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On strong C-integral of Banach-valued functions defined on $\mathbb{R}^m$

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Abstract

In this paper, we define and study the C-integral and strong C-integral of functions mapping a compact interval I_0 of $\mathbb{R}^m$ into a real Banach space X. We prove that the C-integral and strong C-integral are equivalent if and only if X is finite dimensional. We also study the relations between the property S^*C and strong C-integral.

Keywords: C-integral, property S^*C, strong C-integral.
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1. Introduction and preliminaries

Bruckner et al. [5] considered the function

$$F(x) = \begin{cases} x \sin \frac{1}{x^2} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } x = 0. \end{cases}$$

It is a primitive for the Henstock integral, but it is neither a Lebesgue primitive, a differentiable function, nor a sum of a Lebesgue primitive and a differentiable function. The natural question is: "is there a minimal integral including the Lebesgue integral and derivatives?"

To solve this question, Bongiorno [1] provided a minimal constructive integration process of Riemann type, i.e., C-integral, which includes the Lebesgue integral and also integrates the derivatives of differentiable functions. The theory of C-integration has developed rather intensively in the past few years; see, for instance, the papers [2–4, 6–11, 14–18] and the references cited therein.

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In this paper, we define and study the C-integral and strong C-integral of functions mapping a compact interval I_0 of $\mathbb{R}^m$ into a real Banach space X. We prove that the C-integral and strong C-integral are equivalent if and only if X is finite dimensional. Compared with the strong McShane integral, we know that a function $f : I_0 \to X$ has the property S^*M if and only if it is strongly McShane integrable on I_0. Then a question arises naturally: "is there a similar result for the strong C-integral?" The purpose of this paper is to give a negative answer to this question: if a function $f : I_0 \to X$ has the property S^*C, then it is strongly C-integrable, but the converse is not true.

The following conventions and notation will be used, unless stated otherwise. $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^m$ is the ambient space of this paper, I_0 is a compact interval in $\mathbb{R}^m$, and μ stands for the Lebesgue measure, where m is a fixed positive integer. X denotes a real Banach space with norm $\|\cdot\|$ and dual X^*. A partition D is a finite collection of interval-point pairs $\{(I_i, \xi_i)\}_{i=1}^n$, where $\{I_i\}_{i=1}^n$ are nonoverlapping subintervals of I_0. δ is a positive function on I_0, i.e., $\delta : I_0 \to \mathbb{R}^+ = (0, \infty)$. We say that $D = \{(I_i, \xi_i)\}_{i=1}^n$ is

(1) a partial partition of I_0 if $\bigcup_{i=1}^n I_i \subset I_0$;

(2) a partition of I_0 if $\bigcup_{i=1}^n I_i = I_0$;

(3) a δ-fine McShane partition of I_0 if $I_i \subset B(\xi_i, \delta(\xi_i)) = \{t_i \in \mathbb{R}^m ; \mathrm{dist}(\xi_i, t_i) < \delta(\xi_i)\}$ and $\xi_i \in I_0$ for all $i = 1, 2, \ldots, n$, where dist is the metric in $\mathbb{R}^m$;

(4) a δ-fine C-partition of I_0 if it is a δ-fine McShane partition of I_0 satisfying the condition

$$\sum_{i=1}^{n} \mathrm{dist}(\xi_i, I_i) < \frac{1}{\varepsilon}$$

for the given arbitrary $\varepsilon > 0$, where $\mathrm{dist}(\xi_i, I_i)$ denotes the distance of ξ_i from I_i;

(5) a δ-fine Henstock partition of I_0 if $\xi_i \in I_i \subset B(\xi_i, \delta(\xi_i))$ for all $i = 1, 2, \ldots, n$.

Given a δ-fine C-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$, we write

$$S(f, D) = \sum_{i=1}^{n} f(\xi_i)\mu(I_i)$$

for integral sums over D, whenever $f : I_0 \to X$.

Definition 1.1. A function $f : I_0 \to X$ is said to be C-integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta : I_0 \to \mathbb{R}^+$ satisfying

$$\|S(f, D) - A\| < \varepsilon$$

for each δ-fine C-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0. A is called the C-integral of f on I_0 and we write $A = \int_{I_0} f$ or $A = (C)\int_{I_0} f$. The function f is C-integrable on the set $E \subset I_0$ if the function $f\chi_E$ is C-integrable on I_0, we write $\int_E f = \int_{I_0} f\chi_E$.

We can easily obtain the following theorems.

Theorem 1.2. *A function $f : I_0 \to X$ is C-integrable if and only if for each $\varepsilon > 0$ there is a positive function $\delta : I_0 \to \mathbb{R}^+$ such that*

$$\|S(f, D_1) - S(f, D_2)\| < \varepsilon$$

for arbitrary δ-fine C-partitions D_1 and D_2 of I_0.

Theorem 1.3. *Let $f, g : I_0 \to X$.*

(1) *If* f *is* C-*integrable on* I_0*, then* f *is* C-*integrable on every subinterval of* I_0.
(2) *If* f *is* C-*integrable on each of the intervals* I_1 *and* I_2*, where* $I_i, i = 1, 2$ *are nonoverlapping and* $I_1 \cup I_2 = I_0$*, then* f *is* C-*integrable on* I_0 *and* $\int_{I_1} f + \int_{I_2} f = \int_{I_0} f$.
(3) *If* f *and* g *are* C-*integrable on* I_0 *and* α *and* β *are real numbers, then* $\alpha f + \beta g$ *is* C-*integrable on* I_0 *and* $\int_{I_0}(\alpha f + \beta g) = \alpha \int_{I_0} f + \beta \int_{I_0} g$.

Lemma 1.4 (Saks–Henstock)**.** *Let* $f : I_0 \to X$ *be* C-*integrable on* I_0 *and let* $\varepsilon > 0$*. If there is a positive function* $\delta : I_0 \to \mathbb{R}^+$ *such that*

$$\left\| S(f, D) - \int_{I_0} f \right\| < \varepsilon$$

for each δ*-fine* C-*partition* $D = \{(I, \xi)\}$ *of* I_0*, then*

$$\left\| S(f, D') - \sum_{i=1}^{m} \int_{I_i} f \right\| \leqslant \varepsilon$$

for each δ*-fine partial* C-*partition* $D' = \{(I_i, \xi_i)\}_{i=1}^{m}$ *of* I_0.

Proof. The proof is similar to the case for Banach-valued Henstock integrable functions; see [12, Lemma 3.4.1] for details. □

Theorem 1.5. *Let* $f : I_0 \to X$ *be* C-*integrable on* I_0 *and assume that* Y *is a real Banach space.*

(1) *For each* $x^* \in X^*$*, the function* $x^* f$ *is* C-*integrable on* I_0 *and* $\int_{I_0} x^* f = x^*(\int_{I_0} f)$.
(2) *If* $T : X \to Y$ *is a continuous linear operator, then* Tf *is* C-*integrable on* I_0 *and* $\int_{I_0} Tf = T(\int_{I_0} f)$.

Proof. (1) Since $f : I_0 \to X$ is C-integrable on I_0, for each $\varepsilon > 0$ and for each $x^* \in X^*$ there is a positive function $\delta : I_0 \to \mathbb{R}^+$ such that

$$\left\| S(f, D) - \int_{I_0} f \right\| < \frac{\varepsilon}{\|x^*\|}$$

for each δ-fine C-partition $D = \{(I, \xi)\}$ of I_0. Hence, for $x^* \in X^*$, we have

$$\left\| S(x^* f, D) - x^* \left(\int_{I_0} f \right) \right\| \leqslant \|x^*\| \left\| S(f, D) - \int_{I_0} f \right\| < \varepsilon.$$

(2) If $T : X \to Y$ is a continuous linear operator, then there exists a number $M > 0$ such that $\|Tx\| \leqslant M\|x\|$ for each $x \in X$. Since $f : I_0 \to X$ is C-integrable on I_0, for each $\varepsilon > 0$ there is a positive function $\delta : I_0 \to \mathbb{R}^+$ such that

$$\left\| S(f, D) - \int_{I_0} f \right\| < \frac{\varepsilon}{M}$$

for each δ-fine C-partition $D = \{(I, \xi)\}$ of I_0. Hence, we obtain

$$\left\| S(Tf, D) - T \left(\int_{I_0} f \right) \right\| \leqslant M \left\| S(f, D) - \int_{I_0} f \right\| < \varepsilon.$$

The proof is complete. □

2. Strong C-integral

Definition 2.1. A function $f : I_0 \to X$ is said to be strongly C-integrable if there exists an additive function $F : I_0 \to X$ such that for each $\varepsilon > 0$ there is a positive function $\delta : I_0 \to \mathbb{R}^+$ satisfying

$$\sum_{i=1}^{n} \|f(\xi_i)\mu(I_i) - F(I_i)\| < \varepsilon$$

for each δ-fine C-partition $D = \{(I_i, \xi_i)\}_{i=1}^{n}$ of I_0. $F(I_0)$ is termed the strong C-integral of f on I_0 and we write $F(I_0) = \int_{I_0} f$.

Theorem 2.2. *If $f : I_0 \to X$ is strongly C-integrable on I_0, then f is C-integrable on I_0.*

Proof. It follows from the definitions of strong C-integral and C-integral that if f is strongly C-integrable on I_0, then f is C-integrable on I_0. □

Definition 2.3. A function $f : I_0 \to X$ is strongly Henstock (McShane) integrable if there exists an additive function $F : I_0 \to X$ such that for each $\varepsilon > 0$ there is a positive function $\delta : I_0 \to \mathbb{R}^+$ satisfying

$$\sum_{i=1}^{n} \|f(\xi_i)\mu(I_i) - F(I_i)\| < \varepsilon$$

for each δ-fine Henstock (McShane) partition $D = \{(I_i, \xi_i)\}_{i=1}^{n}$ of I_0. $F(I_0)$ is called the strong Henstock (McShane) integral of f on I_0 and we write $F(I_0) = \int_{I_0} f$.

Applying the definitions of strong Henstock integral, strong C-integral, and strong McShane integral and using the fact that each δ-fine Henstock partition is also δ-fine C-partition and δ-fine McShane partition, we get the following theorem immediately.

Theorem 2.4. *Let $f : I_0 \to X$.*

(1) *If f is strongly McShane integrable, then f is strongly C-integrable.*
(2) *If f is strongly C-integrable, then f is strongly Henstock integrable.*

Theorem 2.5. *If the Banach space X is finite dimensional, then a function $f : I_0 \to X$ is C-integrable on I_0 if and only if f is strongly C-integrable on I_0.*

Proof. We only prove that if X is finite dimensional and $f : I_0 \to X$ is C-integrable on I_0, then f is strongly C-integrable on I_0. It follows from the definition of C-integral that for each $\varepsilon > 0$ there is a positive function $\delta : I_0 \to \mathbb{R}^+$ such that

$$\left\| \sum [f(\xi)\mu(I) - F(I)] \right\| < \varepsilon$$

for each δ-fine C-partition $D = \{(I, \xi)\}$ of I_0. Let $\{e_1, e_2, \ldots, e_n\}$ be a base of X and $g_i : I_0 \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ satisfying $f = \sum_{i=1}^{n} g_i e_i$. By the Hahn–Banach theorem, for each e_i there is an $x_i^* \in X^*$ such that

$$x_i^*(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for $i, j = 1, 2, \ldots, n$, and so $x_i^*(f) = \sum_{j=1}^{n} g_j x_i^*(e_j) = g_i$. Therefore, by Theorem 1.5, we conclude that $g_i : I_0 \to \mathbb{R}$ are C-integrable on I_0. Then for each $\varepsilon > 0$ there are positive functions $\delta_i : I_0 \to \mathbb{R}^+$ such that

$$\left| S(g_i, D_i) - \sum \int_I g_i \right| < \frac{\varepsilon}{2}$$

for each δ_i-fine C-partition D_i of I_0. An easy adaptation of Lemma 1.4 yields

$$\sum \left| g_i(\xi)\mu(I) - \int_I g_i \right| < \varepsilon.$$

On the other hand,

$$F(I) = \int_I f = \int_I \sum_{i=1}^{n} g_i e_i = \sum_{i=1}^{n} \int_I g_i e_i = \sum_{i=1}^{n} e_i G_i(I),$$

where $G_i(I) = \int_I g_i$. Let $\delta(\xi) < \delta_i(\xi)$ for $i = 1, 2, \ldots, n$. Consequently,

$$\sum \|f(\xi)\mu(I) - F(I)\| = \sum \left\| \sum_{i=1}^{n} g_i(\xi) e_i \mu(I) - \sum_{i=1}^{n} e_i G_i(I) \right\|$$

$$\leqslant \sum_{i=1}^{n} \|e_i\| \sum |g_i(\xi)\mu(I) - G_i(I)|$$
$$< \varepsilon \sum_{i=1}^{n} \|e_i\|$$

for each δ-fine C-partition $D = \{(I, \xi)\}$ of I_0. Hence, f is strongly C-integrable on I_0. The proof is complete when using Theorem 2.2. □

Theorem 2.6. *The* C*-integral is equivalent to strong* C*-integral on* I_0 *if and only if* X *is finite dimensional.*

Proof. The proof of necessity is similar to the case for Henstock (McShane) integral; see, for example, [13, Theorem 3]. In [13], if X is infinite dimensional, then there exist $x_1^r, x_2^r, \ldots, x_{2^r}^r \in X$ such that

$$\|x_i^r\| = \frac{1}{2^r}$$

for each r, $1 \leqslant i \leqslant 2^r$, and we also have

$$\left\|\sum_{i=1}^{2^r} \theta_i^r x_i^r\right\|^2 \leqslant \frac{3}{2^r}$$

for every θ_i^r with $|\theta_i^r| \leqslant 1$, $1 \leqslant i \leqslant 2^r$. Skvortsov and Solodov [13] defined a function $f : [0, 1] \to X$ by

$$f(t) = \begin{cases} 0 & \text{if } t \in C, \text{ or } t = d_i^r, r \geqslant 0, 1 \leqslant i \leqslant 2^r, \\ 2 \cdot 3^r x_i^r & \text{if } t \in (a_i^r, d_i^r), r \geqslant 0, 1 \leqslant i \leqslant 2^r, \\ -2 \cdot 3^r x_i^r & \text{if } t \in (d_i^r, b_i^r), r \geqslant 0, 1 \leqslant i \leqslant 2^r. \end{cases}$$

Here, C is the Cantor ternary set, $(a_i^r, b_i^r), r \geqslant 0$, $1 \leqslant i \leqslant 2^r$ are the intervals of rank r contiguous to C with middle points d_i^r and satisfy $b_i^r - a_i^r = 3^{-(r+1)}$. The function f is McShane integrable but it is not strongly Henstock integrable, and then we deduce that f is C-integrable but it is not strongly C-integrable. In other words, if C-integral is equivalent to strong C-integral, then X is finite dimensional. □

Definition 2.7. A function $f : I_0 \to X$ has the property S^*C if for each $\varepsilon > 0$ there is a positive function $\delta : I_0 \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \|f(\xi_i) - f(\zeta_j)\|\mu(I_i \cap L_j) < \varepsilon$$

for arbitrary δ-fine C-partitions $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ of I_0.

Theorem 2.8. *If a function* $f : I_0 \to X$ *has the property* S^*C*, then* f *is strongly* C*-integrable on* I_0*.*

Proof. We will prove this theorem in two steps.

Step 1. Assume that $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ are arbitrary δ-fine C-partitions of I_0. Then

$$\begin{aligned} \left\|\sum_{i=1}^{m} f(\xi_i)\mu(I_i) - \sum_{j=1}^{n} f(\zeta_i)\mu(L_j)\right\| &= \left\|\sum_{j=1}^{n}\sum_{i=1}^{m} f(\xi_i)\mu(I_i \cap L_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} f(\zeta_j)\mu(I_i \cap L_j)\right\| \\ &= \left\|\sum_{i=1}^{m}\sum_{j=1}^{n} (f(\xi_i) - f(\zeta_i))\mu(I_i \cap L_j)\right\| \\ &\leqslant \sum_{i=1}^{m}\sum_{j=1}^{n} \|f(\xi_i) - f(\zeta_i)\|\mu(I_i \cap L_j) < \varepsilon. \end{aligned}$$

By virtue of Theorem 1.2, we conclude that f is C-integrable on I_0.

Step 2. By Definition 2.7, for each $\varepsilon > 0$ there is a positive function $\delta : I_0 \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \|f(\xi_i) - f(\zeta_j)\| \mu(I_i \cap L_j) < \frac{\varepsilon}{2}$$

for arbitrary δ-fine C-partitions $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ of I_0. The function f is C-integrable on I_0 and therefore it is C-integrable on I_i for $i = 1, 2, \ldots, m$. Hence, for given $\varepsilon > 0$ there are positive functions $\delta_i : I_0 \to \mathbb{R}^+$ such that $\delta_i(\xi) \leqslant \delta(\xi)$ and for any δ_i-fine C-partition $D_i = \{(L_j^i, \zeta_j^i)\}_{j=1}^{n^i}$ of I_i, $i = 1, 2, \ldots, m$,

$$\left\| \sum_{j=1}^{n^i} f(\zeta_j^i)\mu(L_j^i) - \int_{I_i} f \right\| = \left\| \sum_{j=1}^{n^i} \left[f(\zeta_j^i)\mu(L_j^i) - \int_{L_j^i} f \right] \right\| < \frac{\varepsilon}{2m}.$$

Defining $D = \bigcup_{i=1}^m D_i$, it is a δ-fine C-partition of I_0. Hence, we have

$$\begin{aligned}
\sum_{i=1}^{m} \|f(\xi_i)\mu(I_i) - F(I_i)\| &= \sum_{i=1}^{m} \left\| \sum_{j=1}^{n^i} f(\xi_i)\mu(I_i \cap L_j^i) - \sum_{j=1}^{n^i} F(I_i \cap L_j^i) \right\| \\
&= \sum_{i=1}^{m} \left\| \sum_{j=1}^{n^i} (f(\xi_i) - f(\zeta_j^i))\mu(I_i \cap L_j^i) + \sum_{j=1}^{n^i} [f(\zeta_j^i)\mu(I_i \cap L_j^i) - F(I_i \cap L_j^i)] \right\| \\
&\leqslant \sum_{i=1}^{m}\sum_{j=1}^{n^i} \|f(\xi_i) - f(\zeta_j^i)\|\mu(I_i \cap L_j^i) + \sum_{i=1}^{m} \left\| \sum_{j=1}^{n^i} [f(\zeta_j^i)\mu(I_i \cap L_j^i) - F(I_i \cap L_j^i)] \right\| \\
&< \frac{\varepsilon}{2} + m\frac{\varepsilon}{2m} = \varepsilon.
\end{aligned}$$

By virtue of Definition 2.1, f is strongly C-integrable on I_0. □

Remark 2.9. The converse of Theorem 2.8 is not true. In other words, if f is strongly C-integrable on I_0, then it does not necessarily with the property S^*C.

Proof. Let f be given by

$$f(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } x = 0. \end{cases}$$

It is known that the primitive of f is

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } x = 0, \end{cases}$$

and F is differentiable and $F'(x) = f(x)$ on $[0, 1]$. Hence, f is strongly C-integrable on $[0, 1]$; see [6, p. 146] for details.

Assume that f has the property S^*C. Then for each $\varepsilon > 0$ there is a positive function $\delta : [0, 1] \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n} |f(\xi_i) - f(\zeta_j)|\mu(I_i \cap L_j) < \varepsilon$$

for arbitrary δ-fine C-partitions $D_1 = \{(I_i, \xi_i)\}_{i=1}^m$ and $D_2 = \{(L_j, \zeta_j)\}_{j=1}^n$ of $[0, 1]$. Consequently,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \big| |f(\xi_i)| - |f(\zeta_j)| \big| \mu(I_i \cap L_j) \leqslant \sum_{i=1}^{m}\sum_{j=1}^{n} |f(\xi_i) - f(\zeta_j)|\mu(I_i \cap L_j) < \varepsilon.$$

Hence, $|f|$ has the property S^*C and so $|f|$ is C-integrable on $[0, 1]$ when using Theorems 2.2 and 2.8. It follows easily that f is Lebesgue (McShane) integrable. But F is not absolutely continuous on $[0, 1]$ and thus f is not Lebesgue (McShane) integrable on $[0, 1]$. Therefore, f does not have the property S^*C. $\square$

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References

[1] B. Bongiorno, *A new integral for the problem of primitives*, (Italian) Matematiche (Catania), **51** (1996), 299–313. 1
[2] B. Bongiorno, *On the minimal solution of the problem of primitives*, J. Math. Anal. Appl., **251** (2000), 479–487. 1
[3] D. Bongiorno, *On the problem of nearly derivatives*, Sci. Math. Jpn., **61** (2005), 299–311.
[4] B. Bongiorno, L. Di Piazza, D. Preiss, *A constructive minimal integral which includes Lebesgue integrable functions and derivatives*, J. London Math. Soc., **62** (2000), 117–126. 1
[5] A. M. Bruckner, R. J. Fleissner, J. Foran, *The minimal integral which includes Lebesgue integrable functions and derivatives*, Colloq. Math., **50** (1986), 289–293. 1
[6] L. Di Piazza, *A Riemann-type minimal integral for the classical problem of primitives*, Rend. Istit. Mat. Univ. Trieste, **34** (2002), 143–153. 1, 2
[7] G. S. Eun, J. H. Yoon, Y. K. Kim, B. M. Kim, *On* C*-delta integrals on time scales*, J. Chungcheong Math. Soc., **25** (2012), 349–357.
[8] T. Kawasaki, *Some integrals between the Lebesgue integral and the Denjoy integral*, Sci. Math. Jpn., **29** (e-2016).
[9] T. Kawasaki, I. Suzuki, *Criteria for the C-integral*, Sci. Math. Jpn., **79** (2016), 155–164.
[10] T.-Y. Lee, *Multipliers for generalized Riemann integrals in the real line*, Math. Bohem., **131** (2006), 161–166.
[11] J. M. Park, H. W. Ryu, H. K. Lee, D. H. Lee, *The* M_α *and C-integrals*, Czechoslovak Math. J., **62** (2012), 869–878. 1
[12] S. Schwabik, G.-J. Ye, *Topics in Banach space integration*, Series in Real Analysis, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2005). 1
[13] V. A. Skvortsov, A. P. Solodov, *A variational integral for Banach-valued functions*, Real Anal. Exchange, **24** (1999), 799–805. 2
[14] E. Talvila, *The regulated primitive integral*, Illinois J. Math., **53** (2009), 1187–1219. 1
[15] J. H. Yoon, *On C-delta integral of Banach space valued functions on time scales*, J. Chungcheong Math. Soc., **29** (2016), 157–163.
[16] D.-F. Zhao, C.-S. Hu, *The characterization of the primitives of approximately continuous* C*-integrable functions*, J. Math. (Wuhan), **30** (2010), 427–430.
[17] D.-F. Zhao, B.-W. Li, *A note on the* C*-integral*, J. Math. (Wuhan), **31** (2011), 594–598.
[18] D.-F. Zhao, G.-J. Ye, C*-integral and Denjoy-*C *integral*, Commun. Korean Math. Soc., **22** (2007), 27–39. 1