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Existence and nonexistence of positive solutions for Dirichlet-type boundary value problem of nonlinear fractional differential equation

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Abstract

In this paper, we investigate the existence and nonexistence of positive solutions for nonlinear fractional differential equation boundary value problem. By means of fixed-point theorems on a cone and the properties of Green function, some sufficient criteria are established. Our results can be considered as an extension of some previous results.

Keywords: Positive solution, fractional differential equation, boundary value problem, the Krasnoselskii fixed point theorem.
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1. Introduction

Fractional calculus is a 300-year-old mathematical topic, starting from 30 September 1695, when the derivative of order $\alpha = \frac{1}{2}$ was described by Leibniz [1]. During the last few decades, fractional-order differential equations have been of great interest. The main advantage of fractional-order differential equations in comparison with classical integer-order ones is that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various processes. For an extensive collection of such results, we refer the readers to the monographs by Miller and Ross [8], Oldham and Spanier [9], and Poldubny [10].

Recently, there are some papers dealing with the existence and multiplicity of solutions of nonlinear initial value fractional differential equation by the use of techniques of nonlinear analysis [2, 3, 5, 6]. In [3] and [6], the authors considered the Dirichlet-type boundary value problem for fractional differential equations

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leqslant 2$ is a real number and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. By the use of techniques of fixed-point theorems on cone, the authors

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discussed the existence and multiplicity of positive solutions for (1.1). Though some results have been obtained for system (1.1), such systems are not well studied yet. In this paper, motivated by the work of [4, 11, 12], we proceed to develop more results for the existence and multiplicity of positive solutions of system (1.1). Moreover, we will discuss the nonexistence of positive solutions which is rarely discussed in previous work.

The tree of this paper is organized as follows. In Sections 2, we list some useful definitions and properties, and present the properties of Green Function of fractional differential equations with the boundary value problem. In Section 3, we establish some sufficient conditions for the existence of positive solutions for (1.1). Finally, in Section 4, we discuss the nonexistence of positive solutions of (1.1).

2. Preliminaries

In this section, we will present several foundational definitions of fractional calculus and preliminary results. For more details, one can see [3, 6].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.3. *Let $\alpha > 0$ and $u \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation*

$$D^{\alpha}_{0+}u(t) = 0$$

has solutions

$$u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, and n is the smallest integer greater than or equal to α.

Lemma 2.4. *Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_nt^{\alpha-n},$$

for some $c_i \in \mathbb{R}, i = 1, 2, \cdots, n$, n is the smallest integer greater than or equal to α.

Lemma 2.5. *Given $y \in C[0, 1]$ and $1 < \alpha \leqslant 2$, the unique solution of*

$$\begin{aligned} D^{\alpha}_{0+}u(t) + y(t) = 0, &\quad 0 < t < 1, \\ u(0) = u(1) = 0 & \end{aligned}$$

is

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t, s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2.1}$$

Lemma 2.6. *$u(t)$ is a solution of (1.1), if and only if it is also a solution of the following integral equation*

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds,$$

where $G(t,s)$ is defined in (2.1).

Lemma 2.7. *Let $G^*(t,s) := t^{2-\alpha}G(t,s)$, then*

$$\frac{\alpha-1}{\Gamma(\alpha)}t(1-t)s(1-s)^{\alpha-1} \leqslant G^*(t,s) \leqslant \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-1},$$

for $t, s \in (0,1)$.

Next, we give the definition for cone and the famous fixed point theorem that will be needed in our arguments [7].

Definition 2.8. Let X be a Banach space and E be a closed nonempty subset of X. E is said to be a cone if

(i) $\alpha u + \beta v \in E$ for all $u, v \in E$ and all $\alpha, \beta > 0$;

(ii) $u, -u \in E$ imply $u = 0$.

Theorem 2.9. *Let X be a Banach space, and let $E \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $T : E \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow E$ be a completely continuous operator such that either*

(1) *$\|Ty\| \geqslant \|y\|$ for any $y \in E \cap \partial\Omega_1$, and $\|Ty\| \leqslant \|y\|$ for any $y \in E \cap \partial\Omega_2$, or*

(2) *$\|Ty\| \leqslant \|y\|$ for any $y \in E \cap \partial\Omega_1$, and $\|Ty\| \geqslant \|y\|$ for any $y \in E \cap \partial\Omega_2$.*

Then T has a fixed point in $E \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence of positive solutions of (1.1)

In this section, we establish the positive solutions of (1.1) by applying the fixed point theorems on a cone. In order to explore the existence of positive solutions of (1.1), we suppose the following hypotheses are always satisfied in the sequel.

(A) $f(t,u)$ is continuous on $[0,1] \times [0,\infty)$, and there exist $g \in C([0,+\infty),[0,+\infty))$, $q_1, q_2 \in C((0,+\infty),(0,+\infty))$ such that

$$q_1(t)g(y) \leqslant f(t, t^{\alpha-2}y) \leqslant q_2(t)g(y),\ t \in (0,1),\ y \in [0,\infty), \tag{3.1}$$

where

$$\int_0^1 q_i(s)ds < +\infty,\ i = 1,2.$$

Let $E = C[0,1]$ be endowed with the ordering $u \leqslant v$ if $u(t) \leqslant v(t)$ for all $t \in [0,1]$, and the maximum norm $\|u\| = \max_{0\leqslant t\leqslant 1} |u(t)|$.

Define the cone $P \subset E$ by $P = \{u \in E \mid u(t) \geqslant 0\}$. Then we have the following lemma.

Lemma 3.1. *Let $T : P \to E$ be the operator defined by*

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds,$$

then $T : P \to P$ is completely continuous.

It is clear that $u(t)$ is a positive solution of (1.1), whenever $u(t)$ is a fixed point of T, namely, $u(t) = (Tu)(t)$.

Define the cone $K \subset E$ by

$$K = \{y \in E \mid y(t) \geqslant (\alpha-1)t(1-t)\|y\|\},$$

and an operator $T^* : K \to E$ as follows

$$(T^*y)(t) = \int_0^1 G^*(t,s)f(s, s^{\alpha-2}y(s))ds.$$

Then one has the following lemma.

Lemma 3.2. $T^*(K) \subset K$ *and* $T^* : K \to K$ *is completely continuous.*

For the sake of convenience and simplicity, we introduce the following notations

$$g^0 = \lim_{y\to 0} \max \frac{g(y)}{y}, \qquad g^\infty = \lim_{y\to\infty} \max \frac{g(y)}{y},$$
$$g_0 = \lim_{y\to 0} \min \frac{g(y)}{y}, \qquad g_\infty = \lim_{y\to\infty} \min \frac{g(y)}{y}.$$

Moreover, define, for r a positive number, Ω_r by

$$\Omega_r = \{y \in C[0,1] \mid \|y\| < r\}.$$

Note that $\partial\Omega_r = \{y \in C[0,1] \mid \|y\| = r\}$.

Our first result is as follows:

Theorem 3.3. *Assume that*

$$(P1)\ g_0 = \infty \quad \textit{and} \quad (P2)\ g^\infty = 0$$

hold. Then system (1.1) *has at least one positive solution.*

Proof. First, in view of $g_0 = \infty$, there exist $\rho_0 > 0$, $M > 0$ satisfying

$$\frac{M(\alpha-1)^2}{4\Gamma(\alpha)} \int_0^1 s^2(1-s)^\alpha q_1(s)ds \geqslant 1,$$

such that $g(y) \geqslant My$ for $0 < y \leqslant \rho_0$. Then, for any $y \in \Omega_{\rho_0}$, $0 < y < \rho_0$, by Lemma 2.7 and (3.1), we have

$$\begin{aligned}
\|T^*y\| &\geqslant (T^*y)(\frac{1}{2}) \\
&= \int_0^1 G^*(\frac{1}{2}, s)f(s, s^{\alpha-2}y(s))ds \\
&\geqslant \frac{\alpha-1}{4\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1}q_1(s)g(y(s))ds \\
&\geqslant \frac{M(\alpha-1)}{4\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1}q_1(s)y(s)ds \\
&\geqslant \frac{M(\alpha-1)^2}{4\Gamma(\alpha)}\|y\| \int_0^1 s^2(1-s)^\alpha q_1(s)ds \\
&\geqslant \|y\|,
\end{aligned}$$

which implies that $\|T^*y\| \geqslant \|y\|$ for any $y \in K \cap \partial\Omega_{\rho_0}$.

On the other hand, by using $g^\infty = 0$, there exist $M^* > \rho_0 > 0$ and $\varepsilon > 0$ satisfying

$$\frac{\varepsilon}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \leqslant \frac{1}{2},$$

such that $g(y) \leqslant \varepsilon y$ for $y > M^*$.

Take

$$\rho_1 > M^* + \frac{1}{\varepsilon}\max_{y\in[0,M^*]}\{g(y)\}.$$

Then by Lemma 2.7 and (3.1), we have

$$\begin{aligned}
(T^*y)(t) &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,s^{\alpha-2}y(s))ds\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)g(y(s))ds\\
&\leqslant \frac{1}{\Gamma(\alpha)}\left\{\int_{I_1} s(1-s)^{\alpha-1}q_2(s)\max\{g(y(s))\}ds + \int_{I_2} s(1-s)^{\alpha-1}q_2(s)\varepsilon y(s)ds\right\}\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \max_{y(s)\in I_1}\{g(y(s))\} + \frac{\varepsilon}{\Gamma(\alpha)}\|y\|\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds\\
&\leqslant \frac{\rho_1}{2} + \frac{\|y\|}{2}\\
&= \|y\|,
\end{aligned}$$

where $I_1 = \{s \in [0,1] \mid 0 \leqslant y(s) \leqslant M^*\}$, $I_2 = \{s \in [0,1] \mid y(s) > M^*\}$. This implies that $\|T^*y\| \leqslant \|y\|$ for any $y \in K \cap \partial\Omega_{\rho_1}$.

Thus, by Theorem 2.9, T^* has a fixed point y in $K\cap(\overline{\Omega}_{\rho_1} \setminus \Omega_{\rho_0})$, that is, $y(t) = \int_0^1 G^*(t,s)f(s,s^{\alpha-2}y(s))ds$, $t \in [0,1]$. It is obvious that $u(t) = t^{\alpha-2}y(t)$ is a fixed point of T, and it satisfies $u(t) = \int_0^1 G(t,s)f(s,u(s))ds$, $t \in [0,1]$. Finally, we prove $u(0) = u(1) = 0$.

From $y \in C[0,1]$ and (A), we have

$$\begin{aligned}
\lim_{t\to 0} u(t) &= \lim_{t\to 0}\int_0^1 G(t,s)f(s,u(s))ds\\
&= \lim_{t\to 0}\int_0^1 G(t,s)f(s,s^{\alpha-2}y(s))ds\\
&\leqslant \lim_{t\to 0}\int_0^1 G(t,s)q_2(s)g(y(s))ds\\
&\leqslant \lim_{t\to 0}\int_0^1 G(t,s)q_2(s)ds \max_{0\leqslant\|y\|\leqslant\rho_1} g(y(s))\\
&= 0.
\end{aligned}$$

So, $u(0) = 0$. By (2.1), it is easy to see that $u(1) = y(1) = 0$. Hence, system (1.1) has a positive solution $u(t) = t^{\alpha-2}y(t)$. Therefore the proof is completed. □

Theorem 3.4. *Assume that*

$$(P3)\ g^0 = 0 \quad \textit{and} \quad (P4)\ g_\infty = \infty$$

hold. Then system (1.1) has at least one positive solution.

Proof. Since $g^0 = 0$, for any $\varepsilon > 0$ satisfying

$$\frac{\varepsilon}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \leqslant 1,$$

there exists $\rho_2 > 0$ such that $g(y) \leqslant \varepsilon y$ for $y \leqslant \rho_2$.

Then by Lemma 2.7 and (3.1), we get

$$\begin{aligned}(T^*y)(t) &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,s^{\alpha-2}y(s))ds\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)g(y(s))ds\\ &\leqslant \frac{\varepsilon}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)y(s)ds\\ &\leqslant \frac{\varepsilon}{\Gamma(\alpha)}\|y\|\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds\\ &\leqslant \|y\|,\end{aligned}$$

which shows that $\|T^*y\| \leqslant \|y\|$ for $y \in K \cap \partial\Omega_{\rho_2}$.

Further, since $g_\infty = \infty$, for any $M_1 > 0$ satisfying

$$\frac{3M_1(\alpha-1)^2}{64\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds \geqslant 1,$$

there exists $\rho_3^* > \rho_2 > 0$ such that $g(y) > M_1 y$ for $y \geqslant \rho_3^*$. Let

$$\rho_3 = \frac{16\rho_3^*}{3(\alpha-1)} + \rho_2.$$

Then for any $y \in K \cap \partial\Omega_{\rho_3}$, we have

$$y(t) \geqslant \frac{3(\alpha-1)}{16}\|y\| = \frac{3(\alpha-1)}{16}\rho_3 > \rho_3^*, \quad t \in \left[\frac{1}{4},\frac{3}{4}\right].$$

Consequently,

$$\begin{aligned}\|T^*y\| &\geqslant (T^*y)(\frac{1}{2})\\ &= \int_0^1 G^*(\frac{1}{2},s)f(s,s^{\alpha-2}y(s))ds\\ &\geqslant \frac{\alpha-1}{4\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_1(s)g(y(s))ds\\ &\geqslant \frac{M_1(\alpha-1)}{4\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_1(s)y(s)ds\\ &\geqslant \frac{3M_1(\alpha-1)^2}{64\Gamma(\alpha)}\|y\|\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds\\ &\geqslant \|y\|,\end{aligned}$$

which implies that $\|T^*y\| \geqslant \|y\|$ for any $y \in K \cap \partial\Omega_{\rho_3}$.

Therefore T^* has a fixed point $y(t)$ in $K \cap (\overline{\Omega}_{\rho_3} \setminus \Omega_{\rho_2})$. Clearly, system (1.1) has a positive solution $u(t) = t^{\alpha-2}y(t)$. So the proof is completed. □

Theorem 3.5. *Assume that the following two conditions hold:*

(P5) *there exists an $r_1 > 0$ such that $g(y) \geqslant M_1 r_1$ for $\frac{3(\alpha-1)}{16}r_1 \leqslant y \leqslant r_1$;*

(P6) *there exists an $r_2 > 0$ such that $g(y) \leqslant M_2 r_2$ for $0 < y \leqslant r_2$,* *where*

$$M_1 = \left[\frac{\alpha-1}{4\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds\right]^{-1},$$

$$M_2 = \left[\frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds\right]^{-1}.$$

Then system (1.1) *has at least one positive solution.*

Proof. Without loss of generality, we can assume that $r_1 > r_2$. By (P5), Lemma 2.7, and (3.1), for any $y \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}\|T^*y\| \geqslant (T^*y)(\frac{1}{2}) &= \int_0^1 G^*(\frac{1}{2}, s)f(s, s^{\alpha-2}y(s))ds \\ &\geqslant \frac{\alpha-1}{4\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)g(y(s))ds \\ &\geqslant \frac{M_1 r_1(\alpha-1)}{4\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds \\ &= r_1 = \|y\|,\end{aligned}$$

which leads to $\|T^*y\| \geqslant \|y\|$ for any $y \in K \cap \partial\Omega_{r_1}$.

On the other hand, by (P6), one has

$$\begin{aligned}(T^*y)(t) &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s, s^{\alpha-2}y(s))ds \\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)g(y(s))ds \\ &\leqslant \frac{M_2 r_2}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \\ &= r_2 = \|y\|,\end{aligned}$$

which yields $\|T^*y\| \leqslant \|y\|$ for any $y \in K \cap \partial\Omega_{r_2}$.

Therefore, from Theorem 2.9, T^* has a fixed point in $K \cap (\overline{\Omega_{r_1}} \setminus \Omega_{r_2})$. Further, we can obtain that system (1.1) has at least one positive solution. The proof is completed. □

Theorem 3.6. *Assume that* (P2), (P3), *and* (P5) *hold. Then system* (1.1) *has at least two positive solutions.*

Proof. Firstly, by (P3), for any $\varepsilon > 0$ satisfying

$$\frac{\varepsilon}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \leqslant 1,$$

there exists $\rho_4 \in (0, r_1)$ such that $g(y) \leqslant \varepsilon y$ for $y \leqslant \rho_4$.

Then by Lemma 2.7 and (3.1), we have

$$\begin{aligned}(T^*y)(t) &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s, s^{\alpha-2}y(s))ds \\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)g(y(s))ds\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\varepsilon}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)y(s)ds\\
&\leqslant \|y\|,
\end{aligned}
$$

which shows that $\|T^*y\| \leqslant \|y\|$ for $y \in K \cap \partial\Omega_{\rho_4}$.

Secondly, in view of (P2), for any $\varepsilon > 0$ satisfying

$$\frac{\varepsilon}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds \leqslant \frac{1}{2},$$

there exists an $M_2 > 0$, such that $g(y) \leqslant \varepsilon y$ for $y > M_2$. Let

$$\rho_5 > M_2 + r_1 + \frac{1}{\varepsilon}\max_{y\in[0,M_2]}\{g(y)\}.$$

Then we have

$$
\begin{aligned}
(T^*y)(t) &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,s^{\alpha-2}y(s))ds\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)g(y(s))ds\\
&\leqslant \frac{1}{\Gamma(\alpha)}\left\{\int_{I_1} s(1-s)^{\alpha-1}q_2(s)\max\{g(y(s))\}ds + \int_{I_2} s(1-s)^{\alpha-1}q_2(s)\varepsilon y(s)ds\right\}\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)\max\{g(y(s))\}ds + \frac{\varepsilon}{\Gamma(\alpha)}\|y\|\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds\\
&\leqslant \frac{1}{2\varepsilon}\varepsilon\rho_5 + \frac{\|y\|}{2}\\
&= \|y\|,
\end{aligned}
$$

where $I_1 = \{s \in [0,1] \mid 0 \leqslant y(s) \leqslant M_2\}$, $I_2 = \{s \in [0,1] \mid y(s) > M_2\}$. Hence, $\|T^*y\| \leqslant \|y\|$ for $y \in K \cap \partial\Omega_{\rho_5}$.

Finally, set $\Omega_{r_1} = \{y \in C[0,1] \mid \|y\| < r_1\}$. Then, by (P5), we get

$$
\begin{aligned}
\|T^*y\| &\geqslant (T^*y)(\frac{1}{2})\\
&= \int_0^1 G^*(\frac{1}{2},s)f(s,s^{\alpha-2}y(s))ds\\
&\geqslant \frac{\alpha-1}{4\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)g(y(s))ds\\
&\geqslant \frac{M_1(\alpha-1)}{4\Gamma(\alpha)}r_1\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds\\
&= r_1 = \|y\|,
\end{aligned}
$$

which yields $\|T^*y\| \geqslant \|y\|$ for any $y \in K \cap \partial\Omega_{r_1}$.

Therefore, T^* has a fixed point y_1 in $\overline{\Omega}_{r_1} \setminus \Omega_{\rho_4}$, and a fixed point y_2 in $\overline{\Omega}_{\rho_5} \setminus \Omega_{r_1}$. One can easily see that both $u_1 = t^{\alpha-2}y_1$ and $u_2 = t^{\alpha-2}y_2$, $t \in [0,1]$ are positive solutions of system (1.1). The proof is completed. □

Theorem 3.7. *Assume that* (P1), (P4), *and* (P6) *hold. Then system* (1.1) *has at least two positive solutions.*

Proof. In view of (P1), for any $M_3 > 0$ satisfying

$$\frac{M_3(\alpha-1)^2}{4\Gamma(\alpha)}\int_0^1 s^2(1-s)^{\alpha}q_1(s)ds \geqslant 1,$$

there exists a $\rho_6 \in (0, r_2)$ such that $g(y) \geqslant M_3 y$ for $y \leqslant \rho_6$. Then by Lemma 2.7 and (3.1), we obtain

$$\begin{aligned}
\|T^*y\| &\geqslant (T^*y)(\frac{1}{2})\\
&= \int_0^1 G^*(\frac{1}{2}, s)f(s, s^{\alpha-2}y(s))ds\\
&\geqslant \frac{\alpha-1}{4\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_1(s)g(y(s))ds\\
&\geqslant \frac{M_3(\alpha-1)}{4\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_1(s)y(s)ds\\
&\geqslant \frac{M_3(\alpha-1)^2}{4\Gamma(\alpha)}\|y\|\int_0^1 s^2(1-s)^{\alpha}q_1(s)ds\\
&\geqslant \|y\|,
\end{aligned}$$

which yields $\|T^*y\| \geqslant \|y\|$ for any $y \in K \cap \partial\Omega_{\rho_6}$. In addition, by (P4), for any $M_4 > 0$ satisfying

$$\frac{3M_4(\alpha-1)^2}{64\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds \geqslant 1,$$

there exists $\rho_7^* > 0$ such that $g(y) > M_4 y$ for $y \geqslant \rho_7^*$. Take

$$\rho_7 = \frac{16\rho_7^*}{3(\alpha-1)} + \rho_2,$$

then for any $y \in K \cap \partial\Omega_{\rho_7}$, we have

$$y(t) \geqslant \frac{3(\alpha-1)}{16}\|y\| = \frac{3(\alpha-1)}{16}\rho_7 > \rho_7^*, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Furthermore,

$$\begin{aligned}
\|T^*y\| &\geqslant (T^*y)(\frac{1}{2})\\
&\geqslant \frac{\alpha-1}{4\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_1(s)g(y(s))ds\\
&\geqslant \frac{\alpha-1}{4\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)g(y(s))ds\\
&\geqslant \frac{M_4(\alpha-1)}{4\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)y(s)ds\\
&\geqslant \frac{3M_4(\alpha-1)^2}{64\Gamma(\alpha)}\|y\|\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds\\
&\geqslant \|y\|,
\end{aligned}$$

which implies that $\|T^*y\| \geqslant \|y\|$ for any $y \in K \cap \partial\Omega_{\rho_7}$.

Finally, by (P6), we have

$$\begin{aligned}(T^*y)(t) &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,s^{\alpha-2}y(s))ds\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)g(y(s))ds\\ &\leqslant \frac{M_2r_2}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}q_2(s)ds\\ &= r_2 = \|y\|,\end{aligned}$$

which yields $\|T^*y\| \leqslant \|y\|$ for any $y \in K \cap \partial\Omega_{r_2}$.

Therefore, T^* has a fixed point y_1 in $\overline{\Omega}_{r_2} \setminus \Omega_{\rho_6}$, and a fixed point y_2 in $\overline{\Omega}_{\rho_7} \setminus \Omega_{r_2}$. Thus, system (1.1) has at least two positive solutions $u_1 = t^{\alpha-2}y_1$, $u_2 = t^{\alpha-2}y_2$, $t \in [0,1]$. The proof is completed. □

Theorem 3.8. *Assume that*

$$\text{(P7)}\quad g^0 = \alpha_1 \in [0, M_2) \quad \textit{and} \quad \text{(P8)}\quad g_\infty = \beta_1 \in \left(\frac{16}{3(\alpha-1)}M_1, \infty\right)$$

hold, where M_1, M_2 are defined in Theorem 3.5. Then system (1.1) has at least one positive solution.

Proof. By (P7), for $\varepsilon = M_2 - \alpha_1 > 0$, there exists a sufficiently small $\hat{r}_2 > 0$ such that

$$\max\frac{g(y)}{y} < \alpha_1 + \varepsilon = M_2, \quad \text{for} \quad y \leqslant \hat{r}_2,$$

which yields $g(y) < M_2y \leqslant M_2\hat{r}_2$. Hence, the condition (P6) is satisfied.

By (P8), for $\varepsilon = \beta_1 - \frac{16}{3(\alpha-1)}M_1 > 0$, there exists a sufficiently large $r_1 > 0$, such that

$$\min\frac{g(y)}{y} > \beta_1 - \varepsilon = \frac{16}{3(\alpha-1)}M_1, \quad \text{for} \quad y \geqslant \frac{3(\alpha-1)}{16}r_1.$$

Thus, when $\frac{3(\alpha-1)}{16}r_1 \leqslant y \leqslant r_1$, we have

$$g(y) > \frac{16}{3(\alpha-1)}M_1y \geqslant \frac{16}{3(\alpha-1)}M_1 \cdot \frac{3(\alpha-1)}{16}r_1 = M_1r_1,$$

which implies the condition (P5) hold. By Theorem 3.5, we complete the proof. □

Theorem 3.9. *Assume that*

$$\text{(P9)}\quad g_0 = \alpha_2 \in \left(\frac{16}{3(\alpha-1)}M_1, \infty\right) \quad \textit{and} \quad \text{(P10)}\quad g^\infty = \beta_2 \in [0, M_2)$$

hold, where M_1, M_2 are defined in Theorem 3.5. Then system (1.1) has at least one positive solution.

Proof. By (P9), for $\varepsilon = \alpha_2 - \frac{16}{3(\alpha-1)}M_1 > 0$, there exists a sufficiently small $\hat{r}_1 > 0$ such that

$$\min\frac{g(y)}{y} > \alpha_2 - \varepsilon = \frac{16}{3(\alpha-1)}M_1, \quad \text{for} \quad y \leqslant \hat{r}_1.$$

Thus, when $\frac{3(\alpha-1)}{16}\hat{r}_1 \leqslant y \leqslant \hat{r}_1$, one has,

$$g(y) > \frac{16}{3(\alpha-1)}M_1y \geqslant M_1\hat{r}_1,$$

which yields the condition (P5) holds. In view of (P10), for $\varepsilon = M_2 - \beta_2 > 0$, there exists a sufficiently large $r > 0$ such that

$$\max \frac{g(y)}{y} < \beta_2 + \varepsilon = M_2, \quad \text{for} \quad y > r.$$

In the following, we will show that (P6) holds and the discussion is divided into two cases.

Case one: Suppose that $g(y)$ is unbounded, then exists $g^* \in C([0,+\infty),[0,+\infty))$ such that $g(y) \leqslant g^*(y^*)$ for $y \leqslant y^*$ and when $y^* > r$, $\max \frac{g^*(y^*)}{y^*} < M_2$. If we choose $y^* = r_2 > r$, we will get

$$g(y) \leqslant g^*(y^*) < M_2 y^* = M_2 r_2 \text{ for } y \leqslant y^* = r_2,$$

which yields the condition (P6) holds.

Case two: Suppose that $g(y)$ is bounded, there exists an $M > 0$, such that for any y, we have $g(y) \leqslant M$. In this case, taking sufficiently large $r_2 \geqslant \frac{M}{M_2}$, then $g(y) \leqslant M \leqslant M_2 r_2$ for $0 < y < r_2$, which implies the condition (P6) holds.

Therefore, by Theorem 3.5 we complete the proof. □

Theorem 3.10. *Assume that* (P6), (P8), *and* (P9) *hold. Then system* (1.1) *has at least two positive solutions.*

Proof. From (P8) and the proof of Theorem 3.8, we know that there exists a sufficiently large $r_1 > r_2$ such that

$$g(y) > M_1 r_1, \ \text{ for } \ \frac{3(\alpha-1)}{16} r_1 \leqslant y \leqslant r_1.$$

In view of (P9) and the proof of Theorem 3.9, we see that there exists a sufficiently small $r_1^* \in (0, r_2)$ such that

$$g(y) > M_1 r_1^*, \ \text{ for } \ \frac{3(\alpha-1)}{16} r_1^* \leqslant y \leqslant r_1^*.$$

Noting that (P6) is valid, then from the proof of Theorem 3.5, we know T^* has at least two fixed points. Thus, system (1.1) has at least two positive solutions. The proof is completed. □

Theorem 3.11. *Assume that* (P5), (P7), *and* (P10) *hold. Then system* (1.1) *has at least two positive solutions.*

Proof. From (P7) and the proof of Theorem 3.8, we know that there exists a sufficiently small $r_2 \in (0, r_1)$ such that

$$g(y) \leqslant M_2 r_2, \ \text{ for } \ 0 < y \leqslant r_2.$$

In view of (P10) and the proof of Theorem 3.9, we see that there exists a sufficiently large $r_2^* > r_1$ such that

$$g(y) \leqslant M_2 r_2^*, \ \text{ for } \ 0 < y \leqslant r_2^*.$$

Noting that (P5) is also satisfied, then from the proof of Theorem 3.5, we know T^* has at least two fixed points. Thus, system (1.1) has at least two positive solutions. The proof is completed. □

Theorem 3.12. *Assume that* (P1) *and* (P10) *hold. Then system* (1.1) *has at least one positive solution.*

Proof. In view of (P1), we know from the proof of Theorem 3.3 that, for any $y \in K \cap \partial\Omega_{\rho_0}$, $\|T^* y\| \geqslant \|y\|$.
By (P10), it follows from the proof of Theorem 3.9, there exists a sufficiently large $r_2 > \rho_0$ such that $g(y) \leqslant M_2 r_2$ for $0 < y \leqslant r_2$ and $\|T^* y\| \leqslant \|y\|$. This completes the proof. □

Similar to Theorem 3.12, one immediately has the following consequences.

Theorem 3.13. *Assume that* (P2) *and* (P9) *hold. Then system* (1.1) *has at least one positive solution.*

Theorem 3.14. *Assume that* (P3) *and* (P8) *hold. Then system* (1.1) *has at least one positive solution.*

Theorem 3.15. *Assume that* (P4) *and* (P7) *hold. Then system* (1.1) *has at least one positive solution.*

Theorem 3.16. *Assume that* (P1), (P6), *and* (P8) *hold. Then system* (1.1) *has at least two positive solutions.*

Proof. In view of (P1), we know from the proof of Theorem 3.3 that $\|T^*y\| \geqslant \|y\|$ for any $y \in K \cap \partial\Omega_{\rho_0}$.

Since (P8), from the proof of Theorem 3.8, there exists a sufficiently large $r_1 > r_2$ such that $g(y) \geqslant M_1 r_1$ for $\frac{3(\alpha-1)}{16} r_1 \leqslant y \leqslant r_1$.

For any $y \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned}\|T^*y\| &\geqslant (T^*y)(\frac{1}{2})\\ &= \int_0^1 G^*(\frac{1}{2}, s) f(s, s^{\alpha-2} y(s)) ds\\ &\geqslant \frac{\alpha-1}{4\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) g(y(s)) ds\\ &\geqslant \frac{M_1(\alpha-1)}{4\Gamma(\alpha)} r_1 \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1} q_1(s) ds\\ &= r_1 = \|y\|.\end{aligned}$$

Further, from the proof of Theorem 3.5, for any $y \in K \cap \partial\Omega_{r_2}$, we have $\|T^*y\| \leqslant \|y\|$. Therefore, system (1.1) has at least two positive solutions. The proof is completed. □

The following statements are immediately obtained by applying similar arguments as used in the proof of Theorem 3.16.

Theorem 3.17. *Assume that* (P2), (P5), *and* (P7) *hold. Then system* (1.1) *has at least two positive solutions.*

Theorem 3.18. *Assume that* (P3), (P5), *and* (P10) *hold. Then system* (1.1) *has at least two positive solutions.*

Theorem 3.19. *Assume that* (P4), (P6), *and* (P9) *hold. Then system* (1.1) *has at least two positive solutions.*

4. Nonexistence of positive solutions of (1.1)

In this section, we will explore the nonexistence of positive solutions of system (1.1).

Theorem 4.1. *Assume that* (P8), (P9), *and the following condition hold*

$$\text{(P11)} \qquad \min_{\hat{r}_1 < y < \frac{3(\alpha-1)}{16} r_1} \frac{g(y)}{y} \in \left(\frac{16}{3(\alpha-1)} M_1, \infty \right),$$

where r_1, $\hat{r}_1$ *are defined as the proof of Theorems* 3.6 *and* 3.7, *respectively. Then* (1.1) *has no positive solution.*

Proof. From (P8) and the proof of Theorem 3.8, it follows that there exists a sufficiently large $r_1 > \hat{r}_1$ such that $g(y) \geqslant \frac{16}{3(\alpha-1)} M_1 y$ for $y \geqslant \frac{3(\alpha-1)}{16} r_1$.

From (P9), and the proof of Theorem 3.9, it follows that there exists a sufficiently small $\hat{r}_1 > 0$ such that $g(y) \geqslant \frac{16}{3(\alpha-1)} M_1 y$ for $y \leqslant \hat{r}_1$.

By (P11), we can obtain $g(y) \geqslant \frac{16}{3(\alpha-1)} M_1 y$ for any y.

If T^* has a fixed point y, then

$$\|y\| = \|T^*y\| \geqslant (T^*y)(\frac{1}{2}) = \int_0^1 G^*(\frac{1}{2}, s) f(s, s^{\alpha-2} y(s)) ds$$

$$
\begin{aligned}
&> \frac{\alpha-1}{4\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)g(y(s))ds \\
&\geqslant \frac{\alpha-1}{4\Gamma(\alpha)}\cdot\frac{16}{3(\alpha-1)}M_1\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)y(s)ds \\
&\geqslant \frac{\alpha-1}{4\Gamma(\alpha)}\cdot\frac{16}{3(\alpha-1)}M_1\cdot\frac{3}{16}(\alpha-1)\|y\|\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}q_1(s)ds \\
&= \|y\|,
\end{aligned}
$$

which is a contradiction. The claim is valid. □

The next consequence is presented below whose proof is similar to that of Theorem 4.1, and therefore is omitted.

Theorem 4.2. *Assume that*(P7), (P10), *and the following condition hold*

$$(\mathrm{P12}) \qquad \max_{\hat{r}_2<y<r_2}\frac{g(y)}{y}\in[0,M_2),$$

where $\hat{r}_2$, r_2 are defined as the proof of Theorems 3.8 and 3.9, respectively. Then system (1.1) has no positive solution.

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