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Infinitely many nontrivial solutions for fractional boundary value problems with impulses and perturbation

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Abstract

By the variational methods, the existence criteria of infinitely many nontrivial solutions for fractional differential equations with impulses and perturbation are established. An example is given to illustrate main results. Recent results in the literature are generalized and improved.

Keywords: Fractional differential equations with impulses and perturbation, infinitely many nontrivial solutions, variational methods.
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1. Introduction

In the present paper, we consider the following boundary value problems (BVPs) for impulsive fractional differential equations:

$$\begin{cases} -\frac{1}{2}\frac{d}{dt}({}_0D_t^{-\beta_i} + {}_tD_T^{-\beta_i})u_i'(t) = \rho a_i(t)u_i(t) + \lambda F_{u_i}(t,u(t)), & t \neq t_k,\ \text{a.e.},\ t \in [0,T], \\ \triangle(D_t^{\alpha_i}u_i)(t_k) = I_{ik}(u_i(t_k)), & t_k \in (0,T),\ k = 1,2,\dots,l, \\ u_i(0) = u_i(T) = 0, & 1 \leqslant i \leqslant N, \end{cases} \tag{1.1}$$

where $u = (u_1,\dots,u_N)$, $|u| = \sqrt{\sum_{i=1}^{N} u_i^2}$, $\rho,\lambda > 0$, $\beta_i \in (0,1)$, $\alpha_i = 1 - \frac{\beta_i}{2} \in \left(\frac{1}{2},1\right)$ for $1 \leqslant i \leqslant N$, ${}_0D_t^{-\beta_i}$ and ${}_tD_T^{-\beta_i}$ are the left and right Riemann-Loiuville fractional integrals of order β_i, respectively, ${}_0^cD_t^{\alpha_i}$ and ${}_t^cD_T^{\alpha_i}$ are the left and right Caputo fractional derivatives of order α_i, respectively, $\alpha_i \in L^\infty[0,T], 0 = t_0 <$

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$t_1 < \cdots < t_{l+1} = T, I_{ik} \in C([0,T],R), F\colon [0,T] \times R^N \to R$ is measurable, continuously differentiable, F_{u_i} denotes the partial derivative of F with respect to u_i for $1 \leqslant i \leqslant N$, and

$$\begin{aligned}
(D_t^{\alpha_i}u_i)(t_k) &= \frac{1}{2}\Big\{{}_0D_t^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i) - {}_tD_T^{\alpha_i-1}({}_t^cD_T^{\alpha_i}u_i)\Big\}(t_k),\\
\triangle(D_t^{\alpha_i}u_i)(t_k) &= \frac{1}{2}\Big\{{}_0D_t^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i) - {}_tD_T^{\alpha_i-1}({}_t^cD_T^{\alpha_i}u_i)\Big\}(t_k^+)\\
&\quad - \frac{1}{2}\Big\{{}_0D_t^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i) - {}_tD_T^{\alpha_i-1}({}_t^cD_T^{\alpha_i}u_i)\Big\}(t_k^-),\\
\frac{1}{2}\Big\{{}_0D_t^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i) - {}_tD_T^{\alpha_i-1}({}_t^cD_T^{\alpha_i}u_i)\Big\}(t_k^+) &= \lim_{t\to t_k^+}\frac{1}{2}\Big\{{}_0D_t^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i) - {}_tD_T^{\alpha_i-1}({}_t^cD_T^{\alpha_i}u_i)\Big\}(t),\\
\frac{1}{2}\Big\{{}_0D_t^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i) - {}_tD_T^{\alpha_i-1}({}_t^cD_T^{\alpha_i}u_i)\Big\}(t_k^-) &= \lim_{t\to t_k^-}\frac{1}{2}\Big\{{}_0D_t^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i) - {}_tD_T^{\alpha_i-1}({}_t^cD_T^{\alpha_i}u_i)\Big\}(t),
\end{aligned}$$

for $k = 1, \cdots, l$.

Fractional differential equations have been an area of great interest recently. Fractional calculus provide a powerful tool for the description of hereditary properties of various materials and memory processes. In the past decades, the theory of fractional differential equations has become an important area of investigation because of its wide applicability in many branches of physics, economics and technical sciences, see [18, 20, 30–35, 38] and references therein. The BVPs of fractional differential equations have also attracted more and more attention (for example [2, 10, 12, 17]). The classical approaches to study such problems mainly include fixed point theorems, degree theory, the method of upper and lower solutions and so on.

Concerning the critical point theory, we refer readers to the books and the papers [8, 22, 24] and the references therein. In [15], Jiao and Zhou considered the following fractional boundary value problems

$$\begin{cases} -\dfrac{1}{2}\dfrac{d}{dt}({}_0D_t^{-\beta} + {}_tD_T^{-\beta})u'(t) = \nabla F(t,u(t)), & \text{a.e., } t \in [0,T],\\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where $\beta \in [0,1)$, and ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Loiuville fractional derivatives, respectively. $F\colon [0,T] \times R^N \to R$ (with $N \geqslant 1$) is a suitable given function and $\nabla F(t,x)$ is the gradient of F with respect to x.

By using the least action principle and mountain pass theorem, they obtained some sufficient conditions for the existence of one solution. Since then, the variational methods are applied to deal with the existence of solutions for fractional differential equations, see [1, 7, 14–16, 21, 27, 29, 36].

Differential equations with impulsive effects arise from many phenomena in the real world. Such equations appear in describing processes which experience a suddenly changes of their states in chemical technology, physics phenomena, population dynamics, biotechnology, economics, and etc. ([3]). In recent years, there has been a growing interest in the study of impulsive differential equations, see [9, 19, 26, 28] and the references therein.

Recently, the BVPs of fractional differential equations with impulses have been studied by many authors. But for almost all the work, the main methods are fixed theorems, the coincidence degree theory and the monotone iterative methods. To our best knowledge, the fractional boundary value problems with impulses using variational methods and critical point theory has received considerably less attention ([6, 23, 25, 37]). Bonanno et al. [6], and Rodríguez-López and Tersian [25] studied the following Dirichlet's BVPs for fractional differential equations with impulses:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u(t)), & t \neq t_j, \text{ a.e., } t \in [0,T],\\ \triangle({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), & j = 1,2,\ldots,n,\\ u(0) = u(T) = 0, \end{cases}$$

where $\lambda \in (0,+\infty)$ and $\mu \in (0,+\infty)$ are two parameters. They obtained the existence of triple solutions by using variatioanl methods and a three critical points theorem due to Bonanno and Marano [5].

In [23], the authors investigated the following fractional differential equations with impulses:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = f(t,u(t)), & t \neq t_j, \text{ a.e., } t \in [0,T], \\ \triangle({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = I_j(u(t_j)), & j = 1,2,\dots,n, \\ u(0) = u(T) = 0. \end{cases}$$

Through using critical point theory and variational methods, the authors gave some criteria to guarantee that the above-mentioned impulsive problems have infinitely many solutions.

In [37], the authors studied the following fractional differential systems with impulsive effects:

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t)\,{}_0^cD_t^{\alpha_i}u_i(t)) = \lambda F_{u_i}(t,u) + h_i(u_i(t)), & 0 < t < T, t \neq t_j, \\ \triangle({}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i))(t_j) = I_{ij}(u_i(t_j)), & j = 1,2,\dots,m, \\ u_i(0) = u_i(T) = 0, & 1 \leqslant i \leqslant N. \end{cases}$$

Under suitable hypotheses, by using variational methods and some critical points theorems in [4], they established the criteria of the existence of at least two nontrivial and nonnegative solutions.

The problem (1.2) arises from the phenomena of advection dispersion and has first been investigated by Ervin and Roop in [11]. The authors in [7, 13, 14, 21, 27] further studied the existence and multiplicity of solutions for the above problem (1.2) or related problems with the critical point theory. But as we know, the phenomena of advection dispersion may be effected by the impulses and perturbation. So the BVPs (1.2) can not describe the phenomena of advection dispersion very well. It is necessary to consider the impulses and perturbation, which have more realistic significances. It is more reasonable to extend the problem (1.2) to the n-dimension case. To our best knowledge, the n-dimension case for the problem (1.2) with the impulses and perturbation has not been studied. The purpose of the present paper is to fill the gap.

Motivated by the work above, in this literature, we studied the existence results of solutions for the problem (1.1). The form of (1.1) is completely different from the forms of equations in [6, 23, 25, 37]. The main tool in this paper is the variant fountain theorem, which is also different from the tools in [6, 23, 25, 37]. We considered the problem (1.1) in the domain R^N, which is more universal. We established the existence criteria of infinitely many nontrivial solutions for (1.1). It should also be noted that we obtained some good properties for the solutions of (1.1). Hence, our main results complement and improve the relevant results on fractional problems.

In order to obtain the main results, we state the following assumptions.

(f) $\lambda, \rho > 0, \beta_i \in (0,1), \alpha_i = 1 - \frac{\beta_i}{2} \in \left(\frac{1}{2}, 1\right), a_i \in L^\infty[0,T]$ and $\bar{a}_i := \operatorname*{ess\,inf}_{[0,T]} a_i(t) > 0$ for $1 \leqslant i \leqslant N, I_{ik} \in C([0,T], R)$ satisfy Lipschitz condition, i.e., $|I_{ik}(x_1) - I_{ik}(x_2)| \leqslant L_{ik}|x_1 - x_2|$ for $x_1, x_2 \in R$ with $L_{ik} \geqslant 0$ and $I_{ik}(0) = 0, I_{ik}(-x) = I_{ik}(x)$ for $i = 1,\dots,N, k = 1,\dots,l$, and $F\colon [0,T] \times R^N \to R$ is measurable with respect to t for $u(t) \in R^n$, continuously differentiable in u, for a.e. $t \in [0,T]$, satisfying $\sup_{|u| \leqslant r_0, 1<i<N} \left(\max\{|F(\cdot,u)|, |F_{u_i}(\cdot,u)|\} \in L^1([0,T])\right)$ for $r_0 > 0$, and $F(t,0,\dots,0) = 0, F(t,u) \geqslant 0$ for all $(t,u) \in [0,T] \times R^N$.

(H_1) $\lim_{|u| \to 0} \frac{F(t,u)}{|u|^2} = 0$ uniformly in $t \in [0,T]$, where $u = (u_1,\dots,u_N), |u| = \sqrt{\sum_{i=1}^N u_i^2}$.

(H_2) There exist two constants $b_0 > 0, \nu \in [2, 2^*_{\alpha_0})$ such that

$$|F_{u_i}(t,u)| \leqslant b_0(1 + |u|^{\nu-1}) \text{ for } (t,u) \in [0,T] \times R^N, i = 1,\dots,N,$$

where $2^*_{\alpha_0} = \max_{1 \leqslant i \leqslant N}\{2^*_{\alpha_i}\}, 2^*_{\alpha_i} = \frac{2N}{N - 2\alpha_i}$.

(H$_3$) $\lim\limits_{|u|\to\infty} \frac{F(t,u)}{|u|^2} = \infty$ uniformly in $t \in [0,T]$.

Theorem 1.1. *Let (f) and (H$_1$)-(H$_3$) hold. Assume that* $\sigma = \frac{5c_0}{6} - BlL_0 > 0$ *and* $F(x,u)$ *is even for* $u \in X$*, then for* $\rho \in \left(0, \frac{2\sigma}{A_2 a_0}\right), \lambda \in [1,2]$*, the problem* (1.1) *possesses infinitely many nontrivial solutions* $u^k \in X$ *for all* $k \in \mathbb{N}$*, where* $c_0 = \min\{|\cos(\pi\alpha_i), 1 \leqslant i \leqslant N|\}, a_0 = \max\{\operatorname{ess\,sup}\limits_{[0,T]} a_i(t), 1 \leqslant i \leqslant N\}, A_2 = \max\left\{\frac{T^{2\alpha_i}}{\Gamma^2(\alpha_i+1)}, 1 \leqslant i \leqslant N\right\}, L_0 = \max\limits_{1\leqslant i \leqslant N, 1\leqslant j \leqslant l} L_{ij}, B = \max\left\{\frac{T^{2\alpha_i-1}}{\Gamma^2(\alpha_i)(2\alpha_i-1)}, 1 \leqslant i \leqslant N\right\}$*, and the space* X *will be defined in Section* 2.

We simplify the assumption (f) to (f$'$).

(f$'$) $\lambda > 0, \beta_i \in (0,1), \alpha_i = 1 - \frac{\beta_i}{2} \in \left(\frac{1}{2}, 1\right), I_{ik} \in C([0,T], R)$ satisfy Lipschitz condition, i.e., $|I_{ik}(x_1) - I_{ik}(x_2)| \leqslant L_{ik}|x_1 - x_2|, \forall x_1, x_2 \in R$ with $L_{ik} \geqslant 0$ and $I_{ik}(0) = 0, I_{ik}(-x) = I_{ik}(x)$ for $i = 1, \dots, N, k = 1, \dots, l$, and $F \in C^1([0,T] \times R^N, R)$ is measurable with respect to t for $u(t) \in R^n$ and $F(t, 0, \dots, 0) = 0, F(t,u) \geqslant 0$, for all $(t,u) \in [0,T] \times R^N$.

Corollary 1.2. *Let* (f$'$) *and* (H$_1$)-(H$_3$) *hold. Assume that* $\sigma = \frac{5c_0}{6} - BlL_0 > 0$ *and* $F(x,u)$ *is even for* $u \in X$*, then for* $\lambda \in [1,2]$*, the following problem* (1.3) *has infinitely many nontrivial solutions* $u^k \in X$ *for all* $k \in \mathbb{N}$.

$$\begin{cases} -\dfrac{1}{2}\dfrac{d}{dt}({}_0D_t^{-\beta_i} + {}_tD_T^{-\beta_i})u_i'(t) = \lambda F_{u_i}(t, u(t)), & t \neq t_k, \text{ a.e., } t \in [0,T], \\ \triangle(D_t^{\alpha_i} u_i)(t_k) = I_{ik}(u_i(t_k)), & t_k \in (0,T), k = 1,2,\dots,l, \\ u_i(0) = u_i(T) = 0, & 1 \leqslant i \leqslant N. \end{cases} \tag{1.3}$$

The rest of this paper is organized as follows. In Section 2, some definitions and lemmas which are essential to prove our main results are stated. In Section 3, the existence criteria of infinitely many nontrivial solutions of perturbed fractional differential equations with impulses are established. At last, one example is offered to demonstrate the application of our main results.

2. Preliminaries

We present the necessary definitions for the fractional calculus theory and several lemmas which are used further in this paper.

Definition 2.1 ([18])**.** Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order α for function f are respectively denoted by

$${}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad {}_tD_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)ds,$$

for $t \in [a,b], \alpha > 0$.

The left and right Riemann-Liouville fractional derivatives of order α for function f are respectively defined by

$${}_aD_t^{\alpha} f(t) = \frac{d^n}{dt^n}\, {}_aD_t^{\alpha-n} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n}\left(\int_a^t (t-s)^{n-\alpha-1} f(s)ds\right),$$

$${}_tD_b^{\alpha} f(t) = (-1)^n \frac{d^n}{dt^n}\, {}_tD_b^{\alpha-n} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n}\left(\int_t^b (s-t)^{n-\alpha-1} f(s)ds\right),$$

for $t \in [a,b], n-1 \leqslant \alpha < n, n \in \mathbb{N}$.

Lemma 2.2 ([18])**.** *The left and right RiemannLiouville fractional integral operators have the property of a semi-group, i.e.,*

$${}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2} f(t)) = {}_aD_t^{-\gamma_1-\gamma_2} f(t) \text{ and } {}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2} f(t)) = {}_tD_b^{-\gamma_1-\gamma_2} f(t), \ \forall\, \gamma_1, \gamma_2 > 0,$$

for a.e. $t \in [a,b]$ *and* $f \in L^1([a,b], R^N)$.

Definition 2.3 ([18]). If $\alpha \in (n-1, n)$ and $f \in AC^n([a,b], R)$, then the left and right Caputo fractional derivatives of order α of a function f are respectively defined by

$$ {}_a^cD_t^{\alpha} f(t) = {}_aD_t^{\alpha-n} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, $$
$$ {}_t^cD_b^{\alpha} f(t) = (-1)^n \, {}_tD_b^{\alpha-n} \frac{d^n}{dt^n} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s) ds, $$

for $t \in [a, b], \alpha > 0$.

In view of Definition 2.1 and Lemma 2.2, we can easily transfer problem (1.1) to the following problem

$$ \begin{cases} \dfrac{d}{dt}\Big\{\dfrac{1}{2}\,{}_0D_t^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i(t)) - \dfrac{1}{2}\,{}_tD_T^{\alpha_i-1}({}_t^cD_T^{\alpha_i}u_i(t))\Big\} + \rho a_i(t)u_i(t) + \lambda F_{u_i}(t, u(t)) = 0, \\ \qquad\qquad t \neq t_k, \text{ a.e., } t \in [0, T], \\ \triangle(D_t^{\alpha_i}u_i)(t_k) = \mu_i I_k(u_i(t_k)), \qquad t_k \in (0, T), k = 1, 2, \ldots, l, \\ u_i(0) = u_i(T) = 0, \qquad 1 \leqslant i \leqslant N. \end{cases} \tag{2.1} $$

Then the problem (1.1) is equivalent to the problem (2.1). Hence a solution of the problem (2.1) corresponds to a solution of the BVP (1.1) .

Let us recall that for any fixed $t \in [0, T]$ and $1 \leqslant p \leqslant \infty$,

$$ \|u\|_\infty = \max_{t\in[0,T]} |u(t)|, \quad \|u\|_{L^p} = \Big(\int_0^T |u(s)|^p ds\Big)^{\frac{1}{p}}. $$

For $\alpha_i \in [0, 1), 1 \leqslant i \leqslant N$ we define the fractional derivative spaces $E_0^{\alpha_i}$ by the closure of $C_0^\infty([0, T], R^N)$ with $u(0) = u(T)$ under the norm

$$ \|u_i\|_{\alpha_i} = (\int_0^T |{}_0^cD_t^{\alpha_i}u_i(t)|^2 dt + \int_0^T |u_i(t)|^2 dt)^{\frac{1}{2}}, \quad \forall u_i \in E_0^{\alpha_i}. $$

Clearly, the fractional derivative space $E_0^{\alpha_i}$ is the space of functions $u_I \in L^2(0, T)$ having α_i-order Caputo left and right fractional derivatives and α_i-order Riemann-Loiuville left and right fractional derivatives, ${}_0^cD_t^{\alpha_i}u_i, {}_t^cD_T^{\alpha_i}u_i, {}_0D_t^{\alpha_i}u_i, {}_tD_T^{\alpha_i}u_i \in L^2(0, T)$ and $u_i(0) = u_i(T) = 0$.

Lemma 2.4 ([15]). *Let $\frac{1}{2} < \alpha \leqslant 1$ and $1 < p < \infty$ for all $u \in E_0^\alpha$, one has*

$$ \|u\|_{L^p} \leqslant \frac{T^\alpha}{\Gamma(\alpha+1)} \left\|{}_0^cD_t^\alpha u\right\|_{L^p}. \tag{2.2} $$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ \|u\|_\infty \leqslant \frac{T^{\alpha-1/p}}{\Gamma(\alpha)\big((\alpha-1)q+1\big)^{\frac{1}{q}}} \left\|{}_0^cD_t^\alpha u\right\|_{L^p}. \tag{2.3} $$

It is easy to verify that the norm $\|u_i\|_{\alpha_i} = (\int_0^T |{}_0^cD_t^{\alpha_i}u_i(t)|^2 dt + \int_0^T |u_i(t)|^2 dt)^{\frac{1}{2}}$ is equivalent to $\|u_i\|_{\alpha_i} = (\int_0^T |{}_0^cD_t^{\alpha_i}u_i(t)|^2 dt)^{\frac{1}{2}}$, $\forall u_i \in E_0^{\alpha_i}$. In the sequel, we will consider the fractional derivative spaces $E_0^{\alpha_i}$ with respect to the norm $\|u_i\|_{\alpha_i} = (\int_0^T |{}_0^cD_t^{\alpha_i}u_i(t)|^2 dt)^{\frac{1}{2}}$. By (2.2) and (2.3), we can easily get that for $\alpha_i \in \big[\frac{1}{2}, 1\big), 1 \leqslant i \leqslant N$,

$$ \sum_{i=1}^N \|u_i\|_{L^p}^p \leqslant A_p \sum_{i=1}^N \|u_i\|_{\alpha_i}^p, \quad \sum_{i=1}^N \|u_i\|_\infty^2 \leqslant B \sum_{i=1}^N \|u_i\|_{\alpha_i}^2, \tag{2.4} $$

where $A_p = \max\big\{\frac{T^{p\alpha_i}}{\Gamma^p(\alpha_i+1)}, 1 \leqslant i \leqslant N\big\}, B = \max\big\{\frac{T^{2\alpha_i-1}}{\Gamma^2(\alpha_i)(2\alpha_i-1)}, 1 \leqslant i \leqslant N\big\}$.

Similar to some properties in [15], we have the following results.

Lemma 2.5. *Let $0 < \alpha \leqslant 1$ for $1 \leqslant i \leqslant N$. The fractional derivative space $E_0^{\alpha_i}$ is a reflexive and separable Banach space.*

Lemma 2.6. *Let $\frac{1}{2} < \alpha_i \leqslant 1$ for $1 \leqslant i \leqslant N$. Assume the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha_i}$. Then $u_k \to u$ strongly in $C([0,T],R)$, i.e., $\|u_k - u\|_\infty \to 0$, as $k \to \infty$.*

Lemma 2.7. *Let $\frac{1}{2} < \alpha \leqslant 1$. For any $E_0^{\alpha_i}$, one has*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \leqslant -\int_0^T {}_0^cD_t^\alpha u(t)\, {}_t^cD_T^\alpha u(t)dt \leqslant \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2.$$

By Lemma 2.7, one has

$$\sum_{i=1}^N |\cos(\pi\alpha_i)|\|u_i\|_{\alpha_i}^2 \leqslant \sum_{i=1}^N -\int_0^T {}_0^cD_t^{\alpha_i} u_i(t)\, {}_t^cD_T^{\alpha_i} u_i(t)dt \leqslant \sum_{i=1}^N \frac{1}{|\cos(\pi\alpha_i)|}\|u_i\|_{\alpha_i}^2. \tag{2.5}$$

In the following, we denote $X = E_0^{\alpha_1} \times \cdots \times E_0^{\alpha_N}$, then X is a reflexive and separable Banach space with the norm

$$\|u\|_X = \|(u_1, \dots, u_N)\|_X = \sum_{i=1}^N \|u_i\|_{\alpha_i}. \tag{2.6}$$

Let e_j be a total orthonormal basis of X. We define

$$X_j := \text{span}\{e_j\}, Y_k := \bigoplus_{j=1}^k X_j, \text{ and } Z_k := \overline{\bigoplus_{j=k+1}^\infty X_j}, k \in N,$$

and

$$B_k = \{u \in Y_k : \|u\| \leqslant \rho_k\}, S_k = \{u \in Z_k : \|u\| = r_k\},$$

for $\rho_k > r_k > 0$. Clearly, $X = Y_k \bigoplus Z_k$ with $\dim Y_k < \infty$.

Definition 2.8. A function $u = (u_1, \dots, u_N) \in X$ is called a weak solution of the problem (2.1) if

$$\begin{aligned}\sum_{i=1}^N \int_0^T \Big\{ -\frac{1}{2}\big[{}_0^cD_t^{\alpha_i} u_i(t)\, {}_t^cD_T^{\alpha_i} v_i(t) + {}_t^cD_T^{\alpha_i} u_i(t)\, {}_0^cD_t^{\alpha_i} v_i(t) \big] - \rho a_i(t) u_i(t) v_i(t) \Big\} dt \\ + \sum_{i=1}^N \sum_{j=1}^l I_{ij}(u_i(t_j)) v_i(t_j) - \lambda \sum_{i=1}^N \int_0^T F_{u_i}(t, u(t)) v_i(t) dt = 0,\end{aligned}$$

for all $v = (v_1, \dots, v_N) \in X$.

Similar to the proof of Lemma 2.1 in [6], we can get the following Lemma 2.9.

Lemma 2.9. *Let $\frac{1}{2} \leqslant \alpha_i < 1$ for $1 \leqslant i \leqslant N$. If $u \in X$ is a nontrivial weak solution of (2.1), then u is also a nontrivial solution of (2.1).*

In order to get our main results, we give the following variant fountain theorem.

Lemma 2.10 ([39]). *Let X be a Banach space, assume that $\varphi_{\rho,\lambda}(u)$ satisfies:*

(A_1) *$\varphi_{\rho,\lambda}(u)$ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$, and*

$$\varphi_{\rho,\lambda}(-u) = \varphi_{\rho,\lambda}(u) \text{ for every } (\lambda, u) \in [1,2] \times X;$$

(A_2) *$B(u) \geqslant 0$ for all $u \in X, A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$;*

(A_3) *There exist* $\rho_k > r_k > 0$ *such that*

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\|=r_k} \varphi_{\rho,\lambda}(u) > b_k(\lambda) = \max_{u \in Y_k, \|u\|=\rho_k} \varphi_{\rho,\lambda}(u), \quad \forall \lambda \in [1,2].$$

Then

$$a_k(\lambda) \leqslant c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \varphi_{\rho,\lambda}(\gamma(u)), \quad \forall \lambda \in [1,2],$$

where $\Gamma_k = \{\gamma \in C(B_k, X_\lambda) : \gamma$ *is odd,* $\gamma|_{\partial B_k} = id\}$ $(k \geqslant 2)$*. In addition, for almost every* $\lambda \in [1,2]$*, there exits a sequence* $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ *such that*

$$\sup_n \|u_n^k(\lambda)\| < \infty, \varphi'_{\rho,\lambda}(u_n^k(\lambda)) \to 0 \text{ and } \varphi_{\rho,\lambda}(u_n^k(\lambda)) \to c_k(\lambda), \text{ as } n \to \infty.$$

3. Main results

Define

$$A(u) = \sum_{i=1}^{N} \int_0^T -\frac{1}{2}\, {}_0^cD_t^{\alpha_i} u_i(t)\, {}_t^cD_T^{\alpha_i} u_i(t) dt + \sum_{i=1}^{N} \sum_{j=1}^{l} \int_0^{u_i(t_j)} I_{ij}(s) ds, \tag{3.1}$$

$$B(u) = \int_0^T F(t, u(t)) dt + \frac{\rho}{\lambda} \sum_{i=1}^{N} \frac{a_i(t) u_i^2(t)}{2}, \tag{3.2}$$

$$\begin{aligned} \varphi_{\rho,\lambda}(u) = A(u) - \lambda B(u) &= \sum_{i=1}^{N} \int_0^T -\frac{1}{2}\, {}_0^cD_t^{\alpha_i} u_i(t)\, {}_t^cD_T^{\alpha_i} u_i(t) dt + \sum_{i=1}^{N} \sum_{j=1}^{l} \int_0^{u_i(t_j)} I_{ij}(s) ds \\ &\quad - \lambda \Big(\int_0^T F(t, u(t)) dt + \frac{\rho}{\lambda} \sum_{i=1}^{N} \frac{a_i(t) u_i^2(t)}{2} \Big), \end{aligned} \tag{3.3}$$

for $u = (u_1, \dots, u_N) \in X$ and $\lambda \in [1,2]$.

Clearly, $\varphi_{\rho,\lambda}(u)$ is well-defined and Gâteaux differentiable for $u = (u_1, \dots, u_N) \in X$, and the Gâteaux derivative $\varphi'_{\rho,\lambda}(u) \in X^*$ is given by

$$\begin{aligned} \langle \varphi'_{\rho,\lambda}(u), v \rangle = \sum_{i=1}^{N} \int_0^T \Big\{ -\frac{1}{2} \big[{}_0^cD_t^{\alpha_i} u_i(t)\, {}_t^cD_T^{\alpha_i} v_i(t) + {}_t^cD_T^{\alpha_i} u_i(t)\, {}_0^cD_t^{\alpha_i} v_i(t) \big] - \rho a_i(t) u_i(t) v_i(t) \Big\} dt \\ + \sum_{i=1}^{N} \sum_{j=1}^{l} I_{ij}(u_i(t_j)) v_i(t_j) - \lambda \sum_{i=1}^{N} \int_0^T F_{u_i}(t, u(t)) v_i(t) dt, \end{aligned}$$

for every $v = (v_1, \dots, v_N) \in X$.

Then the critical point of $\Phi - \lambda\Psi$ is exactly the weak solution of the problem (2.1). From the equivalence of (1.1) and (2.1), we know it is also a weak solution of (1.1).

Lemma 3.1. *Let* (f) *and* (H_1)-(H_3) *hold. Assume that* $\sigma = \frac{5c_0}{6} - BlL_0 > 0$*, then for* $\rho \in \Big(0, \frac{2\sigma}{A_2 a_0}\Big), \lambda \in [1,2]$*, there exist* $\rho_k > r_k > 0$ *such that*

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\|=r_k} \varphi_{\rho,\lambda}(u) > b_k(\lambda) = \max_{u \in Y_k, \|u\|=\rho_k} \varphi_{\rho,\lambda}(u), \quad \forall \lambda \in [1,2],$$

where

$$A_2 = \max\Big\{ \frac{T^{2\alpha_i}}{\Gamma^2(\alpha_i + 1)}, 1 \leqslant i \leqslant N \Big\}, \qquad B = \max\Big\{ \frac{T^{2\alpha_i - 1}}{\Gamma^2(\alpha_i)(2\alpha_i - 1)}, 1 \leqslant i \leqslant N \Big\},$$

$$c_0 = \min\{|\cos(\pi\alpha_i), 1 \leqslant i \leqslant N|\}, \qquad a_0 = \max\{\operatorname{ess\,sup}_{[0,T]} a_i(t), 1 \leqslant i \leqslant N\}, \ L_0 = \max_{1 \leqslant i \leqslant N, 1 \leqslant j \leqslant l} L_{ij}.$$

Proof. From (f), we know $I_{ij} \in C([0,T],R)$ satisfy Lipschitz condition and $I_{ij}(0) = 0$, then

$$|I_{ij}(x)| = |I_{ij}(x) - I_{ij}(0)| \leqslant L_{ij}|x|.$$

By the mean value theorem, there exists a point $\xi_i \in [0, u_i]$ such that

$$\left|\int_0^{u_i(t_j)} I_{ij}(s)ds\right| = |I_{ij}(\xi_i)u_i| \leqslant L_{ij}|u_i|^2,$$

which implies

$$-L_{ij}|u_i|^2 \leqslant \int_0^{u_i(t_j)} I_{ij}(s)ds \leqslant L_{ij}|u_i|^2.$$

Then it follows that

$$-\sum_{i=1}^{N}\sum_{j=1}^{l} L_{ij}|u_i|^2 \leqslant \sum_{i=1}^{N}\sum_{j=1}^{l}\int_0^{u_i(t_j)} I_{ij}(s)ds \leqslant \sum_{i=1}^{N}\sum_{j=1}^{l} L_{ij}|u_i|^2 \text{ for } i = 1,\dots,N, j = 1,\dots,l. \tag{3.4}$$

(H$_1$)-(H$_2$) imply that for arbitrary $\delta > 0$ with $\delta A_2 < \frac{c_0}{12}$, there exists a positive constant c_δ which depends on δ such that

$$F(t,u) \leqslant \sum_{i=1}^{N}(\delta|u_i|^2 + c_\delta b_0|u_i|^\nu) \text{ for all } (t,u) \in [0,T]\times R^N. \tag{3.5}$$

By (H$_3$), for any $\theta > 0$ large enough, there exists a constant $\sigma > 0$ such that for all $(t,u) \in [0,T]\times R^N$ and $|u| > \sigma$,

$$F(t,u) \geqslant \theta|u|^2. \tag{3.6}$$

It follows (H$_1$) that there exists $S_0 > 0$ such that for $(t,u) \in [0,T]\times R^N$ and $0 < |u| \leqslant \sigma$,

$$F(t,u) \leqslant S_0|u|^2. \tag{3.7}$$

Combining (3.6) and (3.7), one has

$$F(t,u) \geqslant \theta'|u|^2 \text{ for all } (t,u) \in [0,T]\times R^N, \tag{3.8}$$

where $\theta' = \theta - S_0 - \theta\sigma$.

By choosing suitable θ, it is easy to get that

$$\theta' A_2 > \frac{1}{c_0} + lL_0B. \tag{3.9}$$

From (2.4), (2.5), (3.3), (3.4), and (3.5), for $u \in Z_k$, we have

$$\begin{aligned}
\varphi_{\rho,\lambda}(u) &= \sum_{i=1}^{N}\int_0^T -\frac{1}{2}\,{}_0^cD_t^{\alpha_i}u_i(t)\,{}_t^cD_T^{\alpha_i}u_i(t)dt + \sum_{i=1}^{N}\sum_{j=1}^{l}\int_0^{u_i(t_j)} I_{ij}(s)ds \\
&\quad - \lambda\left(\int_0^T F(t,u(t))dt + \frac{\rho}{\lambda}\sum_{i=1}^{N}\frac{a_i(t)u_i^2(t)}{2}\right) \\
&\geqslant \sum_{i=1}^{N}|\cos(\pi\alpha_i)|\|u\|_{\alpha_i}^2 - \sum_{i=1}^{N}\sum_{j=1}^{l} L_{ij}\|u_i\|_\infty^2 - \lambda\sum_{i=1}^{N}\int_0^T(\delta|u_i|^2 + c_\delta b_0|u_i|^\nu)dt - \rho\sum_{i=1}^{N}\int_0^T\frac{a_i(t)u_i^2(t)}{2}dt \\
&\geqslant \sum_{i=1}^{N}\{\|u\|_{\alpha_i}^2[c_0 - BlL_0 - \lambda\delta A_2 - \frac{\rho A_2 a_0}{2}] - \lambda c_\delta b_0 A_\nu\|u\|_{\alpha_i}^\nu\}.
\end{aligned}$$

Then for $u \in Z_k, \lambda \in [1,2], \rho \in (0, \frac{2\sigma}{A_2 a_0}), \nu \in [2, 2^*_{\alpha_0})$, from $\sigma = \frac{5c_0}{6} - BlL_0 > 0$, one has

$$\varphi_{\rho,\lambda}(u) \geqslant \sum_{i=1}^{N} \|u\|^2_{\alpha_i}(\eta - 2c_\delta b_0 A_\nu \|u\|^{\nu-2}_{\alpha_i}),$$

where $\eta = \sigma - \frac{\rho A_2 a_0}{2} > 0$.

Let $r_k = N(\frac{\eta}{4c_\delta b_0 A_\nu})^{\frac{1}{\nu-2}}$. For $u \in Z_k$, choosing $\|u_i\|_{\alpha_i} = (\frac{\eta}{4c_\delta b_0 A_\nu})^{\frac{1}{\nu-2}}$, for $1 \leqslant i \leqslant N$, by (2.6), we have

$$\|u\|_X = \sum_{i=1}^{N} \|u_i\|_{\alpha_i} = r_k = N(\frac{\eta}{4c_\delta b_0 A_\nu})^{\frac{1}{\nu-2}},$$

Then for $u \in Z_k, \lambda \in [1,2], \rho \in (0, \frac{2\sigma}{A_2 a_0}), \nu \in [2, 2^*_{\alpha_0})$, it follows that

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\| = r_k} \varphi_{\rho,\lambda}(u) \geqslant \sum_{i=1}^{N} \|u\|^2_{\alpha_i}(\eta - 2c_\delta b_0 A_\nu \|u\|^{\nu-2}_{\alpha_i}) = \frac{\eta N}{2}(\frac{\eta}{4c_\delta b_0 A_\nu})^{\frac{1}{\nu-2}} > 0. \tag{3.10}$$

Next we prove $b_k(\lambda) = \max\limits_{u \in Y_k, \|u\| = \rho_k} \varphi_{\rho,\lambda}(u), \quad \forall \lambda \in [1,2]$.

From (f), (2.4), (2.5), (3.3), (3.4), (3.8), and (3.9), for any $\lambda \in [1,2]$ and all $u \in Y_k$ with $\dim Y_k < \infty$, we have

$$\begin{aligned}
\varphi_{\rho,\lambda}(u) &= \sum_{i=1}^{N} \int_0^T -\frac{1}{2}\, {}_0^cD_t^{\alpha_i} u_i(t)\, {}_t^cD_T^{\alpha_i} u_i(t) dt + \sum_{i=1}^{N} \sum_{j=1}^{l} \int_0^{u_i(t_j)} I_{ij}(s) ds \\
&\quad - \lambda \left(\int_0^T F(t, u(t)) dt + \frac{\rho}{\lambda} \sum_{i=1}^{N} \frac{a_i(t) u_i^2(t)}{2} \right) \\
&\leqslant \sum_{i=1}^{N} \frac{1}{|\cos(\pi\alpha_i)|} \|u_i\|^2_{\alpha_i} + \sum_{i=1}^{N} \sum_{j=1}^{l} L_{ij} |u_i|^2 - \int_0^T \theta' |u|^2 dt \\
&\leqslant \sum_{i=1}^{N} \frac{1}{c_0} \|u_i\|^2_{\alpha_i} + \sum_{i=1}^{N} lL_0 \|u_i\|^2_\infty - \int_0^T \theta' \sum_{i=1}^{N} u_i^2 dt \\
&\leqslant \sum_{i=1}^{N} \|u_i\|^2_\alpha (\frac{1}{c_0} + lL_0 B - \theta' A_2) < 0.
\end{aligned}$$

Hence, for any $\|u\| = \rho_k > r_k > 0$, we can obtain

$$b_k(\lambda) := \max_{u \in Y_k, \|u\| = \rho_k} \varphi_{\rho,\lambda}(u), \quad \forall \lambda \in [1,2]. \tag{3.11}$$

From (3.10) and (3.11), we can obtain

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\| = r_k} \varphi_{\rho,\lambda}(u) > 0 > b_k(\lambda) = \max_{u \in Y_k, \|u\| = \rho_k} \varphi_{\rho,\lambda}(u), \quad \forall \lambda \in [1,2].$$

We complete the proof of Lemma 3.1. □

Proof of Theorem 1.1. From (f), (2.5), (3.1), and (3.2), we can get $B(u) \geqslant 0$ for $u \in X$ and $A(u) \geqslant \sum_{i=1}^{N} |\cos(\pi\alpha_i)|$ $\|u_i\|^2_{\alpha_i} \to \infty$ as $\|u\|_X = \|(u_1, \dots, u_N)\|_X = \sum_{i=1}^{N} \|u_i\|_{\alpha_i} \to \infty$, which implies (A_2) of Lemma 2.10 holds.

From (f), (2.4), (3.1), (3.2), and $F(x, u)$ being even for $u \in X$, we can verify $\varphi_{\rho,\lambda}(u)$ maps bounded sets into bounded sets uniformly for $\lambda \in [1, 2]$, and

$$\varphi_{\rho,\lambda}(-u) = \varphi_{\rho,\lambda}(u),$$

which shows (A_1) of Lemma 2.10 is satisfied. Lemma 3.1 implies that (A_3) holds. In virtue of Lemma 2.10, for almost every $\lambda \in [1, 2]$, there exists a sequence $\{u^k_m(\lambda)\}_{n=1}^{\infty} \subset X$ for $k \in N$ such that

$$\sup_m \|u^k_m(\lambda)\|_X < \infty, \varphi'_{\rho,\lambda}(u^k_m(\lambda)) \to 0 \text{ and } \varphi_{\rho,\lambda}(u^k_m(\lambda)) \to c_k(\lambda), \text{ as } m \to \infty. \tag{3.12}$$

From Lemma 2.10, we also have $c_k(\lambda) = \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \varphi_{\rho,\lambda}(\gamma(u)) \geqslant a_k(\lambda)$.

Let $\beta_k = \frac{\eta N}{2}(\frac{\eta}{4c_\delta b_0 A_\nu})^{\frac{2}{\nu-2}}$. For $k \in N$, it follows (3.10) that

$$c_k(\lambda) \geqslant a_k(\lambda) \geqslant \beta_k.$$

Then

$$c_k(\lambda) \in [\beta_k, \beta'_k], \tag{3.13}$$

where $\beta'_k = \max_{u\in B_k} \varphi_{\rho,\lambda}(\gamma(u)), \Gamma_k = \{\gamma \in C(B_k, X_\rho) : \gamma \text{ is odd}, \ \gamma|_{\partial B_{k'}} = \ \text{id} \ (k \geqslant 2)\}$, with $B_k = \{u \in Y_k : \|u\| \leqslant \rho_k\}$.

Owing to (3.12), we can get the boundedness of $\{u^k_m(\lambda)\}$ in X, which shows that $\{u^k_m(\lambda)\}$ has a weakly convergent subsequence. By a standard argument, we can obtain that $\{u^k_m(\lambda)\}$ has a strong convergent subsequence in X. We assume

$$\lim_{m\to\infty} u^k_m(\lambda) = u^k(\lambda) = u^k \in X.$$

Then for $k \in N$, from (3.12) and (3.13), we have

$$\varphi'_{\rho,\lambda}(u^k) = 0, \ \varphi'_{\rho,\lambda}(u^k) \in [\beta_k, \beta'_k],$$

which implies u^k is a nontrivial critical point of $\varphi_{\rho,\lambda}$. Consequently, for $k \in N$ arbitrary, we can get infinitely many nontrivial critical points u^k of $\varphi_{\rho,\lambda}$, which are also the nontrivial weak solutions of (2.1). By the equivalence of (1.1) and (2.1), we know that (1.1) possess infinitely many nontrivial weak solutions. Owing to Lemma 2.9, we know (1.1) has infinitely many nontrivial solutions.

Then we complete the proof of Theorem 1.1. □

Proof of Corollary 1.2. Obviously, (1.3) is the special case of (1.1) for $\rho = 0$. By Theorem 1.1, we can get the conclusion of Corollary 1.2. □

Remark 3.2. If the superlinear case for the nonlinearity F_{u_i} turns to the following sublinear case:

(H_1) $\lim_{|u|\to 0} \frac{F(t,u)}{|u|^2} = 0$ uniformly in $t \in [0, T]$, where $u = (u_1, \dots, u_N), |u| = \sqrt{\sum_{i=1}^{N} u_i^2}$.

(H'_2) There exist constants $a_0 > 0$ and $p \in [1, 2)$ such that

$$|F_{u_i}(t, u)| \leqslant a_0(1 + |u|^{p-1}) \ \text{ for } \ (t, u) \in [0, T] \times R^N, i = 1, \dots, N.$$

(H'_3) There exist $\sigma \in [1, p)$ and $c > 0$ such that $\lim_{|u|\to\infty} \frac{F(t,u)}{|u|^\sigma} \geqslant c$ uniformly for $t \in [0, T]$.

Then by using another variant fountain theorem in [39], we can also establish the existence criteria of infinitely many nontrivial solutions of the problem (1.1).

Remark 3.3. As we know, the classical approaches to study BVPs of fractional differential equations with impulses mainly include fixed point theorems, degree theory, the method of upper and lower solutions and so on. The variational method is an important and efficient method to investigate the integer order differential systems. But for the fractional order problems, the variational method has rarely been used. It is often very difficult to establish a suitable space and variational functional for fractional BVPs for several reasons. First and foremost, the composition rule in general fails to be satisfied by fractional integral and fractional derivative operators. Furthermore, the fractional integral is a singular integral operator and fractional derivative operator is non-local. Besides, the adjoint of a fractional differential operator is not the negative of itself. Compare with the other existing methods, the variational method and the variant fountain theorem used in the present paper are more useful for us to get the main results. In our study, we did not require the energy functional to satisfy the PalaisSmale (P.S) condition, and we can obtain some good properties for the solutions. For examples, the solutions are nontrivial, and have bounded estimation, i.e., $\|u^k(\lambda)\|_X < \infty, \varphi'_{\rho,\lambda}(u^k) \in [\beta_k, \beta'_k]$, for $k \in \mathbb{N}$. For the details, please see the following Example 4.1, which is offered to demonstrate the application of our main results.

4. Example

Example 4.1.

$$\begin{cases} -\dfrac{1}{2}\dfrac{d}{dt}({}_0D_t^{-0.5} + {}_tD_T^{-0.5})u_1'(t) = \rho a_1(t)u_1(t) + \lambda F_{u_1}(t,u(t)), \quad t \in (0,1), \\ -\dfrac{1}{2}\dfrac{d}{dt}({}_0D_t^{-0.4} + {}_tD_T^{-0.4})u_2'(t) = \rho a_2(t)u_2(t) + \lambda F_{u_2}(t,u(t)), \quad t \in (0,1), \\ \triangle(D_t^{0.75}u_1)(0.5) = I_{11}(u_1(0.5)), \\ \triangle(D_t^{0.8}u_2)(0.5) = I_{21}(u_2(0.5)), \\ u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0, \end{cases} \tag{4.1}$$

where $T = 1, N = 2, l = 1, t_1 = 0.5, a_1(t) = \sin^2 t, a_2(t) = \frac{1}{1+t^2}, I_{11}(x) = I_{21}(x) = \frac{x^2}{82}$, and $F(t,u_1,u_2) = e^{-(2.1+\cos t)}(u_1^2+u_2^2)^{1.5}$.

It is easy to see $\beta_1 = 0.5, \beta_2 = 0.4, \alpha_1 = 0.75, \alpha_2 = 0.8$. We can verify that $I_{i1}(x)$ satisfy Lipschitz condition, i.e., $|I_{i1}(x_1) - I_{i1}(x_2)| \leqslant \frac{|x_1+x_2|}{82}|x_1 - x_2|$ for $i = 1,2$. Then we can see that the condition (f) holds.

We can also find that

$$F(t,u) = e^{-(2.1+\cos t)}3|u||u_i| < 3e^{-1}(1+|u|^{3-1}) \quad \text{for } i = 1,2,$$

with $\nu = 3 \in [2, 2^*_{\alpha_0} = 8)$, which implies (H_2) is satisfied. It is also easy to check (H_1) and (H_3) hold.

By a direct calculation, we have $c_0 = |\cos(0.75\pi)| = \frac{\sqrt{2}}{2}, A_2 = \frac{1}{\Gamma^2(1.75)} \approx 1.18, L_0 = \frac{1}{41}, B = \frac{1}{\Gamma^2(0.75)0.5} = 1.33, a_0 = 1$. Then we can verify $\sigma = \frac{5c_0}{6} - BlL_0 = 0.55673 > 0$.

Then all the conditions of Theorem 1.1 are fulfilled. Consequently, for $\rho \in (0, 0.94), \lambda \in [1,2]$, (4.1) possesses infinitely many nontrivial solutions $u^k \in X$ for all $k \in \mathbb{N}$.

For $\beta_k = \frac{\eta N}{2}(\frac{\eta}{4c_\delta b_0 A_\nu})^{\frac{2}{\nu-2}}$, it is easy to see that $c_\delta b_0 \leqslant e^{-3.1} \approx 0.045, N = 2, \nu = 3, \eta = \sigma - \frac{\rho A_2 a_0}{2} \geqslant 0.0023, A_3 \approx 1.29$. By calculating, we know $\beta_k \geqslant 0.0000002255$.

When $\rho_k = 1$, from (3.3), for the functions given in (4.1), we can get the numerical result that $\beta'_k = \max\limits_{u \in B_k} \varphi_{\rho,\lambda}(\gamma(u)) \leqslant 3.156$ by means of Matlab software.

Then, for $\rho \in (0, 0.94), \lambda \in [1,2]$, it follows that (4.1) has infinitely many nontrivial solutions $u^k \in X$ with $\|u^k\|_X < \infty$ and $\varphi_{\rho,\lambda}(u^k) \in [0.0000002255, 3.156]$ for all $k \in \mathbb{N}$.

5. Conclusion

In this paper, we study the existence and multiplicity of solutions for the fractional differential equation (1.1) with impulses and perturbation, which can describe the phenomena of advection dispersion very well. It is also should be noted that we considered (1.1) in the R^N. By the variational methods, the existence criteria of infinitely many nontrivial solutions for the problem (1.1) are established. We also obtain some good properties for the solutions of (1.1). An example is given to demonstrate the usefulness of our main results. Recent results in the literature are generalized and improved.

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