



## Common fixed points via implicit contractions on b-metric-like spaces

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### Abstract

In this paper, we introduce some generalized nonlinear contractions via implicit functions and  $\alpha$ -admissible pair of mappings. We also provide some common fixed point results for above contractions in the class of b-metric-like spaces. We will derive some consequences and corollaries from our obtained results. Some illustrated examples are presented to make effective the concepts and results. ©2017 All rights reserved.

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### 1. Introduction and preliminaries

In 2013, the concept of a b-metric-like space was introduced first by Alghamdi et al. [2] as a generalization of a metric-like space [3, 7, 17, 18, 23], a b-metric space [9, 16, 31, 34, 35] and a partial metric space [1, 6, 15, 25, 30, 32, 33]. For some fixed point results on b-metric-like spaces, see [19] and [13].

The definition of a b-metric-like space is given as follows:

**Definition 1.1.** Let  $X$  be non-empty and  $s \geq 1$ . Let  $\sigma : X \times X \rightarrow [0, \infty)$  be a function which satisfies:

- (d1)  $\sigma(x, y) = 0$  implies that  $x = y$ ;
- (d2)  $\sigma(x, y) = \sigma(y, x)$ ;
- (d3)  $\sigma(x, y) \leq s[\sigma(x, z) + \sigma(z, y)]$ .

Then  $\sigma$  is called a b-metric-like and the pair  $(X, \sigma)$  is called a b-metric-like space.

**Definition 1.2.** A sequence  $\{x_n\}$  converges to  $x \in X$  in the b-metric-like space  $(X, \sigma)$  if and only if

$$\lim_{n \rightarrow \infty} \sigma(x_n, x) = \sigma(x, x).$$

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*Remark 1.3.* In a b-metric-like space, the limit for a convergent sequence is not unique in general.

**Definition 1.4.** A sequence  $\{x_n\}$  is Cauchy in the b-metric-like space  $(X, \sigma)$  if and only if  $\lim_{n,m \rightarrow \infty} d(x_n, x_m)$  exists and is finite.

**Definition 1.5.** The b-metric-like space  $(X, \sigma)$  is complete if and only if each Cauchy sequence in  $X$  is convergent.

**Lemma 1.6.** Let  $(X, \sigma)$  be a b-metric-like space and  $\{x_n\}$  be a sequence that converges to  $u$  with  $\sigma(u, u) = 0$ . Then for each  $y, z \in X$

$$\frac{1}{s} \sigma(u, z) \leq \liminf_{n \rightarrow \infty} \sigma(x_n, z) \leq s \sigma(u, z) \quad \text{and} \quad \sigma(z, z) \leq 2s \sigma(z, y).$$

**Definition 1.7.** Let  $(X, \sigma)$  be a b-metric-like space. The map  $T : X \rightarrow X$  is continuous at  $u \in X$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$T(B_\sigma(u, \delta)) \subseteq B_\sigma(Tu, \varepsilon).$$

**Definition 1.8.** Let  $(X, \sigma)$  be a b-metric-like space and  $T : X \rightarrow X$  be a given mapping.  $T$  is said sequentially continuous at  $u \in X$  if for each sequence  $\{x_n\}$  in  $X$  converging to  $u$ , we have  $Tx_n \rightarrow Tu$ , that is,

$$\lim_{n \rightarrow \infty} \sigma(Tx_n, Tu) = \sigma(Tu, Tu).$$

$T$  is said sequentially continuous on  $X$  if  $T$  is sequentially continuous at each  $u \in X$ .

A simple consequence of above two definitions is stated as follows.

**Lemma 1.9.** Let  $(X, \sigma)$  be a b-metric-like space and  $T : X \rightarrow X$  be a given mapping. If  $T$  is continuous on  $X$ , then it is sequentially continuous on  $X$ .

On the other hand, instead of the standard explicit contractive conditions, Popa [28] used the concept of an implicit function in the contractive condition. This direction was the source of several (common) fixed point and coincidence point theorems in various ambient spaces. For more details, see [5, 10–12, 20].

Now, take  $s \geq 1$  and denote by  $\mathbb{N}$  the set of positive integers. Let  $\Psi_s$  be the set of functions  $\psi : [0, \infty) \rightarrow [0, \infty)$  satisfying:

( $\psi_1$ )  $\psi$  is nondecreasing,

( $\psi_2$ )  $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$  for each  $t \in \mathbb{R}^+$  where  $\psi^n$  is the  $n$ th iterate of  $\psi$ .

*Remark 1.10.* It is easy to see that if  $\psi \in \Psi_s$ , then  $\psi(t) < t$  for any  $t > 0$  and  $\psi(0) = 0$ .

We introduce the following.

**Definition 1.11.** Consider  $s \geq 1$ . Let  $\Gamma_s$  be the set of functions  $F(t_1, \dots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  such that

(F1)  $F$  is nondecreasing in variable  $t_1$  and nonincreasing in variables  $t_5$  and  $t_6$ ;

(F2) there exists  $h_1 \in \Psi_s$  such that for all  $u, v \geq 0$ ,  $F(u, v, v, u, s[u+v], 2su) \leq 0$  implies  $u \leq h_1(v)$ , and  $F(u, v, u, v, 2su, s[u+v]) \leq 0$  implies  $u \leq h_1(v)$ .

We present the following examples.

**Example 1.12.**  $F(t_1, \dots, t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$ , where  $a, b, c \geq 0$  such that  $as + b(1 + s) + cs(s + 3) < 1$ .

**Example 1.13.**  $F(t_1, \dots, t_6) = t_1 - k \max\{t_2, \dots, t_6\}$ , where  $k \in [0, \frac{1}{s+s^2})$ .

In 2012, Samet et al. [29] introduced a new useful concept, named as  $\alpha$ -admissible maps.

**Definition 1.14** ([29]). For a nonempty set  $X$ , let  $T : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, \infty)$  be given mappings. We say that  $T$  is  $\alpha$ -admissible if for all  $x, y \in X$ , we have

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.$$

The above concept opens the door to prove many (common) fixed point results, for example see [4, 21, 22, 24, 26, 27].

Very recently, Aydi [8] generalized Definition 1.14 and introduced the following.

**Definition 1.15.** For a nonempty set  $X$ , let  $A, B : X \rightarrow X$  and  $\alpha : X \times X \rightarrow [0, \infty)$  be mappings. We say that  $(A, B)$  is a generalized  $\alpha$ -admissible pair if for all  $x, y \in X$ , we have

$$\alpha(x, y) \geq 1 \implies \alpha(Ax, By) \geq 1 \quad \text{and} \quad \alpha(By, Ax) \geq 1. \quad (1.1)$$

We can rewrite (1.1) as

$$\alpha(x, y) \geq 1 \implies \min(\alpha(Ax, By), \alpha(By, Ax)) \geq 1.$$

Now, we introduce the concept of  $\alpha$ -implicit contractive pair of mappings in the setting of b-metric-like spaces.

**Definition 1.16.** Let  $(X, \sigma)$  be a b-metric-like space and  $A, B : X \rightarrow X$  be given mappings. We say that  $(A, B)$  is an  $\alpha$ -implicit contractive pair of mappings if there exist two functions  $\alpha : X \times X \rightarrow [0, \infty)$  and  $F \in \Gamma_s$  such that

$$F(\alpha(x, y)\sigma(Ax, By), \sigma(x, y), \sigma(x, Ax), \sigma(y, By), \sigma(x, By), \sigma(y, Ax)) \leq 0 \quad (1.2)$$

for all  $x, y \in X$ .

In this paper, we provide some common fixed point results involving generalized  $\alpha$ -implicit contractions on b-metric-like spaces. As consequences of our obtained results, we prove some existing fixed point results on b-metric spaces and on metric-like spaces. We also provide some examples illustrating our obtained results and the new concepts.

## 2. Fixed point theorems

In this section, we shall state and prove our main results.

**Theorem 2.1.** Let  $(X, \sigma)$  be a complete b-metric-space and  $A, B : X \rightarrow X$  be generalized  $\alpha$ -implicit contractive pair of mappings. Suppose that

- (i)  $(A, B)$  is a generalized  $\alpha$ -admissible pair;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(Ax_0, x_0) \geq 1$  and  $\alpha(x_0, Ax_0) \geq 1$ ;
- (iii)  $A$  and  $B$  are sequentially continuous on  $(X, \sigma)$ .

Then there exists  $u \in X$  such that

$$\frac{1}{s}\sigma(u, Au) \leq \sigma(Au, Au) \leq s\sigma(u, Au), \quad \frac{1}{s}\sigma(u, Bu) \leq \sigma(Bu, Bu) \leq s\sigma(u, Bu), \quad \text{and} \quad \sigma(u, u) = 0. \quad (2.1)$$

Assume in addition that

- (iv)  $\alpha(z, z) \geq 1$  for all  $z$  verifying (2.1);
- (v)  $F$  satisfies

$(F_\gamma)$  if  $F(u, 0, v, w, 2s^2u, 2s^2u) \leq 0$  for all  $u, v, w \geq 0$ , there exists  $\gamma \in [0, \frac{1}{2s^2})$  such that  $u \leq \gamma \max\{v, w\}$ .

Then  $u$  is a common fixed point of  $A$  and  $B$ , that is,  $u = Au = Bu$ .

*Proof.* By assumption (ii), there exists a point  $x_0 \in X$  such that  $\alpha(x_0, Ax_0) \geq 1$  and  $\alpha(Ax_0, x_0) \geq 1$ . Take  $x_1 = Ax_0$  and  $x_2 = Bx_1$ . By induction, we arrive to construct a sequence  $\{x_n\}$  such that

$$x_{2n} = Bx_{2n-1} \quad \text{and} \quad x_{2n+1} = Ax_{2n} \quad \forall n = 1, 2, \dots \quad (2.2)$$

We split the proof into several steps:

Step 1:  $\alpha(x_n, x_{n+1}) \geq 1$  and  $\alpha(x_{n+1}, x_n) \geq 1$  for all  $n = 0, 1, \dots$

We have  $\alpha(x_0, x_1) \geq 1$  and  $\alpha(x_1, x_0) \geq 1$ . Since  $(A, B)$  is a generalized  $\alpha$ -admissible pair,

$$\alpha(x_1, x_2) = \alpha(Ax_0, Bx_1) \geq 1 \quad \text{and} \quad \alpha(x_2, x_1) = \alpha(Bx_1, Ax_0) \geq 1.$$

We also have

$$\alpha(x_3, x_2) = \alpha(Ax_2, Bx_1) \geq 1 \quad \text{and} \quad \alpha(x_2, x_3) = \alpha(Bx_1, Ax_2) \geq 1.$$

By induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_{n+1}, x_n) \geq 1 \quad \text{for all } n = 0, 1, \dots \quad (2.3)$$

Step 2:  $d(x_n, x_{n+1}) \leq h_1^n(d(x_0, x_1))$  for all  $n = 0, 1, \dots$

From (1.2), we have

$$F(\alpha(x_{2n-2}, x_{2n-1})\sigma(Ax_{2n-2}, Bx_{2n-1}), \sigma(x_{2n-2}, x_{2n-1}), \\ \sigma(x_{2n-2}, Ax_{2n-2}), \sigma(x_{2n-1}, Bx_{2n-1}), \sigma(x_{2n-2}, Bx_{2n-1}), \sigma(x_{2n-1}, Ax_{2n-2})) \leq 0,$$

that is,

$$F(\alpha(x_{2n-2}, x_{2n-1})\sigma(x_{2n-1}, x_{2n}), \sigma(x_{2n-2}, x_{2n-1}), \\ \sigma(x_{2n-2}, x_{2n-1}), \sigma(x_{2n-1}, x_{2n}), \sigma(x_{2n-2}, x_{2n}), \sigma(x_{2n-1}, x_{2n-1})) \leq 0.$$

Using (2.3), (d3) in the fifth variable and (F1) in the first, fifth and sixth variables

$$F(\sigma(x_{2n-1}, x_{2n}), \sigma(x_{2n-2}, x_{2n-1}), \sigma(x_{2n-2}, x_{2n-1}), \\ \sigma(x_{2n-1}, x_{2n}), s[\sigma(x_{2n-2}, x_{2n-1}) + \sigma(x_{2n-1}, x_{2n})], 2s\sigma(x_{2n-1}, x_{2n})) \leq 0.$$

By (F2), we obtain

$$\sigma(x_{2n-1}, x_{2n}) \leq h_1(\sigma(x_{2n-2}, x_{2n-1})). \quad (2.4)$$

Similarly, from (1.2), we have

$$F(\alpha(x_{2n}, x_{2n-1})\sigma(Ax_{2n}, Bx_{2n-1}), \sigma(x_{2n}, x_{2n-1}), \\ \sigma(x_{2n}, Ax_{2n}), \sigma(x_{2n-1}, Bx_{2n-1}), \sigma(x_{2n}, Bx_{2n-1}), \sigma(x_{2n-1}, Ax_{2n})) \leq 0,$$

that is,

$$F(\alpha(x_{2n}, x_{2n-1})\sigma(x_{2n+1}, x_{2n}), \sigma(x_{2n}, x_{2n-1}), \\ \sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n-1}, x_{2n}), \sigma(x_{2n}, x_{2n}), \sigma(x_{2n-1}, x_{2n+1})) \leq 0.$$

Again, by (2.3), we apply (F1) in the first variable to get

$$F(\sigma(x_{2n+1}, x_{2n}), \sigma(x_{2n}, x_{2n-1}), \sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n-1}, x_{2n}), \sigma(x_{2n}, x_{2n}), \sigma(x_{2n-1}, x_{2n+1})) \leq 0.$$

Applying again (F1) in the fifth and sixth variables,

$$F(\sigma(x_{2n+1}, x_{2n}), \sigma(x_{2n}, x_{2n-1}), \sigma(x_{2n}, x_{2n+1}), \\ \sigma(x_{2n-1}, x_{2n}), 2s\sigma(x_{2n}, x_{2n+1}), s[\sigma(x_{2n-1}, x_{2n}) + \sigma(x_{2n}, x_{2n+1})]) \leq 0.$$

By (F2),

$$\sigma(x_{2n}, x_{2n+1}) \leq h_1(\sigma(x_{2n-1}, x_{2n})). \quad (2.5)$$

Combining (2.4) and (2.5), we obtain

$$\sigma(x_n, x_{n+1}) \leq h_1(\sigma(x_{n-1}, x_n)) \quad \text{for all } n = 1, 2, \dots$$

We deduce

$$\sigma(x_n, x_{n+1}) \leq h_1^n(\sigma(x_0, x_1)) \quad \text{for all } n = 0, 1, 2, \dots \quad (2.6)$$

Step 3:  $\{x_n\}$  is a Cauchy sequence.

Now, we shall prove that  $\{x_n\}$  is a Cauchy sequence in the b-metric-like space  $(X, \sigma)$ .

Using (d3) and (2.6), we have for all  $n, k \in \mathbb{N}$

$$\begin{aligned} \sigma(x_n, x_{n+k}) &\leq s\sigma(x_n, x_{n+1}) + s^2\sigma(x_{n+1}, x_{n+2}) + \dots + s^{k-1}\sigma(x_{n+k-1}, x_{n+k}) \\ &\leq (sh_1^n + s^2h_1^{n+1} + \dots + s^{k-1}h_1^{n+k-1})[\sigma(x_0, x_1)] \\ &\leq \sum_{i=n}^{n+k-1} s^i h_1^i [\sigma(x_0, x_1)] \leq \sum_{i=n}^{\infty} s^i h_1^i [\sigma(x_0, x_1)]. \end{aligned}$$

Since  $h_1 \in \Psi_s$ ,

$$\sigma(x_n, x_{n+k}) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \text{ for all } k. \quad (2.7)$$

It follows that  $\{x_n\}$  is a Cauchy sequence in the b-metric-like space  $(X, \sigma)$ .

Step 4: Existence of a common fixed point.

Since  $(X, \sigma)$  is complete, the sequence  $\{x_n\}$  converges to some  $u \in X$ , that is,

$$\sigma(u, u) = \lim_{n \rightarrow \infty} \sigma(x_n, u) = \lim_{m, n \rightarrow \infty} \sigma(x_n, x_m).$$

By (2.7), we get

$$\sigma(u, u) = \lim_{n \rightarrow \infty} \sigma(x_n, u) = \lim_{m, n \rightarrow \infty} \sigma(x_n, x_m) = 0. \quad (2.8)$$

This implies that

$$\lim_{n \rightarrow \infty} \sigma(x_{2n+1}, u) = \lim_{n \rightarrow \infty} \sigma(x_{2n+2}, u) = 0.$$

By (2.2),

$$\lim_{n \rightarrow \infty} \sigma(Ax_{2n}, u) = \lim_{n \rightarrow \infty} \sigma(Bx_{2n+1}, u) = 0.$$

We shall prove that  $u = Au = Bu$ , that is,  $u$  is a common fixed point of  $A$  and  $B$ .

Using the sequentially continuity of  $A$  and the fact that  $x_{2n} \rightarrow u$  in  $(X, \sigma)$ ,

$$\lim_{n \rightarrow \infty} \sigma(x_{2n+1}, Au) = \lim_{n \rightarrow \infty} \sigma(Ax_{2n}, Au) = \sigma(Au, Au).$$

Using now the sequentially continuity of  $B$  and the fact that  $x_{2n+1} \rightarrow u$  in  $(X, \sigma)$ ,

$$\lim_{n \rightarrow \infty} \sigma(x_{2n+2}, Bu) = \lim_{n \rightarrow \infty} \sigma(Bx_{2n+1}, Bu) = \sigma(Bu, Bu).$$

On the other side, having  $\lim_{n \rightarrow \infty} \sigma(x_n, u) = 0 = \sigma(u, u)$  together with Lemma 1.6,

$$\frac{1}{s}\sigma(u, Au) \leq \lim_{n \rightarrow \infty} \sigma(x_{2n+1}, Au) \leq s\sigma(u, Au).$$

This yields that

$$\frac{1}{s}\sigma(u, Au) \leq \sigma(Au, Au) \leq s\sigma(u, Au). \quad (2.9)$$

Similarly

$$\frac{1}{s}\sigma(u, Bu) \leq \sigma(Bu, Bu) \leq s\sigma(u, Bu). \quad (2.10)$$

By (2.8), (2.9), and (2.10), the assertion (2.1) holds. So by condition (iv), we have

$$\alpha(u, u) \geq 1.$$

Now, applying (1.2) for  $x = y = u$

$$F(\alpha(u, u)\sigma(Au, Bu), \sigma(u, u), \sigma(u, Au), \sigma(u, Bu), \sigma(u, Bu), \sigma(u, Au)) \leq 0,$$

i.e,

$$F(\alpha(u, u)\sigma(Au, Bu), 0, \sigma(u, Au), \sigma(u, Bu), \sigma(u, Bu), \sigma(u, Au)) \leq 0.$$

Since  $\alpha(u, u) \geq 1$ , applying (F1) for the first variable, we get

$$F(\sigma(Au, Bu), 0, \sigma(u, Au), \sigma(u, Bu), \sigma(u, Bu), \sigma(u, Au)) \leq 0.$$

Remember that

$$\sigma(u, Bu) \leq s\sigma(Bu, Bu) \leq 2s^2\sigma(Au, Bu), \quad (2.11)$$

so applying (F1) in the fifth and sixth variables, we obtain

$$F(\sigma(Au, Bu), 0, \sigma(u, Au), \sigma(u, Bu), 2s^2\sigma(Au, Bu), 2s^2\sigma(Au, Bu)) \leq 0.$$

Recall that  $F$  satisfies property  $(F_\gamma)$ , so there exists  $\gamma \in [0, \frac{1}{2s^2})$  such that

$$\sigma(Au, Bu) \leq \gamma \max\{\sigma(u, Au), \sigma(u, Bu)\} \leq 2\gamma s^2\sigma(Au, Bu),$$

which holds unless  $\sigma(Au, Bu) = 0$ . Thus,  $Au = Bu$ . We deduce from (2.11) that  $\sigma(u, Au) = 0 = \sigma(u, Bu)$ . Hence

$$u = Au = Bu.$$

This completes the proof. □

If we take  $A = B$  in Theorem 2.1, we get the following result.

**Corollary 2.2.** *Let  $(X, \sigma)$  be a complete b-metric-like space and  $A : X \rightarrow X$  be a given mapping. Assume there exists  $F \in \Gamma_s$  such that*

$$F(\alpha(x, y)\sigma(Ax, Ay), \sigma(x, y), \sigma(x, Ax), \sigma(y, Ay), \sigma(x, Ay), \sigma(y, Ax)) \leq 0$$

for all  $x, y \in X$ . Suppose that

- (i)  $(A, A)$  is an  $\alpha$ -admissible;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(Ax_0, x_0) \geq 1$  and  $\alpha(x_0, Ax_0) \geq 1$ ;
- (iii)  $A$  is sequentially continuous on  $(X, \sigma)$ .

Then there exists  $u \in X$  such that

$$\frac{1}{s}\sigma(u, Au) \leq \sigma(Au, Au) \leq s\sigma(u, Au). \quad (2.12)$$

Assume in addition that

- (iv)  $\alpha(z, z) \geq 1$  for all  $z$  verifying (2.12);
- (v)  $F$  satisfies

$(F_\gamma)$  if  $F(u, 0, v, w, 2s^2u, 2s^2u) \leq 0$  for all  $u, v, w \geq 0$ , there exists  $\gamma \in [0, \frac{1}{2s^2})$  such that  $u \leq \gamma \max\{v, w\}$ .

Then  $u$  is a fixed point of  $A$ , that is,  $u = Au$ .

*Proof.* The proof follows from the lines in the proof of Theorem 2.1.  $\square$

Considering the  $b$ -metric case in Theorem 2.1, we have the following.

**Corollary 2.3.** Let  $(X, \sigma)$  be a complete  $b$ -metric space and  $A, B : X \rightarrow X$  be given mappings. Suppose there exists  $F \in \Gamma_s$  such that

$$F(\alpha(x, y)\sigma(Ax, By), \sigma(x, y), \sigma(x, Ax), \sigma(y, By), \sigma(x, By), \sigma(y, Ax)) \leq 0$$

for all  $x, y \in X$ . Assume that

- (i)  $(A, B)$  is an  $\alpha$ -admissible pair;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Ax_0) \geq 1$  and  $\alpha(Ax_0, x_0) \geq 1$ ;
- (iii)  $A$  and  $B$  are sequentially continuous on  $(X, \sigma)$ .

Then there exists  $u \in X$  such that  $u = Au = Bu$ .

*Proof.* Due to conditions (i)-(iii), there exists  $u \in X$  such that from Theorem 2.1, (2.1) becomes

$$u = Au = Bu,$$

that is,  $u$  is a common fixed point of  $A$  and  $B$ . Here, we do not need conditions (iv) and (v) given in Theorem 2.1.  $\square$

**Corollary 2.4.** Let  $(X, \sigma)$  be a complete  $b$ -metric space and  $A : X \rightarrow X$  a given mapping. Assume there exists  $F \in \Gamma_s$  such that

$$F(\alpha(x, y)\sigma(Ax, Ay), \sigma(x, y), \sigma(x, Ax), \sigma(y, Ay), \sigma(x, Ay), \sigma(y, Ax)) \leq 0$$

for all  $x, y \in X$ . Suppose that

- (i)  $(A, A)$  is an  $\alpha$ -admissible mapping;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(Ax_0, x_0) \geq 1$  and  $\alpha(x_0, Ax_0) \geq 1$ ;
- (iii)  $A$  is sequentially continuous on  $(X, \sigma)$ .

Then there exists  $u \in X$  such that  $u = Au$ .

Now, consider in Theorem 2.1 the operator  $F$  presented by Example 1.13. We state the following.

**Corollary 2.5.** Let  $(X, \sigma)$  be a complete  $b$ -metric like metric space and  $A, B : X \rightarrow X$  satisfying

$$\alpha(x, y)\sigma(Ax, By) \leq k \max\{\sigma(x, y), \sigma(x, Ax), \sigma(y, By), \sigma(x, By), \sigma(y, Ax)\}, \quad (2.13)$$

where  $k \in [0, \frac{1}{2s^2})$ . Suppose that

- (i)  $(A, B)$  is an  $\alpha$ -admissible pair;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Ax_0) \geq 1$  and  $\alpha(Ax_0, x_0) \geq 1$ ;
- (iii)  $A$  and  $B$  are sequentially continuous on  $(X, \sigma)$ .

Then, there exists  $u \in X$  such that  $u$  is a common fixed point of  $A$  and  $B$ , that is,  $u = Au = Bu$ .

For  $A = B$ , we have the following result.

**Corollary 2.6.** Let  $(X, \sigma)$  be a complete  $b$ -metric-like space and  $A : X \rightarrow X$  be a mapping such that

$$\alpha(x, y)\sigma(Ax, Ay) \leq k \max\{\sigma(x, y), \sigma(x, Ax), \sigma(y, Ay), \sigma(x, Ay), \sigma(y, Ax)\},$$

where  $k \in [0, \frac{1}{2s^2})$ . Suppose that

- (i)  $(A, A)$  is an  $\alpha$ -admissible mapping;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Ax_0) \geq 1$  and  $\alpha(Ax_0, x_0) \geq 1$ ;
- (iii)  $A$  is continuous on  $(X, \sigma)$ .

Then there exists  $u \in X$  such that  $u$  is a fixed point of  $A$ , that is,  $u = Au$ .

In the next result, we drop the sequentially continuity hypothesis of  $A$  and  $B$  and we replace it by the following.

- (H) If  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$  and  $\alpha(x_{n+1}, x_n) \geq 1$  for all  $n$  and  $x_n \rightarrow x \in X$  as  $n \rightarrow \infty$ , then there exists a subsequence  $\{x_{n(k)}\}$  of  $\{x_n\}$  such that  $\alpha(x_{n(k)}, x) \geq 1$  and  $\alpha(x, x_{n(k)}) \geq 1$  for all  $k$ .

We also replace Definition 1.11 by the following.

**Definition 2.7.** Consider  $s \geq 1$ . Let  $G_s$  be the set of continuous functions  $F(t_1, \dots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  such that

- (G1)  $F$  is nondecreasing in variable  $t_1$  and nonincreasing in variables  $t_5$  and  $t_6$ ;
- (G2) there exists  $h_1 \in \Psi_s$  such that for all  $u, v \geq 0$ ,  $F(\frac{u}{s}, v, v, u, s[u+v], 2su) \leq 0$  implies  $u \leq h_1(v)$ , and  $F(\frac{u}{s}, v, u, v, 2su, s[u+v]) \leq 0$  implies  $u \leq h_1(v)$ .

*Remark 2.8.* If  $F \in \Gamma_s$ , the continuity of  $F$  is not required. But, in Definition 2.7, the continuity is essential. Also, the property (G2) implies (F2). The converse sense is not true. Therefore, the sets  $\Gamma_s$  and  $G_s$  are different to each other.

**Theorem 2.9.** Let  $(X, \sigma)$  be a complete  $b$ -metric-like space and  $A, B : X \rightarrow X$  be generalized  $\alpha$ -implicit contractive pair of mappings (with  $F \in G_s$ ). Suppose that

- (i)  $(A, B)$  is an  $\alpha$ -admissible pair;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Ax_0) \geq 1$  and  $\alpha(Ax_0, x_0) \geq 1$ ;
- (iii) (H) holds.

Then there exists  $u \in X$  such that  $u = Au = Bu$ . We also have  $\sigma(u, u) = 0$ .

*Proof.* Following the proof of Theorem 2.1, the sequence  $\{x_n\}$  defined by (2.2) is Cauchy and converges to some  $u \in X$  in  $(X, \sigma)$ . Remember that (2.3) holds, so from condition (iv), there exists a subsequence  $\{x_{n(k)}\}$  of  $\{x_n\}$  such that  $\alpha(x_{2n(k)}, u) \geq 1$  and  $\alpha(u, x_{2n(k)-1}) \geq 1$  for all  $k$ . We shall show that  $u = Au = Bu$ .

Taking  $x = x_{2n(k)}$  and  $y = u$  in (1.2)

$$\limsup_{k \rightarrow \infty} F(\alpha(x_{2n(k)}, u)\sigma(Ax_{2n(k)}, Bu), \sigma(x_{2n(k)}, u), \sigma(x_{2n(k)}, Ax_{2n(k)}), \sigma(u, Bu), \sigma(x_{2n(k)}, Bu), \sigma(u, Ax_{2n(k)})) \leq 0.$$

Having  $\alpha(x_{2n(k)}, u) \geq 1$ , so applying (G1) in the first variable, we have

$$F(\sigma(x_{2n(k)+1}, Bu), \sigma(x_{2n(k)}, u), \sigma(x_{2n(k)}, x_{2n(k)+1}), \sigma(u, Bu), \sigma(x_{2n(k)}, Bu), \sigma(u, x_{2n(k)+1})) \leq 0.$$

We know that  $\lim_{n \rightarrow \infty} \sigma(x_n, u) = 0 = \sigma(u, u)$ , so Lemma 1.6 implies that

$$\frac{1}{s}\sigma(u, Bu) \leq \limsup_{n \rightarrow \infty} \sigma(x_{2n(k)+1}, Bu) \leq s\sigma(u, Bu).$$

Letting  $k$  to infinity and using continuity of  $F$ , we have

$$F(\frac{1}{s}\sigma(u, Bu), 0, 0, \sigma(u, Bu), s\sigma(u, Bu), 0) \leq 0.$$

Using (G1) in the fifth and sixth variables,

$$F\left(\frac{1}{s}\sigma(u, Bu), 0, 0, \sigma(u, Bu), s\sigma(u, Bu), 2s\sigma(u, Bu)\right) \leq 0.$$

By (G2), it follows that  $p(u, Bu) \leq h_1(0) = 0$ , which implies that  $u = Bu$ .

Similarly, by taking  $x = u$  and  $y = x_{2n(k)-1}$  in (1.2), we have

$$F(\alpha(u, x_{2n(k)-1})\sigma(Au, Bx_{2n(k)-1}), \sigma(u, x_{2n(k)-1}), \sigma(u, Au), \sigma(x_{2n(k)-1}, Bx_{2n(k)-1}), \\ \sigma(u, Bx_{2n(k)-1}), \sigma(x_{2n(k)-1}, Au)) \leq 0.$$

By (G1) and  $\alpha(u, x_{2n(k)-1}) \geq 1$ , we get

$$F(\sigma(Au, x_{2n(k)}), \sigma(u, x_{2n(k)-1}), \sigma(u, Au), \sigma(x_{2n(k)-1}, x_{2n(k)}), \sigma(u, x_{2n(k)}), \sigma(x_{2n(k)-1}, Au)) \leq 0.$$

Again, in view of Lemma 1.6 and  $\lim_{n \rightarrow \infty} \sigma(x_{n(k)}, u) = 0 = \sigma(u, u)$ , we have

$$\frac{1}{s}\sigma(u, Au) \leq \limsup_{n \rightarrow \infty} \sigma(x_{2n(k)}, Au) \leq s\sigma(u, Au).$$

Letting  $k \rightarrow \infty$  and using the continuity of  $F$ , we have

$$F\left(\frac{1}{s}\sigma(Au, u), 0, \sigma(u, Au), 0, 0, s\sigma(u, Au)\right) \leq 0.$$

Using (G1) in the fifth and sixth variables

$$F\left(\frac{1}{s}\sigma(Au, u), 0, \sigma(u, Au), 0, 2s\sigma(u, Au), s\sigma(u, Au)\right) \leq 0.$$

By (G2), we obtain  $\sigma(u, Au) \leq h_1(0) = 0$ , that is,  $u = Au$ . □

We state the following corollaries.

**Corollary 2.10.** *Let  $(X, \sigma)$  be a complete b-metric-like space and  $A : X \rightarrow X$  a given mapping. Assume there exists  $F \in G_s$  such that*

$$F(\alpha(x, y)\sigma(Ax, Ay), \sigma(x, y), \sigma(x, Ax), \sigma(y, Ay), \sigma(x, Ay), \sigma(y, Ax)) \leq 0$$

for all  $x, y \in X$ . Suppose that

- (i)  $(A, A)$  is an  $\alpha$ -admissible pair;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Ax_0) \geq 1$  and  $\alpha(Ax_0, x_0) \geq 1$ ;
- (iii) (H) is satisfied.

Then there exists  $u \in X$  such that  $u = Au$ . We also have  $\sigma(u, u) = 0$ .

*Proof.* Taking  $B = A$  in Theorem 2.9, we get the above result. □

**Corollary 2.11.** *Let  $(X, \sigma)$  be a complete b-metric-like space and  $A, B : X \rightarrow X$  satisfying*

$$\alpha(x, y)\sigma(Ax, By) \leq k \max\{\sigma(x, y), \sigma(x, Ax), \sigma(y, By), \sigma(x, By), \sigma(y, Ax)\},$$

where  $k \in [0, \frac{1}{s^3+s^2})$ . Suppose that

- (i)  $(A, B)$  is an  $\alpha$ -admissible pair;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(Ax_0, x_0) \geq 1$  and  $\alpha(x_0, Ax_0) \geq 1$ ;
- (iii) (H) is satisfied.

Then there exists  $u \in X$  such that  $u = Au = Bu$ .

*Proof.* It suffices to consider in Theorem 2.9 the operator  $F$  given by Example 1.13 with  $k \in [0, \frac{1}{s^3+s^2})$ . Mention that  $F$  is continuous. □

**Corollary 2.12.** Let  $(X, \sigma)$  be a complete  $b$ -metric-like space and  $A : X \rightarrow X$  be a mapping such that

$$\alpha(x, y)\sigma(Ax, Ay) \leq k \max\{\sigma(x, y), \sigma(x, Ax), \sigma(y, Ay), \sigma(x, Ay), \sigma(y, Ax)\}$$

for all  $x, y \in X$ , where  $k \in [0, \frac{1}{s^3+s^2})$ . Suppose that

- (i)  $(A, A)$  is an  $\alpha$ -admissible pair;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Ax_0) \geq 1$  and  $\alpha(Ax_0, x_0) \geq 1$ ;
- (iii) (H) is satisfied.

Then there exists  $u \in X$  such that  $u = Au$ .

*Proof.* Consider in Corollary 2.10 the operator  $F$  given by Example 1.13 with  $k \in [0, \frac{1}{s^3+s^2})$ . □

The following two corollaries are Ćirić [14] type results in the setting of  $b$ -metric-like spaces.

**Corollary 2.13.** Let  $(X, \sigma)$  be a complete  $b$ -metric-like space and  $A, B : X \rightarrow X$  satisfying

$$\sigma(Ax, By) \leq k \max\{\sigma(x, y), \sigma(x, Ax), \sigma(y, By), \sigma(x, By), \sigma(y, Ax)\},$$

where  $k \in [0, \frac{1}{s^3+s^2})$ . Then there exists  $u \in X$  such that  $u = Au = Bu$ .

*Proof.* It suffices to take  $\alpha(x, y) = 1$  in Corollary 2.11. □

**Corollary 2.14.** Let  $(X, \sigma)$  be a complete  $b$ -metric-like space and  $A : X \rightarrow X$  be a mapping such that

$$\sigma(Ax, Ay) \leq k \max\{\sigma(x, y), \sigma(x, Ax), \sigma(y, Ay), \sigma(x, Ay), \sigma(y, Ax)\},$$

where  $k \in [0, \frac{1}{s^3+s^2})$ . Then there exists  $u \in X$  such that  $u = Au$ .

*Proof.* The proof follows easily when taking  $\alpha(x, y) = 1$  in Corollary 2.12. □

To prove uniqueness of the common fixed point given in Theorem 2.1 (resp. Theorem 2.9), we need to take the following additional hypotheses.

(U) For all  $x, y \in CF(A, B)$ , we have  $\alpha(x, y) \geq 1$ , where  $CF(A, B)$  denotes the set of common fixed points of  $A$  and  $B$ , and

(F3) For all  $t > 0$ , we have  $F(t, t, 0, 0, t, t) > 0$ .

**Theorem 2.15.** Adding conditions (U) and (F3) to the hypotheses of Theorem 2.1 (resp. Theorem 2.9), we obtain that  $u$  is the unique common fixed point of  $A$  and  $B$ .

*Proof.* We argue by contradiction, that is, there exist  $u, v \in X$  such that  $u = Au = Bu$  and  $v = Av = Bv$  with  $u \neq v$ . By Theorem 2.1 (resp. Theorem 2.9), such  $u$  and  $v$  satisfy  $\sigma(u, u) = \sigma(v, v) = 0$ . From (1.2), we get

$$F(\alpha(u, v)\sigma(Au, Av), \sigma(u, v), \sigma(u, Au), \sigma(v, Bv), \sigma(u, Bv), \sigma(v, Au)) \leq 0,$$

i.e,

$$F(\alpha(u, v)\sigma(u, v), \sigma(u, v), \sigma(u, u), \sigma(v, v), \sigma(u, v), \sigma(v, u)) \leq 0.$$

Since  $\alpha(u, v) \geq 1$ , by (F1)=(G1) in the first variable, we get

$$F(\sigma(u, v), \sigma(u, v), 0, 0, \sigma(u, v), \sigma(u, v)) \leq 0.$$

Again  $F$  satisfies property (F3), so it is a contradiction. Hence  $u = v$ . □

**Theorem 2.16.** Adding conditions (U) and (F3) to the hypotheses of Corollary 2.2 (resp. Corollary 2.10), we obtain that  $u$  is the unique fixed point of  $A$ .

The following examples illustrate Theorem 2.15 where the considered mappings  $A$  and  $B$  have a unique common fixed point.

**Example 2.17.** Take  $X = [0, \infty)$  endowed with the complete b-metric-like  $\sigma(x, y) = (x + y)^2$  with  $s = 2$ . Consider the mappings  $A, B : X \rightarrow X$  given by

$$Ax = \begin{cases} \frac{x}{9}, & \text{if } x \in [0, 1], \\ (x - 1)^2 + \frac{1}{9}, & \text{if } x > 1, \end{cases} \quad \text{and} \quad Bx = \begin{cases} \frac{x}{9}, & \text{if } x \in [0, 1], \\ \frac{2x^2 - 1}{9}, & \text{if } x > 1. \end{cases}$$

Define the mapping  $\alpha : X \times X \rightarrow [0, \infty)$  by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Let  $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{1}{10}(t_5 + t_6)$ . It is obvious that

- (i)  $(A, B)$  is an  $\alpha$ -admissible pair;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Ax_0) \geq 1$  and  $\alpha(Ax_0, x_0) \geq 1$ ;
- (iii)  $F(u, 0, v, w, 2s^2u, 2s^2v) \leq 0$  for all  $u, v, w \geq 0$ .

Moreover,  $A$  and  $B$  are sequentially continuous on  $(X, \sigma)$ . Indeed, let  $\{x_n\}$  be a sequence in  $X$  such that  $x_n \rightarrow x$  in  $(X, \sigma)$  as  $n \rightarrow \infty$ . It follows that

$$\sigma(x_n, x) = (x_n + x)^2 \rightarrow \sigma(x, x) = 4x^2 \Leftrightarrow x_n + x \rightarrow 2x \Leftrightarrow |x_n - x| \rightarrow 0.$$

$A$  is continuous on  $(X, |\cdot|)$ , so  $|Ax_n - Ax| \rightarrow 0$  as  $n \rightarrow \infty$ . Then

$$\sigma(Ax_n, Ax) \rightarrow \sigma(Ax, Ax).$$

Thus  $A$  is sequentially continuous on  $(X, \sigma)$ . Similarly,  $B$  is sequentially continuous on  $(X, \sigma)$ .

Now, we shall show that  $(A, B)$  is an  $\alpha$ -implicit contractive pair. Let  $x, y \in X$  such that  $\alpha(x, y) = 1$ . So,  $x, y \in [0, 1]$ . We have

$$\begin{aligned} F(\alpha(x, y)\sigma(Ax, By), \sigma(x, y), \sigma(x, Ax), \sigma(y, By), \sigma(x, By), \sigma(y, Ax)) &= \sigma(Ax, By) - \frac{1}{9}(\sigma(x, By) + \sigma(y, Ax)) \\ &= \left(\frac{x}{9} + \frac{y}{9}\right)^2 - \frac{1}{10} \left( \left(x + \frac{y}{9}\right)^2 + \left(y + \frac{x}{9}\right)^2 \right) \\ &= -\frac{1}{810}(71x^2 + 71y^2 + 16xy) \leq 0. \end{aligned}$$

Also, if  $x, y \in X$  such that  $\alpha(x, y) = 0$ , then

$$\begin{aligned} &F(\alpha(x, y)\sigma(Ax, By), \sigma(x, y), \sigma(x, Ax), \sigma(y, By), \sigma(x, By), \sigma(y, Ax)) \\ &= F(0, \sigma(x, y), \sigma(x, Ax), \sigma(y, By), \sigma(x, By), \sigma(y, Ax)) = -\frac{1}{9}(\sigma(x, By) + \sigma(y, Ax)) \leq 0. \end{aligned}$$

All hypotheses of Theorem 2.1 hold and  $0$  is the unique common fixed point of  $A$  and  $B$ .

**Example 2.18.** Take  $X = [0, 2]$  endowed with the b-metric-like  $\sigma(x, y) = x^2 + y^2 + (x - y)^2$ . Consider the mappings  $A, B : X \rightarrow X$  given by

$$Ax = \begin{cases} \frac{x^2}{10(1+x)}, & \text{if } x \in [0, 1], \\ \frac{1}{20} + \ln x, & \text{if } x \in (1, 2], \end{cases} \quad \text{and} \quad Bx = \begin{cases} \frac{x^2}{10(1+x)}, & \text{if } x \in [0, 1], \\ 1, & \text{if } x \in (1, 2]. \end{cases}$$

Define the mapping  $\alpha : X \times X \rightarrow [0, \infty)$  by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Note that  $(X, \sigma)$  is a complete b-metric-like space. Also,  $(A, B)$  is an  $\alpha$ -admissible pair. In fact, let  $x, y \in X$  such that  $\alpha(x, y) \geq 1$ , so  $x, y \in [0, 1]$ . In this case

$$\alpha(Ax, By) = \alpha\left(\frac{x^2}{10(1+x)}, \frac{y^2}{10(1+y)}\right) = 1 \quad \text{and} \quad \alpha(By, Ax) = \alpha\left(\frac{y^2}{10(1+y)}, \frac{x^2}{10(1+x)}\right) = 1.$$

Then  $(A, B)$  is an  $\alpha$ -admissible pair.

Now, we shall show that  $(A, B)$  is an  $\alpha$ -implicit contractive pair. Let  $x, y \in X$  such that  $\alpha(x, y) = 1$ . So  $x, y \in [0, 1]$ . We have

$$\begin{aligned} \alpha(x, y)\sigma(Ax, By) &= \sigma(Ax, By) = (Ax)^2 + (By)^2 + (Ax - By)^2 \\ &= \left(\frac{x^2}{10(1+x)}\right)^2 + \left(\frac{y^2}{10(1+y)}\right)^2 + \left(\frac{x^2}{10(1+x)} - \frac{y^2}{10(1+y)}\right)^2 \\ &\leq \left(\frac{1}{20}\right)^2(x^2 + y^2) + \left(\frac{9}{100}\right)^2(x - y)^2 \\ &\leq \left(\frac{9}{100}\right)^2\sigma(x, y). \end{aligned}$$

Also, if  $x, y \in X$  such that  $\alpha(x, y) = 0$ , then

$$\alpha(x, y)\sigma(Ax, By) = 0 \leq \left(\frac{9}{100}\right)^2\sigma(x, y).$$

Now, we show that (H) is verified. Let  $\{x_n\}$  be a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$ ,  $\alpha(x_{n+1}, x_{n+2}) \geq 1$  for all  $n$  and  $x_n \rightarrow u \in (X, \sigma)$  as  $n \rightarrow \infty$ . Then,  $\{x_n\} \subset [0, 1]$  and it is easy to see that  $x_n \rightarrow u \in (X, |\cdot|)$  as  $n \rightarrow \infty$ . Consequently,  $u \in [0, 1]$  and so  $\alpha(x_n, u) = \alpha(u, x_n) = 1$  for all  $n$ . Thus, all hypotheses of corollary 2.11 are verified and 0 is the unique common fixed point of  $A$  and  $B$ .

**Example 2.19.** Let  $X = \{0, 1, 2\}$  and  $\sigma : X \times X \rightarrow [0, \infty)$  be defined by

$$\sigma(0, 0) = 0, \sigma(1, 1) = 4, \sigma(2, 2) = 64, \sigma(0, 1) = \sigma(1, 0) = 1, \sigma(0, 2) = \sigma(2, 0) = 16 \quad \text{and} \quad \sigma(1, 2) = \sigma(2, 1) = 25.$$

Then,  $(X, \sigma)$  is a complete b-metric-like space with coefficient  $s = 2$ , but neither a b-metric, nor a metric-like since  $\sigma(1, 2) = 25 > 17 = \sigma(1, 0) + \sigma(0, 2)$ .

Define  $A, B : X \rightarrow X$  such that

$$A0 = 0, \quad A1 = A2 = 1, \quad B0 = B1 = 0, \quad \text{and} \quad B2 = 1.$$

Take  $\alpha : X \times X \rightarrow [0, \infty)$  defined by

$$\begin{cases} \alpha(0, 0) = \alpha(0, 1) = 1, \\ 0 \quad \text{otherwise.} \end{cases}$$

Note that  $(A, B)$  is an  $\alpha$ -admissible pair. Indeed, let  $x, y \in X$  such that  $\alpha(x, y) \geq 1$ , then  $(x = y = 0)$  or  $(x = 0, y = 1)$ . It follows that

$$\alpha(A0, B0) = \alpha(B0, A0) = \alpha(0, 0) = 1 \quad \text{and} \quad \alpha(A0, B1) = \alpha(B1, A0) = \alpha(0, 0) = 1.$$

Now, we shall show that the contraction condition (2.13) of Corollary 2.4 is verified for all  $k \in [0, \frac{1}{8})$ . Let  $x, y \in X$  such that  $\alpha(x, y) = 1$ , then  $(x = y = 0)$  or  $(x = 0, y = 1)$ . We have

$$\alpha(0, 0)\sigma(A0, B0) = \sigma(A0, B0) = \sigma(0, 0) = 0 \leq k\sigma(0, 0).$$

Also,

$$\alpha(0, 1)\sigma(A0, B1) = \sigma(0, 0) = 0 \leq k\sigma(0, 1).$$

Moreover, for all  $x, y \in X$  such that  $\alpha(x, y) = 0$ , we have

$$\alpha(x, y)\sigma(Ax, By) = 0 \leq k\sigma(x, y).$$

The condition contraction (2.13) holds.

For  $x_0 = 0$ , we have  $\alpha(Ax_0, x_0) \geq 1$  and  $\alpha(x_0, Ax_0) \geq 1$ . On the other hand,  $A$  and  $B$  are sequentially continuous on  $(X, \sigma)$ . In fact, if  $\{x_n\}$  is a sequence such that  $x_n \rightarrow x$  in  $(X, \sigma)$ , it is easy to see that there exists  $N \in \mathbb{N}$  such that  $x_n = x$  for all  $n \geq N$  and so  $Ax_n = Ax$  for all  $n \geq N$ . It follows that  $\sigma(Ax_n, Ax) = \sigma(Ax, Ax)$  for all  $n \geq N$ . Thus  $Ax_n \rightarrow Ax$  in  $(X, \sigma)$ , so  $A$  is sequentially continuous on  $(X, \sigma)$ . Similarly,  $B$  is sequentially continuous on  $(X, \sigma)$ . Therefore, all the hypothesis of Corollary 2.4 are satisfied. Here,  $0$  is the unique common fixed point of  $A$  and  $B$ .

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