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Stability of additive-quadratic ρ-functional equations in Banach spaces: a fixed point approach

Choonkil Park[a], Sang Og Kim[b,*], Cihangir Alaca[c]

[a]Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea.
[b]Department of Mathematics, Hallym University, Chuncheon 24252, Republic of Korea.
[c]Department of Mathematics, Celal Bayar University, Muradiye Campus 45140 Manisa, Turkey.

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Abstract

Let

$$M_1f(x,y) := \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) + \frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y),$$
$$M_2f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

We solve the additive-quadratic ρ-functional equations

$$M_1f(x,y) = \rho M_2f(x,y), \tag{1}$$

and

$$M_2f(x,y) = \rho M_1f(x,y), \tag{2}$$

where ρ is a fixed nonzero number with $\rho \neq 1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equations (1) and (2) in Banach spaces.

Keywords: Hyers-Ulam stability, additive-quadratic ρ-functional equation, fixed point method, Banach space.
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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [10] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [6] gave a first affirmative partial

*Corresponding author
Email addresses: baak@hanyang.ac.kr (Choonkil Park), sokim@hallym.ac.kr (Sang Og Kim), cihangiralaca@yahoo.com.tr (Cihangir Alaca)

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answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The stability of quadratic functional equation was proved by Skof [9] for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

We recall a fundamental result in fixed point theory.

Theorem 1.1 ([2, 4]). *Let (X, d) be a complete generalized metric space and let $J: X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty,$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leqslant \frac{1}{1-\alpha} d(y, Jy)$, *for all $y \in Y$.*

In Section 2, we solve the additive-quadratic functional equation (1) and prove the Hyers-Ulam stability of the additive-quadratic functional equation (1) in Banach spaces.

In Section 3, we solve the additive-quadratic ρ-functional equation (2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (2) in Banach spaces.

Throughout this paper, assume that X is a normed space and that Y is a Banach space. Let ρ be a nonzero number with $\rho \neq 1$.

2. Additive-quadratic ρ-functional equation (1) in Banach spaces

We solve and investigate the additive-quadratic ρ-functional equation (1) in normed spaces.

Lemma 2.1.

(i) *If a mapping $f: X \to Y$ satisfies $M_1 f(x,y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

(ii) *If a mapping $f: X \to Y$ satisfies $M_2 f(x,y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

Proof.

(i)

$$M_1 f_o(x,y) = f_o(x+y) - f_o(x) - f_o(y) = 0,$$

for all $x, y \in X$. So f_o is the Cauchy additive mapping.

$$M_1 f_e(x,y) = \frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y) = 0,$$

for all $x, y \in X$. So f_o is the quadratic mapping.

(ii)
$$M_2f_o(x,y) = 2f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y) = 0,$$
for all $x, y \in X$. Since $M_2f(0,0) = 0$, $f(0) = 0$ and f_o is the Cauchy additive mapping.
$$M_2f_e(x,y) = 2f_e\left(\frac{x+y}{2}\right) + 2f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y) = 0,$$
for all $x, y \in X$. Since $M_2f(0,0) = 0$, $f(0) = 0$ and f_e is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping. □

From now on, for a given mapping $f : X \to Y$, define $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then f_o is an odd mapping and f_e is an even mapping.

Lemma 2.2. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*
$$M_1f(x,y) = \rho M_2f(x,y), \tag{2.1}$$
for all $x, y \in X$, then $f : X \to Y$ is the sum of the Cauchy additive mapping f_o and the quadratic mapping f_e.

Proof. Letting $y = x$ in (2.1) for f_o, we get $f_o(2x) - 2f_o(x) = 0$ and so $f_o(2x) = 2f_o(x)$ for all $x \in X$. Thus
$$f_o\left(\frac{x}{2}\right) = \frac{1}{2}f_o(x), \tag{2.2}$$
for all $x \in X$.

It follows from (2.1) and (2.2) that
$$f_o(x+y) - f_o(x) - f_o(y) = \rho\left(2f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y)\right) = \rho(f_o(x+y) - f_o(x) - f_o(y)),$$
and so
$$f_o(x+y) = f_o(x) + f_o(y),$$
for all $x, y \in X$.

Letting $y = x$ in (2.1) for f_e, we get $\frac{1}{2}f_e(2x) - 2f_e(x) = 0$ and so $f_e(2x) = 4f_e(x)$ for all $x \in X$. Thus
$$f_e\left(\frac{x}{2}\right) = \frac{1}{4}f_e(x), \tag{2.3}$$
for all $x \in X$.

It follows from (2.1) and (2.3) that
$$\begin{aligned}\frac{1}{2}f_e(x+y) + \frac{1}{2}f_e(x-y) - f_e(x) - f_e(y) &= \rho\left(2f_e\left(\frac{x+y}{2}\right) + 2f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y)\right)\\ &= \rho\left(\frac{1}{2}f_e(x+y) + \frac{1}{2}f_e(x-y) - f_e(x) - f_e(y)\right),\end{aligned}$$
and so
$$f_e(x+y) + f_e(x-y) = 2f_e(x) + 2f_e(y),$$
for all $x, y \in X$.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping f_o and the quadratic mapping f_e. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (2.1) in Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leqslant \frac{L}{4}\varphi(x,y), \tag{2.4}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|M_1 f(x,y) - \rho M_2 f(x,y)\| \leqslant \varphi(x,y), \tag{2.5}$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_o(x) - A(x)\| \leqslant \frac{L}{4(1-L)}(\varphi(x,x) + \varphi(-x,-x)),$$
$$\|f_e(x) - Q(x)\| \leqslant \frac{L}{4(1-L)}(\varphi(x,x) + \varphi(-x,-x)),$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5) for f_o, we get

$$\|f_o(2x) - 2f_o(x)\| \leqslant \frac{1}{2}\varphi(x,x) + \frac{1}{2}\varphi(-x,-x), \tag{2.6}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\},$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leqslant \mu(\varphi(x,x) + \varphi(-x,-x)), \quad \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [7]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right),$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leqslant \varepsilon(\varphi(x,x) + \varphi(-x,-x)),$$

for all $x \in X$. Since $\frac{L}{4}\varphi(x,y) \leqslant \frac{L}{2}\varphi(x,y)$ for all $x, y \in X$,

$$\begin{aligned}\|Jg(x) - Jh(x)\| = \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| &\leqslant 2\varepsilon\left(\varphi\left(\frac{x}{2},\frac{x}{2}\right) + \varphi\left(-\frac{x}{2},-\frac{x}{2}\right)\right)\\ &\leqslant 2\varepsilon\frac{L}{2}(\varphi(x,x) + \varphi(-x,-x)) = L\varepsilon(\varphi(x,x) + \varphi(-x,-x)),\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leqslant L\varepsilon$. This means that

$$d(Jg,Jh) \leqslant Ld(g,h),$$

for all $g, h \in S$.

It follows from (2.6) that

$$\left\|f_o(x) - 2f_o\left(\frac{x}{2}\right)\right\| \leqslant \frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2}\right) + \frac{1}{2}\varphi\left(-\frac{x}{2},-\frac{x}{2}\right) \leqslant \frac{L}{8}(\varphi(x,x) + \varphi(-x,-x)),$$

for all $x \in X$. So $d(f_o, Jf_o) \leqslant \frac{L}{8} \leqslant \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right), \tag{2.7}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f_o(x) - A(x)\| \leqslant \mu(\varphi(x, x) + \varphi(-x, -x)),$$

for all $x \in X$;

(2) $d(J^l f_o, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f_o\left(\frac{x}{2^n}\right) = A(x),$$

for all $x \in X$;

(3) $d(f_o, A) \leqslant \frac{1}{1-L} d(f_o, Jf_o)$, which implies

$$\|f_o(x) - A(x)\| \leqslant \frac{L}{4(1-L)}(\varphi(x, x) + \varphi(-x, -x)),$$

for all $x \in X$.

It follows from (2.4) and (2.5) that

$$\begin{aligned}
&\left\|A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\
&\quad = \lim_{n\to\infty}\left\|2^n\left(f_o\left(\frac{x+y}{2^n}\right) - f_o\left(\frac{x}{2^n}\right) - f_o\left(\frac{y}{2^n}\right)\right) - 2^n\rho\left(2f_o\left(\frac{x+y}{2^{n+1}}\right) - f_o\left(\frac{x}{2^n}\right) - f_o\left(\frac{y}{2^n}\right)\right)\right\| \\
&\quad \leqslant \frac{1}{2}\lim_{n\to\infty}\left(2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 2^n\varphi\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right)\right) = 0,
\end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right),$$

for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive.

Letting $y = x$ in (2.5) for f_e, we get

$$\left\|\frac{1}{2}f_e(2x) - 2f_e(x)\right\| \leqslant \frac{1}{2}\varphi(x, x) + \frac{1}{2}\varphi(-x, -x), \tag{2.8}$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right),$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leqslant \varepsilon(\varphi(x, x) + \varphi(-x, -x)),$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \leqslant 4\varepsilon\left(\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(-\frac{x}{2}, -\frac{x}{2}\right)\right)\\ &\leqslant 4\varepsilon\frac{L}{4}(\varphi(x,x) + \varphi(-x,-x)) = L\varepsilon(\varphi(x,x) + \varphi(-x,-x)),\end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leqslant L\varepsilon$. This means that

$$d(Jg, Jh) \leqslant Ld(g, h),$$

for all $g, h \in S$.

It follows from (2.8) that

$$\left\|f_e(x) - 4f_e\left(\frac{x}{2}\right)\right\| \leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(-\frac{x}{2}, -\frac{x}{2}\right) \leqslant \frac{L}{4}(\varphi(x,x) + \varphi(-x,-x)),$$

for all $x \in X$. So $d(f_e, Jf_e) \leqslant \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right), \tag{2.9}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.9) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f_e(x) - Q(x)\| \leqslant \mu(\varphi(x,x) + \varphi(-x,-x)),$$

for all $x \in X$;

(2) $d(J^l f_e, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n f_e\left(\frac{x}{2^n}\right) = Q(x),$$

for all $x \in X$;

(3) $d(f_e, Q) \leqslant \frac{1}{1-L} d(f_e, Jf_e)$, which implies

$$\|f_e(x) - Q(x)\| \leqslant \frac{L}{4(1-L)}(\varphi(x,x) + \varphi(-x,-x)),$$

for all $x \in X$.

It follows from (2.4) and (2.5) that

$$\begin{aligned}&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) - \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\|\\ &= \lim_{n\to\infty}\left\|4^n\left(\frac{1}{2}f_e\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f_e\left(\frac{x-y}{2^n}\right) - f_e\left(\frac{x}{2^n}\right) - f_e\left(\frac{y}{2^n}\right)\right)\right.\\ &\quad\left. - 4^n\rho\left(2f_e\left(\frac{x+y}{2^{n+1}}\right) + 2f_e\left(\frac{x-y}{2^{n+1}}\right) - f_e\left(\frac{x}{2^n}\right) - f_e\left(\frac{y}{2^n}\right)\right)\right\|\\ &\leqslant \frac{1}{2}\lim_{n\to\infty}\left(4^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 4^n\varphi\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right)\right) = 0,\end{aligned}$$

for all $x, y \in X$. So

$$\frac{1}{2}Q\left(\frac{x+y}{2}\right)+\frac{1}{2}Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)=\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right),$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic. □

Corollary 2.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|M_1f(x,y)-\rho M_2f(x,y)\| \leqslant \theta(\|x\|^r+\|y\|^r), \tag{2.10}$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_o(x)-A(x)\| \leqslant \frac{2\theta}{2^r-2}\|x\|^r,$$
$$\|f_e(x)-Q(x)\| \leqslant \frac{4\theta}{2^r-4}\|x\|^r,$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ for f_o (respectively, $L = 2^{2-r}$ for f_e) and we get the desired result. □

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leqslant 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right),$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.5). Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_o(x)-A(x)\| \leqslant \frac{1}{4(1-L)}(\varphi(x,x)+\varphi(-x,-x)),$$
$$\|f_e(x)-Q(x)\| \leqslant \frac{1}{4(1-L)}(\varphi(x,x)+\varphi(-x,-x)),$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

It follows from (2.6) that

$$\left\|f_o(x)-\frac{1}{2}f_o(2x)\right\| \leqslant \frac{1}{4}\varphi(x,x)+\frac{1}{4}\varphi(-x,-x),$$

for all $x \in X$.

For f_o, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x),$$

for all $x \in X$.

It follows from (2.8) that

$$\left\|f_e(x)-\frac{1}{4}f_e(2x)\right\| \leqslant \frac{1}{4}\varphi(x,x)+\frac{1}{4}\varphi(-x,-x),$$

for all $x \in X$.

For f_e, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x),$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10). Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_o(x) - A(x)\| \leqslant \frac{2\theta}{2-2^r}\|x\|^r,$$
$$\|f_e(x) - Q(x)\| \leqslant \frac{4\theta}{4-2^r}\|x\|^r,$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ for f_o (respectively, $L = 2^{r-2}$ for f_e) and we get the desired result. □

3. Additive-quadratic ρ-functional equation (2) in Banach spaces

We solve and investigate the additive-quadratic ρ-functional equation (2) in normed spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$M_2 f(x, y) = \rho M_1 f(x, y), \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is the sum of the Cauchy additive mapping f_o and the quadratic mapping f_e.

Proof. Letting $y = 0$ in (3.1) for f_o, we get

$$f_o\left(\frac{x}{2}\right) = \frac{1}{2} f_o(x), \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$f_o(x+y) - f_o(x) - f_o(y) = 2f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y) = \rho(f_o(x+y) - f_o(x) - f_o(y)),$$

and so

$$f_o(x+y) = f_o(x) + f_o(y),$$

for all $x, y \in X$.

Letting $y = 0$ in (3.1) for f_e, we get

$$f_e\left(\frac{x}{2}\right) = \frac{1}{4} f_e(x), \tag{3.3}$$

for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}\frac{1}{2}f_e(x+y) + \frac{1}{2}f_e(x-y) - f_e(x) - f_e(y) &= 2f_e\left(\frac{x+y}{2}\right) + 2f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y)\\ &= \rho\left(\frac{1}{2}f_e(x+y) + \frac{1}{2}f_e(x-y) - f_e(x) - f_e(y)\right),\end{aligned}$$

and so

$$f_e(x+y) + f_e(x-y) = 2f_e(x) + 2f_e(y),$$

for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (3.1) in Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leqslant \frac{L}{4}\varphi(x,y),$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|M_2 f(x,y) - \rho M_1 f(x,y)\| \leqslant \varphi(x,y), \tag{3.4}$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}\|f_o(x) - A(x)\| &\leqslant \frac{1}{2(1-L)}(\varphi(x,0) + \varphi(-x,0)),\\ \|f_e(x) - Q(x)\| &\leqslant \frac{1}{2(1-L)}(\varphi(x,0) + \varphi(-x,0)),\end{aligned}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4) for f_o, we get

$$\left\|f_o(x) - 2f_o\left(\frac{x}{2}\right)\right\| = \left\|2f_o\left(\frac{x}{2}\right) - f_o(x)\right\| \leqslant \frac{1}{2}\varphi(x,0) + \frac{1}{2}\varphi(-x,0), \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \ \ h(0) = 0\},$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leqslant \mu(\varphi(x,0) + \varphi(-x,0)), \ \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [7]).

For f_o, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right),$$

for all $x \in X$.

Letting $y = 0$ in (3.4) for f_e, we get

$$\left\|f_e(x) - 4f_e\left(\frac{x}{2}\right)\right\| = \left\|4f_e\left(\frac{x}{2}\right) - f_e(x)\right\| \leqslant \frac{1}{2}\varphi(x,0) + \frac{1}{2}\varphi(-x,0), \tag{3.6}$$

for all $x \in X$.

For f_e, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right),$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\|M_2 f(x,y) - \rho M_1 f(x,y)\| \leqslant \theta(\|x\|^r + \|y\|^r), \tag{3.7}$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$

such that

$$\|f_o(x) - A(x)\| \leqslant \frac{2^r\theta}{2^r - 2}\|x\|^r,$$
$$\|f_e(x) - Q(x)\| \leqslant \frac{2^r\theta}{2^r - 4}\|x\|^r,$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ for f_o (respectively, $L = 2^{2-r}$ for f_e) and we get the desired result. □

Theorem 3.4. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leqslant 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right),$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (3.4). *Then there exist a unique additive mapping* $A : X \to Y$ *and a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f_o(x) - A(x)\| \leqslant \frac{L}{2(1-L)}(\varphi(x, 0) + \varphi(-x, 0)),$$
$$\|f_e(x) - Q(x)\| \leqslant \frac{L}{2(1-L)}(\varphi(x, 0) + \varphi(-x, 0)),$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$\left\|f_o(x) - \frac{1}{2}f_o(2x)\right\| \leqslant \frac{1}{4}\varphi(2x, 0) + \frac{1}{4}\varphi(-2x, 0) \leqslant \frac{L}{2}\varphi(x, 0) + \frac{L}{2}\varphi(-x, 0),$$

for all $x \in X$.

For f_o, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x),$$

for all $x \in X$.

It follows from (3.6) that

$$\left\|f_e(x) - \frac{1}{4}f_e(2x)\right\| \leqslant \frac{1}{8}\varphi(2x, 0) + \frac{1}{8}\varphi(-2x, 0) \leqslant \frac{L}{4}\varphi(x, 0) + \frac{L}{4}\varphi(-x, 0) \leqslant \frac{L}{2}\varphi(x, 0) + \frac{L}{2}\varphi(-x, 0),$$

for all $x \in X$, since $\frac{L}{4}\varphi(x, 0) + \frac{L}{4}\varphi(-x, 0) \leqslant \frac{L}{2}\varphi(x, 0) + \frac{L}{2}\varphi(-x, 0)$ for all $x \in X$.

For f_e, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x),$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.5. *Let* $r < 1$ *and* θ *be positive real numbers, and let* $f : X \to Y$ *be a mapping satisfying* (3.7). *Then* *there exist a unique additive mapping* $A : X \to Y$ *and a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f_o(x) - A(x)\| \leqslant \frac{2^r\theta}{2-2^r}\|x\|^r,$$
$$\|f_e(x) - Q(x)\| \leqslant \frac{2^r\theta}{4-2^r}\|x\|^r,$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ for f_o (respectively, $L = 2^{r-2}$ for f_e) and we get the desired result. □

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