, \infty)$ be an injective and nondecreasing mapping such that

$$\phi(x + y + z) \leqslant \phi(x) + \phi(y) + \phi(z) - 2\phi(0).$$

Then $m_s(x, y, z) = \phi(s(x, y, z))$ is an M_s-metric on X.

Proof. It is easy to prove that the conditions $(m_s 1)$, $(m_s 2)$, and $(m_s 3)$ are satisfied. For the condition $(m_s 4)$, it follows that, for for all $x, y, z, t \in X$,

$$\phi(s(x, y, z)) \leqslant \phi(s(x, x, t) + s(y, y, t) + s(z, z, t))$$
$$\leqslant \phi(s(x, x, t)) + \phi(s(y, y, t)) + \phi(s(z, z, t)) - 2\phi(0),$$
$$\phi(s(x, y, z)) - \phi(0) \leqslant (\phi(s(x, x, t)) - \phi(0)) + (\phi(s(y, y, t)) - \phi(0)) + (\phi(s(z, z, t)) - \phi(0)),$$
$$(m_s(x, y, z) - m_{s_{x,y,z}}) \leqslant (m_s(x, x, t) - m_{s_{x,x,t}}) + (m_s(y, y, t) - m_{s_{y,y,t}}) + (m_s(z, z, t) - m_{s_{z,z,t}}).$$

This completes the proof. $\qquad \square$

Example 1.12. Let (X, s) be an S-metric space with the S-metric s. Then the mapping m_s defined by $m_s(x, y, z) = as(x, y, z) + b$ for all $x, y, z \in X$, where $a, b > 0$, is an M_s-metric since we can put $\phi(t) = at + b$.

Remark 1.13. According to Example 1.12, from Banach's contraction, that is, there exists $k \in [0, 1)$ such that

$$m_s(Tx, Ty, Tz) \leqslant km_s(x, y, z),$$

for all $x, y, z \in X$, we have

$$m_s(Tx, Ty, Tz) = as(Tx, Ty, Tz) + b \leqslant kas(x, y, z) + kb \Rightarrow s(Tx, Ty, Tz) \leqslant ks(x, y, z) + \frac{b(k-1)}{a},$$

which does not imply the ordinary Banach contraction in S-metric spaces (see [11]), that is, there exists $k \in [0, 1)$ such that

$$s(Tx, Tx, Ty) \leqslant ks(x, x, y),$$

for all $x, y \in X$, where T is a self-mapping on X.

Conversely, the ordinary Banach contraction in S-metric spaces implies Banach's contraction in M_s-metric spaces. Thus this states that, even if the M_s-metric m_s and the ordinary S-metric s have the same topology, Banach's contraction for the M_s-metric m_s does not imply Banach's contraction for the ordinary S-metric s.

For example, let X be a nonempty set and d be the ordinary metric on X. Define two mappings $s : X^3 \mapsto [0, \infty)$ and $m_s : X^3 \mapsto [0, \infty)$ by

$$s(x, y, z) = d(x, z) + d(y, z), \quad m_s(x, y, z) = d(x, y) + d(x, z) + d(y, z)$$

for all $x, y, z \in X$, respectively. Then s and m_s are an S-metric and an M_s-metric, respectively. Obviously, it follows that the S-metric s and the M_s-metric m_s generate two topologies τ_s and τ_{m_s} on X, respectively, where

$$B_s(x, \epsilon) := \{y \in X : s(x, x, y) < \epsilon\} = \{y \in X : d(x, y) < \epsilon\}$$

and

$$B_{m_s}(x, \epsilon) := \{y \in X : m_s(x, x, y) - m_{s_{x,x,y}} < \epsilon\} = \{y \in X : 2d(x, y) < \epsilon\}$$

form the base of τ_s and τ_{m_s}, respectively. Notice that τ_s and τ_{m_s} are the same topology. However, by Banach's contraction in M_s-metric spaces, we have

$$m_s(Tx, Tx, Ty) < km_s(x, x, y) \Rightarrow d(Tx, Ty) < 2kd(x, y),$$

for some $k \in [0, 1)$, which does not imply the ordinary Banach contraction in S-metric, that is,

$$s(Tx, Tx, Ty) < ks(x, x, y) \Rightarrow d(Tx, Ty) < kd(x, y),$$

for some $k \in [0, 1)$.

Lemma 1.14. *Let* (X, m_s) *be an* M_s-*metric space. Then, for all* $x, y, z, t \in X$,

(1) $m_s(x, y, y) \leqslant m_s(x, x, y)$;
(2) $m_s(x, y, x) \leqslant 2m_s(x, x, y)$;
(3) $m_s(x, y, z) - m_{s_{x,y,z}} \leqslant (m_s(x, x, z) - m_{s_{x,x,z}}) + (m_s(y, y, z) - m_{s_{y,y,z}})$;
(4) $m_s(x, y, z) - m_{s_{x,y,z}} \leqslant (m_s(x, x, y) - m_{s_{x,x,y}}) + (m_s(z, z, y) - m_{s_{z,z,y}})$;
(5) $m_s(x, y, z) - m_{s_{x,y,z}} \leqslant (m_s(y, y, x) - m_{s_{y,y,x}}) + (m_s(z, z, x) - m_{s_{z,z,x}})$;
(6) $m_s(x, y, z) - m_{s_{x,y,z}} \leqslant \frac{2}{3}[(m_s(x, x, z) - m_{s_{x,x,z}}) + (m_s(z, z, y) - m_{s_{z,z,y}}) + (m_s(y, y, x) - m_{s_{y,y,x}})]$;
(7) $(M_{s_{x,y,z}} - m_{s_{x,y,z}}) \leqslant (M_{s_{x,x,t}} - m_{s_{x,x,t}}) + (M_{s_{y,y,t}} - m_{s_{y,y,t}}) + (M_{s_{z,z,t}} - m_{s_{z,z,t}})$.

Proof. (1)-(7) can be directly obtained from Definition 1.7. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 1.15. If m_s is an M_s-metric on a nonempty set X, then two mappings $m_s^w, m_s^* : X^2 \mapsto \mathbb{R}^+$ defined by

$$m_s^w(x, y, z) = m_s(x, y, z) - 2m_{s_{x,y,z}} + M_{s_{x,y,z}}$$

and

$$m_s^*(x, y, z) = \begin{cases} m_s(x, y, z) - m_{s_{x,y,z}}, & x \neq y \neq z, \\ 0, & x = y = z = 0, \end{cases}$$

for all $x, y, z \in X$ are two ordinary S-metrics on X. In fact, if $m_s^w(x, y, z) = 0$, then we have

$$m_s(x, y, z) = 2m_{s_{x,y,z}} - M_{s_{x,y,z}}. \tag{1.1}$$

But, from equation (1.1) and $(m_s 2)$, it follows that

$$m_{s_{x,y,z}} = M_{s_{x,y,z}} = m_s(x, x, x) = m_s(y, y, y) = m_s(z, z, z).$$

So, by equation (1.1), we have that $m_s(x, y, z) = m_s(x, x, x) = m_s(y, y, y) = m_s(z, z, z)$ and so $x = y = z$. We can get the inequality property in the definition of an S-metric from Lemma 1.14 (7) and $(m_s 4)$.

2. Topologies for M_s-metrics

It is clear that each M_s-metric m_s on X generates a topology τ_{m_s} on X. The set

$$\{B_{m_s}(x, \epsilon) : x \in X, \epsilon > 0\},$$

where

$$B_{m_s}(x, \epsilon) := \{y \in X : m_s(x, x, y) - m_{s_{x,x,y}} < \epsilon\},$$

for all $x \in X$ and $\epsilon > 0$, forms a base of τ_{m_s}.

Definition 2.1. Let (X, m_s) be an M_s-metric space. Then:

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point x if
$$\lim_{n \to \infty} (m_s(x_n, x_n, x) - m_{s_{x_n,x_n,x}}) = 0.$$

(2) A sequence $\{x_n\}$ in X is called an M_s-*Cauchy sequence* if $\lim_{n,m \to \infty} (m_s(x_n, x_n, x_m) - m_{s_{x_n,x_n,x_m}})$ and $\lim_{n,m \to \infty} (M_{s_{x_n,x_n,x_m}} - m_{s_{x_n,x_n,x_m}})$ exist and finite.

(3) An M_s-metric space is said to be *complete* if every m_s-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_{m_s}, to a point $x \in X$ such that
$$\lim_{n \to \infty} (m_s(x_n, x_n, x) - m_{s_{x_n,x_n,x}}) = 0, \quad \lim_{n \to \infty} (M_{s_{x_n,x_n,x}} - m_{s_{x_n,x_n,x}}) = 0.$$

Lemma 2.2. *Let (X, m_s) be an M_s-metric space. Then:*

(1) *$\{x_n\}$ is an M_s-Cauchy sequence in (X, m_s) if and only if it is an S-Cauchy sequence in the S-metric space (X, m_s^w).*

(2) *An M_s-metric space (X, m_s) is complete if and only if the S-metric space (X, m_s^w) is complete. Furthermore,*
$$\lim_{n \to \infty} m_s^w(x_n, x_n, x) = 0 \implies \lim_{n \to \infty} (m_s(x_n, x_n, x) - m_{s_{x_n,x_n,x}}) = 0, \quad \lim_{n \to \infty} (M_{s_{x_n,x_n,x}} - m_{s_{x_n,x_n,x}}) = 0.$$

Proof. It obviously follows from the definitions of M_s-Cauchy sequence, M_s-completeness, S-Cauchy sequence and S-completeness. □

Meanwhile, the above assertions are true for m_s^*.

Lemma 2.3. *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M_s-metric space (X, m_s). Then*
$$\lim_{n \to \infty} m_s(x_n, x_n, y_n) - m_{s_{x_n,x_n,y_n}} = m_s(x, x, y) - m_{s_{x,x,y}}.$$

Proof. We have
$$|(m_s(x_n, x_n, y_n) - m_{s_{x_n,x_n,y_n}}) - (m_s(x, x, y) - m_{s_{x,x,y}})|$$
$$\leqslant 2|m_s(x_n, x_n, x) - m_{s_{x_n,x_n,x}}| + 2|m_s(y_n, y_n, y) - m_{s_{y_n,y_n,y}}|.$$
□

From Lemma 2.3, we can deduce the following lemma.

Lemma 2.4. *Assume that $x_n \to x$ as $n \to \infty$ in an M_s-metric space (X, m_s). Then*
$$\lim_{n \to \infty} m_s(x_n, x_n, y) - m_{s_{x_n,x_n,y}} = m_s(x, x, y) - m_{s_{x,x,y}},$$
for all $y \in X$.

Lemma 2.5. *Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in an M_s-metric space (X, m_s). Then, $m_s(x, x, y) = m_{s_{x,x,y}}$. Furthermore, if $m_s(x, x, x) = m_s(y, y, y)$, then $x = y$.*

Proof. By Lemma 2.3, we have
$$0 = \lim_{n \to \infty} m_s(x_n, x_n, x_n) - m_{s_{x_n,x_n,x_n}} = m_s(x, x, y) - m_{s_{x,x,y}}.$$
□

Lemma 2.6. *Let $\{x_n\}$ be a sequence in M_s-metric space (X, m_s) such that there exists $r \in [0, 1)$ such that*
$$m_s(x_{n+1}, x_{n+1}, x_n) \leqslant r m_s(x_n, x_n, x_{n-1}), \tag{2.1}$$
for all $n \in \mathbb{N}$. Then we have

(1) $\lim_{n \to \infty} m_s(x_n, x_n, x_{n-1}) = 0$;

(2) $\lim_{n \to \infty} m_s(x_n, x_n, x_n) = 0$;

(3) $\lim\limits_{n,m\to\infty} m_{s_{x_n,x_n,x_m}} = 0$;

(4) $\{x_n\}$ *is an* M_s-*Cauchy sequence.*

Proof. From the equation (2.1), we have

$$m_s(x_n,x_n,x_{n-1}) \leqslant rm_s(x_{n-1},x_{n-1},x_{n-2}) \leqslant r^2 m_s(x_{n-2},x_{n-2},x_{n-3}) \leqslant \cdots \leqslant r^n m_s(x_1,x_1,x_0)$$

and so $\lim\limits_{n\to\infty} m_s(x_n,x_n,x_{n-1}) = 0$, which implies that (A) holds. From $(m_s 2)$ and (1), we have

$$\lim_{n\to\infty} m_{s_{x_n,x_n,x_{n-1}}} \leqslant \lim_{n\to\infty} m_s(x_n,x_n,x_{n-1}) = 0,$$

that is, (2) holds. Clearly, (3) and (4) hold. □

Theorem 2.7. *The topology* τ_{m_s} *is not Hausdorff.*

Proof. Let $x,y,z \in X$ be such that $a := m_s(x,x,x) < m_s(z,z,z) = \frac{a+b}{2} < b := m_s(y,y,y)$ with

$$\frac{b}{2} < \frac{k}{2} < m_s(y,y,y) < M_{s_{x,x,y}} = b, \ \ r = 2m_s(x,x,y) - a - b > 0,$$

$$\max\{m_s(x,x,z), m_s(z,z,y)\} \leqslant (2m_s(x,x,y) - k)\frac{\epsilon}{r}.$$

Without loss of generality, we assume that, for each $\epsilon > 0$, $\epsilon < r$. Now, we need to prove that the intersection of the following neighborhoods is not empty:

$$U_x = \{z \in X : m_s(x,x,z) - m_{s_{x,x,z}} < \epsilon\}, \ \ V_y = \{z \in X : m_s(y,y,z) - m_{s_{y,y,z}} < \epsilon\}.$$

To prove $z \in U_x$, we have

$$m_s(x,x,z) < (2m_s(x,x,y) - k)\frac{\epsilon}{r},$$

$$m_s(x,x,z) - m_{s_{x,x,z}} < (2m_s(x,x,y) - k)\frac{\epsilon}{r} - a < (2m_s(x,x,y) - k - a)\frac{\epsilon}{r} < (2m_s(x,x,y) - a - b)\frac{\epsilon}{r} = \epsilon$$

and, for any $z \in V_y$, we also have

$$m_s(y,y,z) < (2m_s(x,x,y) - k)\frac{\epsilon}{r},$$

$$m_s(y,y,z) - m_{s_{y,y,z}} < (2m_s(x,x,y) - k)\frac{\epsilon}{r} - \frac{a+b}{2}$$

$$< (2m_s(x,x,y) - k)\frac{\epsilon}{r} - \frac{a+b}{2}\frac{\epsilon}{r}$$

$$< (2m_s(x,x,y) - k - \frac{a+b}{2})\frac{\epsilon}{r} < (2m_s(x,x,y) - a - b)\frac{\epsilon}{r} = \epsilon.$$

So, we can find $x,y \in X$ such that, for all nonempty neighborhoods U_x of x and V_y of y, $U_x \cap V_y \neq \emptyset$. This completes the proof. □

3. Main results

3.1. On fixed point problems under Banach's contractive condition in M_s-metric spaces

In this section, we introduce the concepts of the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem in M_s-metric spaces. Then we study the corresponding results for the fixed point problem under Banach's contractive condition in M_s-metric spaces.

Definition 3.1. Let (X, m_s) be an M_s-metric space and $T : X \mapsto X$ be a mapping. The fixed point problem

$$x = Tx \tag{3.1}$$

is said to be *Ulam-Hyers stable* if there exists $c > 0$ such that, for any $\epsilon > 0$ and for each $w^* \in X$ which is an ϵ-solution of the fixed point problem (3.1), i.e., w^* satisfies the inequality

$$m_s(w^*, w^*, Tw^*) \leqslant \epsilon,$$

there exists a solution $x^* \in X$ of the equation (3.1) such that

$$m_s(x^*, w^*, w^*) \leqslant c\epsilon.$$

Definition 3.2. Let (X, m_s) be an M_s-metric space and $T : X \mapsto X$ be a mapping. The fixed point problem of T is said to be *well-posed* if the following conditions hold:
(1) T has a unique fixed point x^* of T;
(2) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} m_s(x_n, Tx_n, Tx_n) = 0$, we have $\lim_{n\to\infty} m_s(x^*, x_n, x_n) = 0$.

Definition 3.3. Let (X, m_s) be an M_s-metric space and $T : X \mapsto X$ be a mapping. The fixed point problem of T is said to have the *limit shadowing property* in X if, for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} m_s(x_n, Tx_n, Tx_n) = 0$, it follows that there exists $z \in X$ such that $\lim_{n\to\infty} m_s(T^n z, x_n, x_n) = 0$.

Theorem 3.4. *Let (X, m_s) be an M_s-metric space and $T : X \mapsto X$ be a Banach contractive mapping with constant $k \in [0, 1)$. Then the following assertions hold:*
(1) *the fixed point problem of T is Ulam-Hyers stable;*
(2) *the fixed point problem of T is well-posed;*
(3) *the fixed point problem of T has the limit shadowing property in X.*

Proof. From Theorem 1.3, it follows that T has a unique fixed point $x^* \in X$ with $m_s(x^*, x^*, x^*) = 0$.

Next, we first claim that the fixed point problem of T is Ulam-Hyers stable. Let $\frac{\epsilon}{2} > 0$ and $w^* \in X$ be an $\frac{\epsilon}{2}$-solution of (3.1), i.e.,

$$m_s(w^*, w^*, Tw^*) \leqslant \frac{\epsilon}{2}. \tag{3.2}$$

From (m_s4) together with (3.2), we obtain

$$
\begin{aligned}
m_s(x^*, w^*, w^*) &\leqslant [m_s(x^*, x^*, Tw^*) - m_{s_{x^*,x^*,Tw^*}}] + 2[m_s(w^*, w^*, Tw^*) - m_{s_{w^*,w^*,Tw^*}}] + m_{s_{x^*,w^*,w^*}} \\
&= [m_s(Tx^*, Tx^*, Tw^*) - m_{s_{x^*,x^*,Tw^*}}] + 2[m_s(w^*, w^*, Tw^*) - m_{s_{w^*,w^*,Tw^*}}] + m_{s_{x^*,w^*,w^*}} \\
&\leqslant [km_s(x^*, x^*, w^*) - m_{s_{x^*,x^*,Tw^*}}] + 2[m_s(w^*, w^*, Tw^*) - m_{s_{w^*,w^*,Tw^*}}] + m_{s_{x^*,w^*,w^*}} \\
&= [km_s(x^*, x^*, w^*) - m_{s_{x^*,x^*,Tw^*}}] + 2m_s(w^*, w^*, Tw^*) + \min\{m_s(x^*, x^*, x^*), m_s(w^*, w^*, w^*)\} \\
&\quad - 2\min\{m_s(w^*, w^*, w^*), m_s(Tw^*, Tw^*, Tw^*)\} \\
&= [km_s(x^*, x^*, w^*) - m_{s_{x^*,x^*,Tw^*}}] + 2m_s(w^*, w^*, Tw^*) - 2m_s(Tw^*, Tw^*, Tw^*) \\
&\leqslant km_s(x^*, x^*, w^*) + \epsilon,
\end{aligned}
$$

which implies $m_s(x^*, x^*, w^*) \leqslant c\epsilon$, where $c = \frac{1}{1-k}$. Therefore, the fixed point of T is Ulam-Hyers stable.

Next, we prove that the fixed point problem of T is well-posed. Assume that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} m_s(x_n, Tx_n, Tx_n) = 0$. Now, we show that $\lim_{n\to\infty} m_s(x^*, x_n, x_n) = 0$. By (m_s4), we have

$$
\begin{aligned}
m_s(x^*, x_n, x_n) &\leqslant [m_s(x^*, x^*, Tx_n) - m_{s_{x^*,x^*,Tx_n}}] + 2[m_s(x_n, x_n, Tx_n) - m_{s_{x_n,x_n,Tx_n}}] + m_{s_{x^*,x_n,x_n}} \\
&= [m_s(Tx^*, Tx^*, Tx_n) - m_{s_{x^*,x^*,Tx_n}}] + 2[m_s(x_n, x_n, Tx_n) - m_{s_{x_n,x_n,Tx_n}}] + m_{s_{x^*,x_n,x_n}} \\
&\leqslant [km_s(x^*, x^*, x_n) - m_{s_{x^*,x^*,Tx_n}}] + 2m_s(x_n, x_n, Tx_n) + m_{s_{x^*,x_n,x_n}} - 2m_{s_{x_n,x_n,Tx_n}} \\
&= [km_s(x^*, x^*, x_n) - m_{s_{x^*,x^*,Tx_n}}] + 2m_s(x_n, x_n, Tx_n) \\
&\quad - 2\min\{m_s(x_n, x_n, x_n), m_s(Tx_n, Tx_n, Tx_n)\} \\
&\leqslant km_s(x^*, x^*, x_n) + 2m_s(x_n, x_n, Tx_n),
\end{aligned}
$$

for all $n \in \mathbb{N}$, which implies that

$$m_s(x^*, x^*, x_n) \leqslant \frac{2}{1-k} m_s(x_n, x_n, Tx_n), \tag{3.3}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in (3.3), we have $\lim\limits_{n\to\infty} m_s(x^*, x^*, x_n) = 0$ and hence the fixed point problem of T is well-posed.

Finally, we prove that T has the limit shadowing property. Let $\{x_n\}$ be any sequence in X such that $\lim\limits_{n\to\infty} m_s(x_n, Tx_n, Tx_n) = 0$. From the discussion above, we have $\lim\limits_{n\to\infty} m_s(x^*, x_n, x_n) = 0$. Since x^* is a fixed point of T, we have

$$\lim_{n\to\infty} m_s(x_n, x_n, T^n x^*) = \lim_{n\to\infty} m_s(x_n, x_n, x^*) = 0.$$

Therefore, T has the limit shadowing property. This completes the proof. □

Now, we give an example to illustrate the validity of Theorem 3.4.

Example 3.5. Let $X = [0, \infty)$ and $m_s : X^3 \mapsto [0, \infty)$ be a mapping defined by

$$m_s(x, y, z) = \max\{x, y, z\} - \min\{x, y, z\},$$

for all $x, y, z \in X$. Then (X, m_s) is a complete M_s-metric space. Define a mapping $T : X \mapsto X$ by $Tx = \frac{x}{2}$ for all $x \in X$. For each $x, y, z \in X$, we obtain

$$m_s(Tx, Ty, Tz) = \frac{1}{2}(\max\{x, y, z\} - \min\{x, y, z\}) = \frac{1}{2} m_s(x, y, z).$$

It follows that T is the Banach contraction with constant $k = \frac{1}{2} \in [0, 1)$.

First, we claim that the fixed point problem of T is Ulam-Hyers stable. Assume that $\epsilon > 0$ and $w^* \in X$ is an ϵ-solution of the fixed point problem of T, that is,

$$m_s(w^*, w^*, Tw^*) \leqslant \epsilon \implies \frac{1}{2}\left(\max\left\{w^*, w^*, \frac{w^*}{2}\right\} - \min\left\{w^*, w^*, \frac{w^*}{2}\right\}\right) = \frac{w^*}{4} \leqslant \epsilon.$$

It is easy to see that $x^* = 0$ is a solution of the fixed point problem of T and

$$m_s(x^*, x^*, w^*) = \frac{1}{2}(\max\{0, 0, w^*\} - \min\{0, 0, w^*\}) = \frac{w^*}{2} \leqslant 2\epsilon$$

with the constant c in Definition 3.1 taken by $2 > 0$. Hence the fixed point problem of T is Ulam-Hyers stable.

Next, we prove that the fixed point problem of T is well-posed. We can see that $x^* = 0$ is a unique fixed point of T. Now, we assume that $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} m_s(x_n, Tx_n, Tx_n) = 0$, that is,

$$\lim_{n\to\infty} \frac{1}{2}\left(\max\left\{x_n, \frac{x_n}{2}, \frac{x_n}{2}\right\} - \min\left\{x_n, \frac{x_n}{2}, \frac{x_n}{2}\right\}\right) = 0 \implies \lim_{n\to\infty} x_n = 0.$$

Then we obtain that $\lim\limits_{n\to\infty} m_s(x^*, x_n, x_n) = \lim\limits_{n\to\infty} m_s(0, x_n, x_n) = \lim\limits_{n\to\infty} x_n = 0$ and so the fixed point problem of T is well-posed.

Finally, we state that the fixed point problem of T has the limit shadowing property in X. Suppose that $\{x_n\}$ is any sequence in X such that $\lim\limits_{n\to\infty} m_s(x_n, Tx_n, Tx_n) = 0$. It follows that $\lim\limits_{n\to\infty} x_n = 0$. We can see that there exists $z = 0 \in X$ such that

$$\lim_{n\to\infty} m_s(T^n z, x_n, x_n) = \lim_{n\to\infty} m_s(0, x_n, x_n) = \lim_{n\to\infty} x_n = 0,$$

which implies that the fixed point problem of T has the limit shadowing property in X.

3.2. On fixed point problems under Kannan's contractive condition in M_s-metric spaces

The purpose of this section is to introduce another type of the Ulam-Hyers stability, the well-posedness and the limit shadowing property of the fixed point problem in M_s-metric spaces. Then we also give the analogous results for the fixed point problem of Kannan contractive mappings in M_s-metric spaces.

Definition 3.6. Let (X, m_s) be an M_s-metric space and $T : X \mapsto X$ be a mapping. The fixed point problem (3.1) is said to be *Ulam-Hyers stable type* (K) if there exists $c > 0$ such that, for any $\epsilon > 0$ and for each $w^* \in X$ which is an ϵ-solution of the fixed point problem (3.1), i.e., w^* satisfies the inequality

$$m_s(w^*, w^*, Tw^*) \leqslant \epsilon,$$

there exists a solution $x^* \in X$ of the (3.1) such that

$$m_s(x^*, w^*, w^*) - c m_s(x^*, x^*, x^*) \leqslant c\epsilon.$$

Remark 3.7. It is obvious that the Ulam-Hyers stability of the fixed problem implies the Ulam-Hyers stability type (K).

Definition 3.8. Let (X, m_s) be an M_s-metric space and $T : X \mapsto X$ be a mapping. The fixed point problem of T is said to be *well-posed type* (K) if the following conditions hold:

(1) T has a unique fixed point x^* of T;
(2) there exists $c > 0$ such that, for any sequence $\{x_n\}$ in X with $\lim\limits_{n \to \infty} m_s(x_n, Tx_n, Tx_n) = 0$, we have $\lim\limits_{n \to \infty} m_s(x_n, x^*, x^*) = c m_s(x^*, x^*, x^*).$

Definition 3.9. Let (X, m_s) be an M_s-metric space and $T : X \mapsto X$ be a mapping. The fixed point problem of T is said to have the *limit shadowing property type* (K) in X if there exists $c > 0$ such that for any sequence $\{x_n\}$ in X with $\lim\limits_{n \to \infty} m_s(x_n, Tx_n, Tx_n) = 0$, it follows that there exists $z \in X$ such that $\lim\limits_{n \to \infty} m_s(T^n z, x_n, x_n) = c m_s(z, z, z).$

Theorem 3.10. *Let (X, m_s) be an M_s-metric space and $T : X \mapsto X$ be a Kannan contractive mapping with constant $k \in [0, \frac{1}{2})$. Then the following assertions hold:*

(1) *The fixed point problem of T is Ulam-Hyers stable type* (K).
(2) *The fixed point problem of T is well-posed type* (K).
(3) *The fixed point problem of T has the limit shadowing property type* (K) *in X.*

Proof. From Theorem 1.4, it follows that T has a unique fixed point $x^* \in X$ with $m_s(x^*, x^*, x^*) = 0$.

Now, we first claim that the fixed point problem of T is Ulam-Hyers stable type (K). Let $\frac{\epsilon}{2} > 0$ and $w^* \in X$ be an $\frac{\epsilon}{2}$-solution of the equation (3.1), i.e.,

$$m_s(w^*, w^*, Tw^*) \leqslant \frac{\epsilon}{2}. \tag{3.4}$$

From $(m_s 4)$ together with (3.4), we obtain

$$
\begin{aligned}
m_s(x^*, w^*, w^*) &\leqslant [m_s(x^*, x^*, Tw^*) - m_{s_{x^*,x^*,Tw^*}}] + 2[m_s(w^*, w^*, Tw^*) - m_{s_{w^*,w^*,Tw^*}}] + m_{s_{x^*,w^*,w^*}} \\
&= [m_s(Tx^*, Tx^*, Tw^*) - m_{s_{x^*,x^*,Tw^*}}] + 2[m_s(w^*, w^*, Tw^*) - m_{s_{w^*,w^*,Tw^*}}] + m_{s_{x^*,w^*,w^*}} \\
&\leqslant m_s(Tx^*, Tx^*, Tw^*) + 2m_s(w^*, w^*, Tw^*) + m_{s_{x^*,w^*,w^*}} \\
&\leqslant k[m_s(x^*, x^*, Tx^*) + m_s(w^*, w^*, Tw^*)] + 2m_s(w^*, w^*, Tw^*) + m_{s_{x^*,w^*,w^*}} \\
&= k[m_s(x^*, x^*, x^*) + m_s(w^*, w^*, Tw^*)] + 2m_s(w^*, w^*, Tw^*) + m_s(x^*, x^*, x^*) \\
&\leqslant (k+1)m_s(x^*, x^*, x^*) + (2+k)\epsilon \\
&\leqslant (k+2)[m_s(x^*, x^*, x^*) + \epsilon],
\end{aligned}
$$

which implies $m_s(x^*, x^*, w^*) - c m_s(x^*, x^*, x^*) \leqslant c\epsilon$, where $c = k + 2 > 0$. Thus the fixed point of T is Ulam-Hyers stable (K).

Next, we prove that the fixed point problem of T is well-posed type (K). Assume that $\{x_n\}$ is a sequence in X such that $\lim\limits_{n \to \infty} m_s(x_n, Tx_n, Tx_n) = 0$. Now, we show that $\lim\limits_{n \to \infty} m_s(x^*, x_n, x_n) = 0$. By $(m_s 4)$, it follows that

$$
\begin{aligned}
m_s(x^*, x_n, x_n) &\leqslant [m_s(x^*, x^*, Tx_n) - m_{s_{x^*, x^*, Tx_n}}] + 2[m_s(x_n, x_n, Tx_n) - m_{s_{x_n, x_n, Tx_n}}] + m_{s_{x^*, x_n, x_n}} \\
&= [m_s(Tx^*, Tx^*, Tx_n) - m_{s_{x^*, x^*, Tx_n}}] + 2[m_s(x_n, x_n, Tx_n) - m_{s_{x_n, x_n, Tx_n}}] + m_{s_{x^*, x_n, x_n}} \\
&\leqslant m_s(Tx^*, Tx^*, Tx_n) + 2m_s(x_n, x_n, Tx_n) + m_{s_{x^*, x_n, x_n}} \\
&\leqslant k[m_s(x^*, x^*, Tx^*) + m_s(x_n, x_n, Tx_n)] + 2m_s(x_n, x_n, Tx_n) + m_{s_{x^*, x_n, x_n}} \\
&= k[m_s(x^*, x^*, x^*) + m_s(x_n, x_n, Tx_n)] + 2m_s(x_n, x_n, Tx_n) + m_s(x^*, x^*, x^*) \\
&\leqslant (k+2)[m_s(x^*, x^*, x^*) + m_s(x_n, x_n, Tx_n)],
\end{aligned}
$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the last inequality above, we have

$$
\lim_{n \to \infty} m_s(x_n, x_n, T^n x^*) = \lim_{n \to \infty} m_s(x_n, x_n, x^*) = (k+2)m_s(x^*, x^*, x^*). \tag{3.5}
$$

Hence the fixed point problem of T is well-posed type (K).

Finally, we prove that T has the limit shadowing property type (K). Let $\{x_n\}$ be any sequence in X such that $\lim\limits_{n \to \infty} m_s(x_n, Tx_n, Tx_n) = 0$. Since x^* is a fixed point of T, from (3.5), we have $\lim\limits_{n \to \infty} m_s(x^*, x_n, x_n) = 0$. Since x^* is a fixed point of T, we have

$$
\lim_{n \to \infty} m_s(x_n, x_n, T^n x^*) = \lim_{n \to \infty} m_s(x_n, x_n, x^*) = (k+2)m_s(x^*, x^*, x^*).
$$

Therefore, T has the limit shadowing property type (K). This completes the proof. □

Now, we present an example to illustrate Theorem 3.10.

Example 3.11. Let $X = [0, \infty)$ and define a mapping $m_s : X^3 \mapsto [0, \infty)$ by

$$
m_s(x, y, z) = \max\{x^2, y^2, z^2\} - \min\{x^2, y^2, z^2\},
$$

for all $x, y, z \in X$. Then (X, m_s) is a complete M_s-metric space. Define a mapping $T : X \mapsto X$ by $Tx = \frac{x}{2}$ for all $x \in X$. For each $x, y \in X$, we obtain

$$
\begin{aligned}
m_s(Tx, Tx, Ty) &= \left(\max \left\{ \frac{x^2}{4}, \frac{x^2}{4}, \frac{y^2}{4} \right\} - \min \left\{ \frac{x^2}{4}, \frac{x^2}{4}, \frac{y^2}{4} \right\} \right) \\
&= \left| \frac{x^2}{4} - \frac{y^2}{4} \right| \\
&\leqslant \left| \frac{x^2}{4} \right| + \left| \frac{y^2}{4} \right| \\
&= \frac{1}{3} \left[\left(\max \left\{ \frac{x^2}{4}, \frac{x^2}{4}, x^2 \right\} - \min\{ \frac{x^2}{4}, \frac{x^2}{4}, x^2 \} \right) + \left(\max \left\{ \frac{y^2}{4}, \frac{y^2}{4}, y^2 \right\} - \min \left\{ \frac{y^2}{4}, \frac{y^2}{4}, y^2 \right\} \right) \right] \\
&= \frac{1}{3} (m_s(x, x, Tx) + m_s(y, y, Ty).
\end{aligned}
$$

It follows that T is a Kannan contractive mapping with constant $k = \frac{1}{3} \in [0, \frac{1}{2})$.

First, we claim that the fixed point problem of T is Ulam-Hyers stable type (K). Assume that $\epsilon > 0$ and $w \in X$ is an ϵ-solution of the fixed point problem of T, that is,

$$
m_s(w, w, Tw) \leqslant \epsilon \implies \left(\max \left\{ w^2, w^2, \frac{w^2}{4} \right\} - \min \left\{ w^2, w^2, \frac{w^2}{4} \right\} \right) = \frac{3w^2}{4} \leqslant \epsilon.
$$

It is easy to see that $x^* = 0$ is a solution of the fixed point problem of T and

$$m_s(x^*, x^*, w) = (\max\{0, 0, w^2\} - \min\{0, 0, w^2\}) = w^2 \leqslant \frac{4}{3}\epsilon$$

with the constant c in Definition 3.6 taken by $\frac{4}{3} > 0$. Hence the fixed point problem of T is Ulam-Hyers stable type (K). Now, we prove that the fixed point problem of T is well-posed type (K). We can see that $x^* = 0$ is a unique fixed point of T. We assume that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} m_s(x_n, Tx_n, Tx_n) = 0$, that is,

$$\lim_{n\to\infty} \left(\max\left\{ x_n^2, \frac{x_n^2}{4}, \frac{x_n^2}{4} \right\} - \min\left\{ x_n^2, \frac{x_n^2}{4}, \frac{x_n^2}{4} \right\} \right) = \lim_{n\to\infty} \frac{3x_n^2}{4} = 0 \implies \lim_{n\to\infty} x_n^2 = 0.$$

Then we obtain

$$\lim_{n\to\infty} m_s(x^*, x_n, x_n) = \lim_{n\to\infty} m_s(0, x_n, x_n) = \lim_{n\to\infty} x_n^2 = 0$$

and so the fixed point problem of T is well-posed type (K).

Finally, we state that the fixed point problem of T has the limit shadowing property type (K) in X. Suppose that $\{x_n\}$ is any sequence in X such that $\lim_{n\to\infty} m_s(x_n, Tx_n, Tx_n) = 0$. It follows that $\lim_{n\to\infty} x_n^2 = 0$. We can see that there exists $z = 0 \in X$ such that

$$\lim_{n\to\infty} m_s(T^n z, x_n, x_n) = \lim_{n\to\infty} m_s(0, x_n, x_n) = \lim_{n\to\infty} x_n^2 = 0,$$

which implies that the fixed point problem of T has the limit shadowing property type (K) in X.

Acknowledgment

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