



Uniqueness results for nonlinear fractional differential equations with infinite-point integral boundary conditions

Suli Liu^a, Junpeng Liu^a, Qun Dai^b, Huilai Li^{a,*}

^aDepartment of Mathematics, Jilin University, Changchun, 130012, P. R. China.

^bCollege of Science, Changchun University of Science and Technology, Changchun, 130022, P. R. China.

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Abstract

In this paper, we consider a class of nonlinear fractional differential equations involving the Riemann-Liouville fractional derivative with infinite-point integral boundary conditions. Our analysis relies on the fixed point index theory and u_0 -positive operator. An example is given for the illustration of the main work. ©2017 All rights reserved.

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1. Introduction

Fractional calculus developed since 17th century through the pioneering works of Leibniz, Euler, Lagrange, Liouville and many other researchers and has been investigated in diverse. The recent development covers the theoretical as well as potential applications of the subject in physical and technical science. Recently, Atangana and Baleanu proposed a derivative with fractional order based upon the Mittag-Leffler function which has a non-singular and nonlocal kernel, see [2, 3] and the references therein. Fractional differential equations have been of great interest recently, see [6, 10, 11]. There are many results dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by the means of techniques of nonlinear analysis, see [1, 4, 8, 9, 12].

In [13], Zhang studied the existence of positive solutions of the following nonlinear fractional differential equations with infinite-point boundary value conditions:

$$\begin{cases} D_{0+}^{\alpha} u(t) + q(t)f(t, u(t)) = 0, & 0 < t < 1, \quad n-1 < \alpha \leq n, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases}$$

*Corresponding author

Email addresses: lius115@mails.jlu.edu.cn (Suli Liu), liujp14@mails.jlu.edu.cn (Junpeng Liu), daiqun1130@163.com (Qun Dai), lihuilai@jlu.edu.cn (Huilai Li)

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where $\alpha > 2$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \dots < \xi_{j-1} < \xi_j < \dots < 1$ ($j = 1, 2, \dots$), $\Delta - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} > 0$, $\Delta = (\alpha-1)(\alpha-2)\dots(\alpha-i)$, and D_{0+}^{α} is the standard Riemann-Liouville derivative. By using the fixed point theorem, several local existence and multiplicity of positive solutions were obtained.

Motivated by the work mentioned above, in this paper, we study the following nonlinear fractional differential equations with infinite-point integral boundary value conditions:

$$\begin{cases} D_{0+}^{\alpha} u(t) + q(t)f(t, u(t)) = 0, & 0 < t < 1, \quad n-1 < \alpha \leq n, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j I_{0+}^{\beta} u(\xi_j), \end{cases} \quad (1.1)$$

where $\alpha > 2$, $\beta \geq 0$, $i \in [0, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \dots < \xi_{j-1} < \xi_j < \dots < 1$ ($j = 1, 2, \dots$), $\Delta - \sum_{j=1}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \alpha_j \xi_j^{\alpha+\beta-1} > 0$ (Δ will be defined in Lemma 2.3 below), and D_{0+}^{α} is the standard Riemann-Liouville differential operator.

Here, we emphasize that the infinite-point integral boundary value condition of (1.1) can be understood in the sense that the value of the unknown function $u^{(i)}(t)$ at the position $t = 1$ is proportional to the infinite-point Riemann-Liouville integral of the unknown function $\sum_{j=1}^{\infty} \alpha_j I_{0+}^{\beta} u(\xi_j)$. Furthermore, for $\beta = 0$, the integral boundary value condition of (1.1) reduces to the usual form of infinite-point boundary value condition $u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j)$, which has been considered in [13].

The rest of the paper is organized as follows. In Section 2, we list some lemmas. The uniqueness results are given in Section 3. In Section 4, an example is presented to illustrate the main results.

2. Preliminaries

We list the following assumptions adopted in this paper:

- (A₁) $q : (0, 1) \rightarrow [0, +\infty)$ is continuous and $0 < \int_0^1 p(s)ds < +\infty$;
 (A₂) $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

For the convenience of the reader, we present here some necessary definitions of the fractional calculus which can be found in the recent literature [6, 10, 11].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where $n-1 \leq \alpha < n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.3. Assume that $y(t) \in C([0, 1])$, then the problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & 0 < t < 1, \quad n-1 < \alpha \leq n, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j I_{0+}^{\beta} u(\xi_j), \end{cases} \quad (2.1)$$

where $\Delta - \sum_{j=1}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \alpha_j \xi_j^{\alpha+\beta-1} > 0$, $i \in [0, n-2]$ is a fixed integer, is equivalent to

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.2)$$

with $p(s) := \Delta - \sum_{s \leq \xi_j} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \alpha_j (\xi_j - s)^{\alpha+\beta-1} (1-s)^{i+1-\alpha}$ and

$$\Delta = \begin{cases} 1, & i = 0, \\ (\alpha-1)(\alpha-2) \cdots (\alpha-i), & i \geq 1. \end{cases}$$

$G(t,s)$ is called Green's function of FBVP (2.1). Obviously, $G(t,s)$ is a continuous function on $[0,1] \times [0,1]$.

Proof. It is easy to see that problem (2.1) is equivalent to the following integral equation

$$u(t) = -I_{0+}^{\alpha} y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}.$$

By $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we have

$$C_2 = C_3 = \cdots = C_n = 0.$$

Then, we get

$$u(t) = -I_{0+}^{\alpha} y(t) + C_1 t^{\alpha-1}.$$

By $u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j I_{0+}^{\beta} u(\xi_j)$, we have

$$-I_{0+}^{\alpha-i} y(1) + C_1 \Delta = \sum_{j=1}^{\infty} \alpha_j \left\{ -I_{0+}^{\alpha+\beta} y(\xi_j) + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \xi_j^{\alpha+\beta-1} \right\}.$$

When $\Delta - \sum_{j=1}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \alpha_j \xi_j^{\alpha+\beta-1} \neq 0$, we obtain that

$$C_1 = \frac{1}{\Delta - \sum_{j=1}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \alpha_j \xi_j^{\alpha+\beta-1}} \left\{ I_{0+}^{\alpha-i} y(1) - \sum_{j=1}^{\infty} \alpha_j I_{0+}^{\alpha+\beta} y(\xi_j) \right\}.$$

Let $p(s) := \Delta - \sum_{s \leq \xi_j} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \alpha_j (\xi_j - s)^{\alpha+\beta-1} (1-s)^{i+1-\alpha}$, then $p(0) = \Delta - \sum_{j=1}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \alpha_j \xi_j^{\alpha+\beta-1}$. Therefore, the solution to problem (2.1) is

$$\begin{aligned} u(t) &= -I_{0+}^{\alpha} y(t) + \frac{t^{\alpha-1}}{p(0)} \left\{ I_{0+}^{\alpha-i} y(1) - \sum_{j=1}^{\infty} \alpha_j I_{0+}^{\alpha+\beta} y(\xi_j) \right\} \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \\ &\quad + \frac{t^{\alpha-1}}{p(0)} \left\{ \frac{1}{\Gamma(\alpha-i)} \int_0^1 (1-s)^{\alpha-i-1} y(s) ds - \sum_{j=1}^{\infty} \frac{\alpha_j}{\Gamma(\alpha+\beta)} \int_0^{\xi_j} (\xi_j - s)^{\alpha+\beta-1} y(s) ds \right\} \\ &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ -p(0) \int_0^t (t-s)^{\alpha-1} y(s) ds + \Delta t^{\alpha-1} \left\{ \int_0^t + \int_t^1 \right\} (1-s)^{\alpha-i-1} y(s) ds \right\} \end{aligned}$$

$$\left. - \sum_{j=1}^{\infty} \frac{\alpha_j \Gamma(\alpha)}{\Gamma(\alpha + \beta)} t^{\alpha-1} \int_0^{\xi_j} (\xi_j - s)^{\alpha+\beta-1} y(s) ds \right\}$$

$$= \int_0^1 G(t, s) y(s) ds.$$

The proof is finished. \square

Lemma 2.4. Suppose that $p(0) > 0$, then $p(s) > 0$, $s \in [0, 1]$, and $p(s)$ is nondecreasing on $[0, 1]$.

Proof. By simple computation, we know that $p'(s) > 0$ which implies that $p(s)$ is nondecreasing on $[0, 1]$. We omit the details. \square

It is clear that $p(s) - p(0) > 0$, $\forall s \in (0, 1]$, and $\frac{p(s)-p(0)}{s}$ is continuous for all $s \in (0, 1]$. Then, there exists $M_1 \geq m_1 > 0$, such that $m_1 s + p(0) \leq p(s) \leq M_1 s + p(0)$, where

$$m_1 := \inf_{0 < s \leq 1} \frac{p(s) - p(0)}{s}, \quad M_1 := \sup_{0 < s \leq 1} \frac{p(s) - p(0)}{s}.$$

Lemma 2.5. The function $G(t, s)$ defined by (2.2) satisfies the following conditions:

- (i) $p(0)\Gamma(\alpha)G(t, s) \geq m_1 s(1-s)^{\alpha-1-i} t^{\alpha-1}$, $\forall t, s \in [0, 1]$;
- (ii) $p(0)\Gamma(\alpha)G(t, s) \leq [M_1 + p(0)](1-s)^{\alpha-1-i} t^{\alpha-1}$, $\forall t, s \in [0, 1]$;
- (iii) $G(t, s) > 0$, $\forall t, s \in (0, 1)$.

Proof. For $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned} p(0)\Gamma(\alpha)G(t, s) &= t^{\alpha-1} p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1} \\ &= [p(s) - p(0)](1-s)^{\alpha-1-i} t^{\alpha-1} + p(0)[(1-s)^{\alpha-1-i} t^{\alpha-1} - (t-s)^{\alpha-1}] \\ &\geq m_1 s(1-s)^{\alpha-1-i} t^{\alpha-1} + p(0)[(1-s)^{\alpha-1-i} t^{\alpha-1} - (t-s)^{\alpha-1}] \\ &\geq m_1 s(1-s)^{\alpha-1-i} t^{\alpha-1}, \end{aligned}$$

and

$$\begin{aligned} p(0)\Gamma(\alpha)G(t, s) &= t^{\alpha-1} p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1} \\ &= [p(s) - p(0)](1-s)^{\alpha-1-i} t^{\alpha-1} + p(0)[(1-s)^{\alpha-1-i} t^{\alpha-1} - (t-s)^{\alpha-1}] \\ &\leq M_1 s(1-s)^{\alpha-1-i} t^{\alpha-1} + p(0)(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\leq [M_1 + p(0)](1-s)^{\alpha-1-i} t^{\alpha-1}. \end{aligned}$$

For $0 \leq t \leq s \leq 1$, we have

$$p(0)\Gamma(\alpha)G(t, s) = t^{\alpha-1} p(s)(1-s)^{\alpha-1-i} \geq [p(s) - p(0)](1-s)^{\alpha-1-i} t^{\alpha-1} \geq m_1 s(1-s)^{\alpha-1-i} t^{\alpha-1},$$

and

$$\begin{aligned} p(0)\Gamma(\alpha)G(t, s) &= t^{\alpha-1} p(s)(1-s)^{\alpha-1-i} \\ &= [p(s) - p(0)](1-s)^{\alpha-1-i} t^{\alpha-1} + p(0)(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\leq M_1 s(1-s)^{\alpha-1-i} t^{\alpha-1} + p(0)(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\leq [M_1 + p(0)](1-s)^{\alpha-1-i} t^{\alpha-1}. \end{aligned}$$

The proof is finished. \square

Let Banach space $E = C([0, 1])$ be endowed with the norm $\|u\|_\infty = \max_{0 \leq t \leq 1} |u(t)|$, and θ is the zero function in E . Define a closed cone $P \subset E$ by $P = \{u \in E \mid u(t) \geq 0, t \in [0, 1]\}$.

Lemma 2.6 (Krein-Rutmann theorem [5]). *Suppose that $T : E \rightarrow E$ is a completely continuous linear operator and $T(P) \subseteq P$. If there exist $\psi \in C[0, 1] \setminus (-P)$ and a constant $c > 0$ such that $cT\psi \geq \psi$, then the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Definition 2.7 ([7]). We say that a bounded linear operator $T : E \rightarrow E$ is u_0 -positive on the cone P if there exists $u_0 \in P \setminus \{\theta\}$ such that for every $x \in P \setminus \{\theta\}$ there exist a natural number n and positive functions $\alpha(x) > 0$, $\beta(x) > 0$ such that

$$\alpha(x)u_0 \leq T^n x \leq \beta(x)u_0,$$

where θ is the zero function in E . Furthermore, if $u_0 = \varphi_1$, the positive eigenfunction of T corresponding to its first eigenvalue λ_1 , then T is a φ_1 -positive operator.

Define the operator $A : E \rightarrow E$ by

$$(Au)(t) = \int_0^1 G(t, s)q(s)f(s, u(s))ds, \quad t \in [0, 1].$$

It is not hard to see that the fixed points of operator A coincide with the solutions to the FBVP (1.1).

Define the operator $T : E \rightarrow E$ by

$$(Tu)(t) = \int_0^1 G(t, s)q(s)u(s)ds, \quad t \in [0, 1]. \quad (2.3)$$

It is not difficult to verify that $T : P \rightarrow P$ is a completely continuous linear operator. By virtue of the Krein-Rutmann theorem, we have the following lemma.

Lemma 2.8. *Suppose T is defined by (2.3), then the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Proof. By Lemma 2.5, $G(t, s) > 0$ for all $t, s \in (0, 1)$. Take $[t_1, t_2] \subset (0, 1)$, $q(t) > 0, \forall t \in [t_1, t_2]$, choose $\psi \in E$ such that $\psi(t) \geq 0$ for all $t \in [0, 1]$, $\psi(t) > 0$ for all $t \in [t_1, t_2]$, $\psi(t) = 0$ for all $t \in [0, t_1) \cup (t_2, 1]$. Thus, we have

$$(T\psi)(t) = \int_0^1 G(t, s)q(s)\psi(s)ds = \int_{t_1}^{t_2} G(t, s)q(s)\psi(s)ds > 0.$$

So, there exists a constant $c > 0$ such that $c(T\psi)(t) \geq \psi(t), \forall t \in [0, 1]$. By Lemma 2.6, we complete the proof. \square

Lemma 2.9. *T is u_0 -positive operator with $u_0(t) = t^{\alpha-1}$. In addition, T is φ_1 -positive operator, where φ_1 is the positive eigenfunction corresponding to its first eigenvalue.*

Proof. For any $x \in P \setminus \{\theta\}$, by Lemma 2.5, we have

$$(Tx)(t) = \int_0^1 G(t, s)q(s)x(s)ds \leq \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 [M_1 + p(0)](1-s)^{\alpha-1-i}q(s)x(s)ds \cdot t^{\alpha-1}.$$

On the other hand, we have

$$(Tx)(t) = \int_0^1 G(t, s)q(s)x(s)ds \geq \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 m_1 s(1-s)^{\alpha-1-i}q(s)x(s)ds \cdot t^{\alpha-1}.$$

Therefore, T is u_0 -positive operator with $u_0(t) = t^{\alpha-1}$, i.e.,

$$\alpha(x)u_0 \leq Tx \leq \beta(x)u_0, \quad \forall x \in P \setminus \{\theta\}.$$

Let φ_1 be the positive eigenfunction of T corresponding to λ_1 , i.e., $\varphi_1 = \lambda_1 T\varphi_1$. Then, there exist $\tilde{\alpha}(\varphi_1), \tilde{\beta}(\varphi_1) > 0$ such that

$$\tilde{\alpha}(\varphi_1)u_0 \leq T\varphi_1 = \frac{1}{\lambda_1}\varphi_1 \leq \tilde{\beta}(\varphi_1)u_0.$$

Hence, we obtain that T is φ_1 -positive operator. The proof is completed. \square

3. Main results

Theorem 3.1. *Suppose that there exists $K \in [0, \lambda_1)$ such that*

$$|f(t, u) - f(t, v)| \leq K|u - v|, \quad \forall u, v \in E,$$

where λ_1 is the first eigenvalue of T . Then the FBVP (1.1) has a unique solution u^* in E , moreover, for any $u_0 \in E$, there exists iterative sequence $\{u_n\}_{n=0}^{\infty}$ with

$$u_{n+1} = Au_n, \quad \lim_{n \rightarrow \infty} u_n = u^*, \quad n = 0, 1, 2, \dots$$

Proof. Firstly, it is not hard to see that the fixed points of operator A coincide with the solutions to the problem (1.1).

Secondly, we will show that A has fixed points in E . For any given $u_0 \in E$, let $u_{n+1} = Au_n$. By Lemma 2.9, there exists $\beta = \beta(|u_1 - u_0|) > 0$, such that

$$(T|u_1 - u_0|)(t) \leq \beta\varphi_1(t), \quad t \in [0, 1].$$

Let $K = k\lambda_1$, $k \in [0, 1)$, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} |u_{m+1} - u_m| &= |(Au_m)(t) - (Au_{m-1})(t)| \\ &= \left| \int_0^1 G(t, s)q(s)[f(s, u_m(s)) - f(s, u_{m-1}(s))]ds \right| \\ &\leq \int_0^1 G(t, s)q(s)|f(s, u_m(s)) - f(s, u_{m-1}(s))|ds \\ &\leq k\lambda_1 \int_0^1 G(t, s)q(s)|u_m - u_{m-1}|ds \\ &= k\lambda_1 T(|u_m - u_{m-1}|)(t) \\ &\leq \dots \leq k^m \lambda_1^m T^m(|u_1 - u_0|)(t) \\ &\leq k^m \lambda_1^m T^{m-1} \beta \varphi_1(t) \\ &= \beta k^m \lambda_1^{m-1} T^{m-2} \varphi_1(t) = \beta k^m \lambda_1 \varphi_1(t). \end{aligned}$$

Thus, for $n, m \in \mathbb{N}$, we have

$$\begin{aligned} |u_{n+m+1} - u_n| &= |u_{n+m+1} - u_{n+m} + \dots + u_{n+1} - u_n| \\ &\leq |u_{n+m+1} - u_{n+m}| + \dots + |u_{n+1} - u_n| \\ &\leq \beta[k^{n+m} + \dots + k^n]\lambda_1 \varphi_1(t) \\ &= \beta\lambda_1 \frac{k^n(1 - k^{m+1})}{1 - k} \varphi_1(t). \end{aligned}$$

Therefore,

$$0 \leq \|u_{n+m+1} - u_n\| \leq \beta\lambda_1 \frac{k^n(1 - k^{m+1})}{1 - k} \|\varphi_1(t)\| \rightarrow 0, \quad \text{as } n, m \rightarrow \infty.$$

By the completeness of E , there exists a $u^* \in E$ such that $\lim_{n \rightarrow \infty} u_n = u^*$.

Thus, $u^* = \lim_{n \rightarrow \infty} u_{n+1} = \lim_{n \rightarrow \infty} Au_n = Au^*$, A have fixed points in E .

Finally, we will show that A has at most one fixed point in E . Suppose there exist two fixed points $u, v \in E$, $u = Au, v = Av$. By Lemma 2.9, there exists $\beta = \beta(|u - v|) > 0$, such that

$$(\mathbb{T}|u - v|)(t) \leq \beta \varphi_1(t), \quad t \in [0, 1].$$

Then for all $n \in \mathbb{N}$, the following holds

$$|u - v| = |A^n u - A^n v| \leq \beta k^n \lambda_1 \varphi_1.$$

This means that $u = v$, and A has at most one fixed point in E . The proof is completed. □

Remark 3.2. The iterative sequence in Theorem 3.1 starting with a simple function is helpful for calculating.

By the same method of [4], we have the following results. We omit the details.

Theorem 3.3. *Suppose that there exist $u_0 \in E, K \in [0, \lambda_1)$ such that*

$$\begin{cases} D_{0+}^\alpha u_0(t) + q(t)f(t, u_0(t)) \geq 0, & 0 < t < 1, \quad n - 1 < \alpha \leq n, \\ u_0(0) = u_0'(0) = \dots = u_0^{(n-2)}(0) = 0, \quad u_0^{(i)}(1) \leq \sum_{j=1}^\infty \alpha_j I_{0+}^\beta u_0(\xi_j), \\ 0 \leq f(t, u) - f(t, v) \leq K(u - v), \quad u(t) \geq v(t), \quad \forall t \in [0, 1], \quad u, v \in \Omega, \end{cases}$$

where λ_1 is the first eigenvalue of \mathbb{T} , $\Omega = \{u \in E | u \geq u_0\}$. Then the FBVP (1.1) has a unique solution u^* in Ω .

Theorem 3.4. *Suppose that there exist $u_0 \in E, K \in [0, \lambda_1)$ such that*

$$\begin{cases} D_{0+}^\alpha u_0(t) + q(t)f(t, u_0(t)) \leq 0, & 0 < t < 1, \quad n - 1 < \alpha \leq n, \\ u_0(0) = u_0'(0) = \dots = u_0^{(n-2)}(0) = 0, \quad u_0^{(i)}(1) \geq \sum_{j=1}^\infty \alpha_j I_{0+}^\beta u_0(\xi_j), \\ 0 \leq f(t, u) - f(t, v) \leq K(u - v), \quad u(t) \geq v(t), \quad \forall t \in [0, 1], \quad u, v \in \Omega, \end{cases}$$

where λ_1 is the first eigenvalue of \mathbb{T} , $\Omega = \{u \in E | u \leq u_0\}$. Then the FBVP (1.1) has a unique solution u^* in Ω .

4. An example

Example 4.1. Consider the following problem

$$\begin{cases} D_{0+}^{7/2} u(t) + \lambda_0(1 - t)^{-1/2} [\frac{1}{3}u + 2 + 5t^3 + \sin t + \frac{1}{2}t^2u] = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \quad u'(1) = \sum_{j=1}^\infty \frac{8}{j^2} I_{0+}^{3/2} u \left(\frac{1}{2j} \right), \end{cases} \tag{4.1}$$

where $\alpha = \frac{7}{2}$, $i = 1$, $\alpha_j = \frac{8}{j^2}$, $\beta = \frac{3}{2}$, $\xi_j = \frac{1}{2j}$, $\Gamma(\frac{7}{2}) \approx 3.32335$, $\Delta = 2.5$, $p(0) = \Delta - \sum_{j=1}^\infty \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta)} \alpha_j \xi_j^{\alpha + \beta - 1} \approx 2.38611 > 0$, and $q(t) = (1 - t)^{-1/2}$, $f(t, u(t)) = \lambda_0[\frac{1}{3}u + 2 + 5t^3 + \sin t + \frac{1}{2}t^2u]$, $0 < \lambda_0 < \lambda_1$, λ_1 is the first eigenvalue of the operator \mathbb{T} .

It is clear that (A_1) , (A_2) hold. For all $u, v \in E$, we have

$$|f(t, u(t)) - f(t, v(t))| \leq \lambda_0 \left| \left[\frac{1}{3}(u - v) + \frac{1}{2}t^2(u - v) \right] \right| \leq \lambda_1 \frac{5}{6} |u - v|.$$

It follows from Theorem 3.1 that FBVP (4.1) has a unique solution, moreover, for any $u_0 \in E$, there exists iterative sequence $\{u_n\}_{n=0}^\infty$ with

$$u_{n+1} = Au_n, \quad \lim_{n \rightarrow \infty} u_n = u^*, \quad n = 0, 1, 2, \dots$$

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