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On some inequalities for generalized s-convex functions and applications on fractal sets

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Abstract

The authors present some new inequalities of generalized Hermite-Hadamard's type for the class of functions whose second local fractional derivatives of order α in absolute value at certain powers are generalized s-convex functions in the second sense. Moreover, some applications are given.

Keywords: s-convex functions, fractal space, local fractional derivative.
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1. Introduction and preliminaries

Let $f\colon U \subset \mathbb{R} \longrightarrow \mathbb{R}^{\alpha}$. For any $x_1, x_2 \in U$ and $\gamma \in [0,1]$ if the inequality

$$f(\gamma x_1 + (1-\gamma)x_2) \leqslant \gamma^{\alpha} f(x_1) + (1-\gamma)^{\alpha} f(x_2)$$

holds, then f is called a generalized convex function on U [14]. In $\alpha = 1$, we have convex function, which means that if P_1, P_2, and P_3 are three distinct points on the graph of f with P_2 between P_1 and P_3, then P_2 is on or below the chord P_1P_3 [7].

The convexity of functions plays a significant role in many fields, for example, in biological system, economy, optimization, and so on [6, 17].

Let $f \in {}_{a_1}I_{a_2}^{(\alpha)}$ be a generalized convex function on $[a_1, a_2]$ with $a_1 < a_2$. Then,

$$f\left(\frac{a_1+a_2}{2}\right) \leqslant \frac{\Gamma(1+\alpha)}{(a_2-a_1)^{\alpha}} \ {}_{a_1}I_{a_2}^{(\alpha)} f(x) \leqslant \frac{f(a_1)+f(a_2)}{2^{\alpha}}. \tag{1.1}$$

is known as generalized Hermite-Hadmard's inequality [14]. Many authers paid attention to the study of

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generalized Hermite-Hadmard's inequality and generalized convex function, see [9, 13]. If $\alpha = 1$ in (1.1), then [5]

$$f\left(\frac{a_1+a_2}{2}\right) \leqslant \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)dx \leqslant \frac{f(a_1)+f(a_2)}{2},$$

which is known as classical Hermite-Hadamard inequality, for more properties about this inequality we refer the interested readers to [4, 8].

In [12], Mo and Sui introduced the definitions of two kinds of generalized s-convex functions on fractal sets such as follows.

Definition 1.1.

(i) A function $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}^\alpha$, is called a generalized s-convex $(0 < s < 1)$ in the first sense if

$$f(\gamma_1 x_1 + \gamma_2 x_2) \leqslant \gamma_1^{s\alpha} f(x_1) + \gamma_2^{s\alpha} f(x_2), \tag{1.2}$$

$\forall x_1, x_2 \in \mathbb{R}_+$, $\forall \gamma_1, \gamma_2 \geqslant 0$ with $\gamma_1^s + \gamma_2^s = 1$.

(ii) A function $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}^\alpha$, is called a generalized s-convex $(0 < s < 1)$ in the second sense if (1.2) holds $\forall x_1, x_2 \in \mathbb{R}_+$, $\forall \gamma_1, \gamma_2 \geqslant 0$ with $\gamma_1 + \gamma_2 = 1$.

In the same paper, [12], Mo and Sui proved that all functions which are generalized s-convex in the second sense, for $s \in (0, 1)$, are non-negative.

If $\alpha = 1$ in Definition 1.1, then we have the classical s-convex functions in the first sense (second sense) see [5].

Also, in [5], Dragomir and Fitzatrick demonstrated a variation of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1.2. *Assume that $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a s-convex function in the second sense, $0 < s < 1$ and $a_1, a_2 \in \mathbb{R}_+$, $a_1 < a_2$. If $f \in L^1([a_1, a_2])$, then*

$$2^{s-1} f\left(\frac{a_1+a_2}{2}\right) \leqslant \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)dx \leqslant \frac{f(a_1)+f(a_2)}{s+1}. \tag{1.3}$$

If we set $c = \dfrac{1}{s+1}$, then it is the best possible in the second inequality in (1.3).

A variation of generalized Hadamard's inequality holds for generalized s-convex functions in the second sense.

Theorem 1.3. *Assume that $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+^\alpha$ is a generalized s-convex function in the second sense where $0 < s < 1$ and $a_1, a_2 \in \mathbb{R}_+$ with $a_1 < a_2$. If $f \in L^1([a_1, a_2])$, then*

$$2^{\alpha(s-1)} f\left(\frac{a_1+a_2}{2}\right) \leqslant \frac{\Gamma(1+\alpha)}{(a_2-a_1)^\alpha}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x) \leqslant \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\left(f(a_1)+f(a_2)\right). \tag{1.4}$$

If we set $c = \dfrac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}$ for $s \in (0, 1)$, then it is the best possible in the second inequality (1.4), the proof was started by Kiliçman and Saleh [10].

There are many researchers studied the properties of functions on fractal space and constructed many kinds of fractional calculus by using different approaches; see [3, 18, 21].

This work is organized as follows. After this introduction, in Section 2, we discuss some generalized Hermite-Hadamard type inequalities for the class of functions whose second local fractional derivatives in absolute value at certain powers are generalized s-convex functions in the second sense. Finally, some applications of the results from Section 2 for special mean are considered.

2. Main results

We need the following lemma to prove our main results:

Lemma 2.1. *Assume that* $f\colon [a_1, a_2] \subset \mathbb{R} \longrightarrow \mathbb{R}^\alpha$ *is a local fractional derivative of order* α *(*$f \in D_\alpha$*) on* (a_1, a_2) *with* $a_1 < a_2$*. If* $f^{(2\alpha)} \in C_\alpha[a_1, a_2]$*, then the following equality holds:*

$$\begin{aligned}
&\frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(a_2-a_1)^\alpha}\ {}_{a_1}I_{a_2}^{(\alpha)}f(x) - \frac{\Gamma(1+2\alpha)}{2^\alpha}f\left(\frac{a_1+a_2}{2}\right)\\
&= \frac{(a_2-a_1)^{2\alpha}}{16^\alpha}\Bigg[\ {}_0I_1^{(\alpha)}\gamma^{2\alpha}f^{(2\alpha)}\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1\right)\\
&\quad + {}_0I_1^{(\alpha)}(\gamma-1)^{2\alpha}f^{(2\alpha)}\left(\gamma a_2+(1-\gamma)\frac{a_1+a_2}{2}\right)\Bigg].
\end{aligned}$$

Proof. From the local fractional integration by parts, we get

$$\begin{aligned}
A_1 &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \gamma^{2\alpha}f^{(2\alpha)}\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1\right)(d\gamma)^\alpha\\
&= (\frac{2}{a_2-a_1})^\alpha f^{(\alpha)}\left(\frac{a_1+a_2}{2}\right) - \Gamma(1+2\alpha)\left(\frac{2}{a_2-a_1}\right)^{2\alpha}\gamma^\alpha f\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1\right)|_0^1\\
&\quad + \Gamma(1+2\alpha)\Gamma(1+\alpha)\left(\frac{2}{b-a}\right)^{2\alpha}\int_0^1 f\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1\right)(d\gamma)^\alpha\\
&= \left(\frac{2}{a_2-a_1}\right)^\alpha f^{(\alpha)}\left(\frac{a_1+a_2}{2}\right) - \Gamma(1+2\alpha)\left(\frac{2}{a_2-a_1}\right)^{2\alpha} f\left(\frac{a_1+a_2}{2}\right)\\
&\quad + \Gamma(1+2\alpha)\Gamma(1+\alpha)\left(\frac{2}{b-a}\right)^{2\alpha}\int_0^1 f\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1\right)(d\gamma)^\alpha.
\end{aligned}$$

Setting $x = \gamma\frac{a_1+a_2}{2} + (1-\gamma)a_1$, for $\gamma \in [0,1]$ and multiplying both sides in the last equation by $\frac{(a_2-a_1)^{2\alpha}}{16^\alpha}$, we get

$$\begin{aligned}
A_1 &= \frac{(a_2-a_1)^{2\alpha}}{16^\alpha}\,{}_0I_1^{(\alpha)}\gamma^{2\alpha}f^{(2\alpha)}\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1\right)\\
&= \frac{(a_2-a_1)^\alpha}{8^\alpha}f^{(\alpha)}\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)}{4^\alpha}f\left(\frac{a_1+a_2}{2}\right) + \frac{\Gamma(1+2\alpha)\Gamma(1+\alpha)}{2^\alpha(a_2-a_1)^\alpha}\int_{a_1}^{\frac{a_1+a_2}{2}} f(x)(dx)^\alpha.
\end{aligned}$$

By the similar way, also we have

$$\begin{aligned}
A_2 &= \frac{(a_2-a_1)^{2\alpha}}{16^\alpha}\,{}_0I_1^{(\alpha)}(\gamma-1)^{2\alpha}f^{(2\alpha)}\left(\gamma a_2+(1-\gamma)\frac{a_1+a_2}{2}\right)\\
&= -\frac{(a_2-a_1)^\alpha}{8^\alpha}f^{(\alpha)}\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)}{4^\alpha}f\left(\frac{a_1+a_2}{2}\right) + \frac{\Gamma(1+2\alpha)\Gamma(1+\alpha)}{2^\alpha(a_2-a_1)^\alpha}\int_{\frac{a_1+a_2}{2}}^{a_2} f(x)(dx)^\alpha.
\end{aligned}$$

Thus, adding A_1 and A_2, we get the desired result. □

Theorem 2.2. *Assume that* $f : U \subset [0,\infty) \longrightarrow \mathbb{R}^\alpha$ *such that* $f \in D_\alpha$ *on* $\mathrm{Int}(U)$ *(*$\mathrm{Int}(U)$ *is the interior of* U*) and* $f^{(2\alpha)} \in C_\alpha[a_1, a_2]$*, where* $a_1, a_2 \in U$ *with* $a_1 < a_1$*. If* $|f|$ *is generalized* s*-convex on* $[a_1, a_2]$ *for some fixed* $0 < s \leqslant 1$*, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}} f\left(\frac{a_1+a_2}{2}\right)-\frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{2^{\alpha}\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|+\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right.\right. \\
&\quad \left.\left.-2^{\alpha}\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\right]\left[\left|f^{(2\alpha)}(a_1)\right|+\left|f^{(2\alpha)}(a_2)\right|\right]\right\} \qquad (2.1)\\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{2^{\alpha(2-s)}\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}+\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right. \\
&\quad \left.-\frac{2^{\alpha}\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\right\}\left[\left|f^{(2\alpha)}(a_1)\right|+\left|f^{(2\alpha)}(a_2)\right|\right]. \qquad (2.2)
\end{aligned}$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}} f\left(\frac{a_1+a_2}{2}\right)-\frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left[{}_0I_1^{(\alpha)}\gamma^{2\alpha}\left|f^{(2\alpha)}(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1)\right|\right. \\
&\quad \left.+{}_0I_1^{(\alpha)}(\gamma-1)^{2\alpha}\left|f^{(2\alpha)}\left(\gamma a_2+(1-\gamma)\frac{a_1+a_2}{2}\right)\right|\right] \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\ {}_0I_1^{(\alpha)}\gamma^{2\alpha}\left[\gamma^{\alpha s}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|+(1-\gamma)^{\alpha s}\left|f^{(2\alpha)}(a_1)\right|\right] \\
&\quad +\frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\ {}_0I_1^{(\alpha)}(\gamma-1)^{2\alpha}\left[\gamma^{\alpha s}\left|f^{(2\alpha)}(a_2)\right|+(1-\gamma)^{\alpha s}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|\right] \\
&=\frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|\right. \\
&\quad \left.+\left[\frac{\Gamma(1+\alpha s)}{\Gamma(1+(s+1)\alpha)}-2^{\alpha}\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\right]\left|f^{(2\alpha)}(a_1)\right|\right\} \\
&\quad +\frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|\right. \\
&\quad +\left[\frac{\Gamma(1+\alpha s)}{\Gamma(1+(s+1)\alpha)}-2^{\alpha}\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\right]\left|f^{(2\alpha)}(a_2)\right| \\
&\quad \left.+\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|\right\} \\
&=\frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{2^{\alpha}\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|+\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right.\right. \\
&\quad \left.\left.-2^{\alpha}\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+3)\alpha)}\right]\left[\left|f^{(2\alpha)}(a_1)\right|+\left|f^{(2\alpha)}(a_2)\right|\right]\right\}.
\end{aligned}$$

This proves inequality (2.1). Since

$$2^{\alpha(s-1)}f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\leqslant\frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\left(f^{(2\alpha)}(a_1)+f^{(2\alpha)}(a_2)\right),$$

then

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}} f\left(\frac{a_1+a_2}{2}\right)-\frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{2^{\alpha}\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|+\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\left.-2^{\alpha}\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\right]\left[\left|f^{(2\alpha)}(a_1)\right|+\left|f^{(2\alpha)}(a_2)\right|\right]\Bigg\}\\
\leqslant&\ \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{2^{\alpha}\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\frac{2^{-\alpha(s-1)}\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\left|f^{(2\alpha)}(a_1)\right|+\left|f^{(2\alpha)}(a_2)\right|\right]\right.\\
&\left.+\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{2^{\alpha}\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\right]\left[\left|f^{(2\alpha)}(a_1)\right|+\left|f^{(2\alpha)}(a_2)\right|\right]\right\}\\
=&\ \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{2^{\alpha(2-s)}\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}+\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right.\\
&\left.-\frac{2^{\alpha}\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}\left[\left|f^{(2\alpha)}(a_1)\right|+\left|f^{(2\alpha)}(a_2)\right|\right]\right\}.
\end{aligned}
$$

Thus, we get the inequality (2.2) and the proof is complete. □

Remark 2.3.

1. When $\alpha=1$, Theorem 2.2 reduces to Theorem 2 in [15].
2. If $s=1$ in Theorem 2.2, then

$$
\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}}f\left(\frac{a_1+a_2}{2}\right)-\frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)}f(x)\right|\\
&\leqslant\frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{2^{\alpha}\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|+\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right.\right.\\
&\left.\left.\quad-2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right]\left[\left|f^{(2\alpha)}(a_1)\right|+\left|f^{(2\alpha)}(a_2)\right|\right]\right\}\\
&\leqslant\frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left\{\frac{2^{\alpha}\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\frac{[\Gamma(1+\alpha)]^2}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right.\\
&\left.\quad-\frac{2^{\alpha}\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right\}\left[\left|f^{(2\alpha)}(a_1)\right|+\left|f^{(2\alpha)}(a_2)\right|\right].
\end{aligned}
$$

3. If $s=1$ and $\alpha=1$ in Theorem 2.2, then

$$
\begin{aligned}
\left|f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(x)dx\right|&\leqslant\frac{(a_2-a_1)^2}{192}\left\{6\left|f''\left(\frac{a_1+a_2}{2}\right)\right|+\left|f''(a_1)\right|+\left|f''(a_2)\right|\right\}\\
&\leqslant\frac{(a_2-a_1)^2}{48}\left\{\left|f''(a_1)\right|+\left|f''(a_2)\right|\right\}.
\end{aligned}
$$

We give a new upper bound of the left generalized Hadamard's inequality for generalized s-convex functions in the following theorem:

Theorem 2.4. *Assume that* $f:U\subset[0,\infty)\longrightarrow\mathbb{R}^{\alpha}$ *such that* $f\in D_{\alpha}$ *on* $\mathrm{Int}(U)$ *and* $f^{(2\alpha)}\in C_{\alpha}[a_1,a_2]$, *where* $a_1,a_2\in U$ *with* $a_1<a_1$. *If* $|f^{(2\alpha)}|^{p_2}$ *is generalized s-convex on* $[a_1,a_2]$ *for some fixed* $0<s\leqslant 1$ *and* $p_2>1$ *with* $\frac{1}{p_1}+\frac{1}{p_2}=1$, *then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}}f\left(\frac{a_1+a_2}{2}\right)-\frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)}f(x)\right|\\
&\leqslant\frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right]^{\frac{1}{p_2}}\left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}}\\
&\quad\times\left[\left(\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|^{p_2}+\left|f^{(2\alpha)}(a_1)\right|^{p_2}\right)^{\frac{1}{p_2}}+\left(\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|^{p_2}+\left|f^{(2\alpha)}(a_2)\right|^{p_2}\right)^{\frac{1}{p_2}}\right].
\end{aligned}
\tag{2.3}
$$

Proof. Let $p_1 > 1$, then from Lemma 2.1 and using generalized Hölder's inequality [20], we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^\alpha} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(a_2-a_1)^\alpha}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^\alpha}\left\{ {}_0I_1^{(\alpha)} \gamma^{2\alpha}\left|f^{(2\alpha)}\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1\right)\right|\right. \\
&\quad \left. + {}_0I_1^{(\alpha)}(\gamma-1)^{2\alpha}\left|f^{(2\alpha)}\left(\gamma a_2+(1-\gamma)\frac{a_1+a_2}{2}\right)\right|\right\} \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^\alpha}\left({}_0I_1^{(\alpha)}\gamma^{2p_1\alpha}\right)^{\frac{1}{p_1}}\left({}_0I_1^{(\alpha)}\left|f^{(2\alpha)}\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1\right)\right|^{p_2}\right)^{\frac{1}{p_2}} \\
&\quad + \frac{(a_2-a_1)^{2\alpha}}{16^\alpha}\left({}_0I_1^{(\alpha)}(1-\gamma)^{2p_1\alpha}\right)^{\frac{1}{p_1}}\left({}_0I_1^{(\alpha)}\left|f^{(2\alpha)}\left(\gamma a_2+(1-\gamma)\frac{a_1+a_2}{2}\right)\right|^{p_2}\right)^{\frac{1}{p_2}}.
\end{aligned}$$

Since $\left|f^{(2\alpha)}\right|^{p_2}$ is generalized s-convex, then

$$\begin{aligned}
&{}_0I_1^{(\alpha)}\left|f^{(2\alpha)}\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)a_1\right)\right|^{p_2} \\
&\qquad \leqslant \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|^{p_2} + \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left|f^{(2\alpha)}(a_1)\right|^{p_2}, \\
&{}_0I_1^{(\alpha)}\left|f^{(2\alpha)}\left(\gamma a_2+(1-\gamma)\frac{a_1+a_2}{2}\right)\right|^{p_2} \\
&\qquad \leqslant \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left|f^{(2\alpha)}(a_2)\right|^{p_2} + \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|^{p_2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^\alpha} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(a_2-a_1)^\alpha}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^\alpha}\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right]^{\frac{1}{p_2}}\left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}} \\
&\quad \times\left\{\left[\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|^{p_2}+\left|f^{(2\alpha)}(a_1)\right|^{p_2}\right]^{\frac{1}{p_2}}+\left[\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|^{p_2}+\left|f^{(2\alpha)}(a_2)\right|^{p_2}\right]^{\frac{1}{p_2}}\right\}.
\end{aligned}$$

The proof is complete. □

Remark 2.5. If $s = 1$ in Theorem 2.4, then

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^\alpha} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(a_2-a_1)^\alpha}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]^{\frac{1}{p_2}}\left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}} \\
&\quad \times\left\{\left[\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|^{p_2}+\left|f^{(2\alpha)}(a_1)\right|^{p_2}\right]^{\frac{1}{p_2}}+\left[\left|f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right)\right|^{p_2}+\left|f^{(2\alpha)}(a_2)\right|^{p_2}\right]^{\frac{1}{p_2}}\right\}.
\end{aligned} \tag{2.4}$$

Corollary 2.6. *Assume that* $f : U \subset [0, \infty) \longrightarrow \mathbb{R}^\alpha$ *such that* $f \in D_\alpha$ *on* $\mathrm{Int}(U)$ *and* $f^{(2\alpha)} \in C_\alpha[a_1, a_2]$, *where* $a_1, a_2 \in U$ *with* $a_1 < a_1$. *If* $|f^{(2\alpha)}|^{p_2}$ *is generalized s-convex on* $[a_1, a_2]$ *for some fixed* $0 < s \leqslant 1$ *and* $p_2 > 1$ *with* $\frac{1}{p_1} + \frac{1}{p_2} = 1$, *then the following inequality holds:*

$$\left|\frac{\Gamma(1+2\alpha)}{2^\alpha} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^\alpha(a_2-a_1)^\alpha}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right|$$

$$\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}} \frac{[\Gamma(1+s\alpha)]^{\frac{1}{p_2}}}{[\Gamma(1+(s+1)\alpha)]^{\frac{2}{p_2}}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}} \left\{\left[\left(2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha) +\Gamma(1+(s+1)\alpha)\right)^{\frac{1}{p_2}} + 2^{\frac{\alpha(1-s)}{p_2}} [\Gamma(1+\alpha)]^{\frac{1}{p_2}} [\Gamma(1+\alpha)]^{\frac{1}{p_2}}\right] \left[\left|f^{(2\alpha)}(a_1)\right| + \left|f^{(2\alpha)}(a_2)\right|\right]\right\}.$$

Proof. Since $\left|f^{(2\alpha)}\right|^{p_2}$ is generalized s-convex, then

$$2^{\alpha(s-1)} f^{(2\alpha)}\left(\frac{a_1+a_2}{2}\right) \leqslant \frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\left(f^{(2\alpha)}(a_1)+f^{(2\alpha)}(a_2)\right).$$

Hence using (2.3), we get

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}} \left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right]^{\frac{1}{p_2}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}} \\
&\quad \times \left\{\left[2^{\alpha(1-s)}\frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\left(\left|f^{(2\alpha)}(a_1)\right|^{p_2} + \left|f^{(2\alpha)}(a_2)\right|^{p_2}\right) + \left|f^{(2\alpha)}(a_1)\right|^{p_2}\right]^{\frac{1}{p_2}}\right. \\
&\quad \left. + \left[2^{\alpha(1-s)}\frac{\Gamma(1+s\alpha)\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}\left(\left|f^{(2\alpha)}(a_1)\right|^{p_2} + \left|f^{(2\alpha)}(a_2)\right|^{p_2}\right) + \left|f^{(2\alpha)}(a_2)\right|^{p_2}\right]^{\frac{1}{p_2}}\right\} \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}} \frac{[\Gamma(1+s\alpha)]^{\frac{1}{p_2}}}{[\Gamma(1+(s+1)\alpha)]^{\frac{2}{p_2}}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}} \\
&\quad \times \left\{\left[\left(2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha)+\Gamma(1+(s+1)\alpha)\right)|f^{(2\alpha)}(a_1)|^{p_2}\right.\right. \\
&\quad \left. +2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha)|f^{(2\alpha)}(a_2)|^{p_2}\right]^{\frac{1}{p_2}} + \left[2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha)|f^{(2\alpha)}(a_1)|^{p_2}\right. \\
&\quad \left.\left. + \left(2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha)+\Gamma(1+(s+1)\alpha)\right)|f^{(2\alpha)}(a_2)|^{p_2}\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

and since $\sum_{i=1}^{k}(x_i+y_i)^{\alpha n} \leqslant \sum_{i=1}^{k} x_i^{\alpha n} + \sum_{i=1}^{k} y_i^{\alpha n}$ for $0<n<1, x_i, y_i \geqslant 0$, for all $1 \leqslant i \leqslant k$, then we have

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}} \frac{[\Gamma(1+s\alpha)]^{\frac{1}{p_2}}}{[\Gamma(1+(s+1)\alpha)]^{\frac{2}{p_2}}} \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}} \\
&\quad \times \left\{\left[\left(2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha)+\Gamma(1+(s+1)\alpha)\right)^{\frac{1}{p_2}}\left|f^{(2\alpha)}(a_1)\right|\right.\right. \\
&\quad \left. +2^{\frac{\alpha(1-s)}{p_2}}[\Gamma(1+s\alpha)]^{\frac{1}{p_2}}[\Gamma(1+\alpha)]^{\frac{1}{p_2}}\left|f^{(2\alpha)}(a_2)\right|\right] \\
&\quad + \left[2^{\frac{\alpha(1-s)}{p_2}}[\Gamma(1+s\alpha)]^{\frac{1}{p_2}}[\Gamma(1+\alpha)]^{\frac{1}{p_2}}\left|f^{(2\alpha)}(a_1)\right|\right. \\
&\quad \left.\left. + \left(2^{\alpha(1-s)}\Gamma(1+s\alpha)\Gamma(1+\alpha)+\Gamma(1+(s+1)\alpha)\right)^{\frac{1}{p_2}}\left|f^{(2\alpha)}(a_2)\right|\right]\right\},
\end{aligned}$$

where $0<\frac{1}{p_2}<1$ for $p_2>1$. By a simple calculation, we obtain the required result. □

Now, the generalized Hadamard's type inequality for generalized s-concave functions.

Theorem 2.7. *Assume that* $f : U \subset [0,\infty) \longrightarrow \mathbb{R}^{\alpha}$ *such that* $f \in D_{\alpha}$ *on* $\mathrm{Int}(U)$ *and* $f^{(2\alpha)} \in C_{\alpha}[a_1, a_2]$, *where* $a_1, a_2 \in U$ *with* $a_1 < a_1$. *If* $|f^{(2\alpha)}|^{p_2}$ *is generalized* s*-convex on* $[a_1, a_2]$ *for some fixed* $0 < s \leqslant 1$ *and* $p_2 > 1$ *with* $\frac{1}{p_1} + \frac{1}{p_2} = 1$, *then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)\,[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\ &\quad\leqslant \frac{2^{\frac{\alpha(s-1)}{p_2}}(a_2-a_1)^{2\alpha}}{16^{\alpha}\,(\Gamma(1+\alpha))^{\frac{1}{p_2}}}\left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}}\left[\left|f^{(2\alpha)}\left(\frac{3a_1+a_2}{4}\right)\right| + \left|f^{(2\alpha)}\left(\frac{a_1+3a_2}{4}\right)\right|\right].\end{aligned}$$

Proof. From Lemma 2.1 and using the generalized Hölder inequality for $p_2 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$, we get

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)\,[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\ &\quad\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left[\ {}_0I_1^{(\alpha)}\gamma^{2\alpha}\left|f^{(2\alpha)}\left(\gamma\frac{a_1+a_2}{2} + (1-\gamma)a_1\right)\right|\right. \\ &\quad\quad + \left.\ {}_0I_1^{(\alpha)}(\gamma-1)^{2\alpha}\left|f^{(2\alpha)}\left(\gamma a_2 + (1-\gamma)\frac{a_1+a_2}{2}\right)\right|\right] \\ &\quad\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left(\ {}_0I_1^{(\alpha)}\gamma^{2p_1\alpha}\right)^{\frac{1}{p_1}}\left(\ {}_0I_1^{(\alpha)}\left|f^{(2\alpha)}\left(\gamma\frac{a_1+a_2}{2} + (1-\gamma)a_1\right)\right|^{p_2}\right)^{\frac{1}{p_2}} \\ &\quad\quad + \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left(\ {}_0I_1^{(\alpha)}(\gamma-1)^{2p_1\alpha}\right)^{\frac{1}{p_1}}\left(\ {}_0I_1^{(\alpha)}\left|f^{(2\alpha)}\left(\gamma a_2 + (1-\gamma)\frac{a_1+a_2}{2}\right)\right|^{p_2}\right)^{\frac{1}{p_2}}.\end{aligned}$$

Since $\left|f^{(2\alpha)}\right|^{p_2}$ is generalized s-concave, then

$${}_0I_1^{(\alpha)}\left|f^{(2\alpha)}\left(\gamma\frac{a_1+a_2}{2} + (1-\gamma)a_1\right)\right|^{p_2} \leqslant \frac{2^{\alpha(s-1)}}{\Gamma(1+\alpha)}\left|f^{(2\alpha)}\left(\frac{3a_1+a_2}{4}\right)\right|^{p_2}, \tag{2.5}$$

also

$${}_0I_1^{(\alpha)}\left|f^{(2\alpha)}\left(\gamma a_2 + (1-\gamma)\frac{a_1+a_2}{2}\right)\right|^{p_2} \leqslant \frac{2^{\alpha(s-1)}}{\Gamma(1+\alpha)}\left|f^{(2\alpha)}\left(\frac{a_1+3a_2}{4}\right)\right|^{p_2}. \tag{2.6}$$

From (2.5) and (2.6), we observe that

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)\,[\Gamma(1+\alpha)]^2}{2^{\alpha}(a_2-a_1)^{\alpha}}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x)\right| \\ &\quad\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}}\frac{2^{\frac{\alpha(s-1)}{p_2}}}{(\Gamma(1+\alpha))^{\frac{1}{p_2}}}\left|f^{(2\alpha)}\left(\frac{3a_1+a_2}{4}\right)\right| \\ &\quad\quad + \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}}\frac{2^{\frac{\alpha(s-1)}{p_2}}}{(\Gamma(1+\alpha))^{\frac{1}{p_2}}}\left|f^{(2\alpha)}\left(\frac{a_1+3a_2}{4}\right)\right| \\ &\quad= \frac{2^{\frac{\alpha(s-1)}{p_2}}(a_2-a_1)^{2\alpha}}{16^{\alpha}\,(\Gamma(1+\alpha))^{\frac{1}{p_2}}}\left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}}\left[\left|f^{(2\alpha)}\left(\frac{3a_1+a_2}{4}\right)\right| + \left|f^{(2\alpha)}\left(\frac{a_1+3a_2}{4}\right)\right|\right].\end{aligned}$$

So, the proof is complete. □

Remark 2.8.

1. If $\alpha = 1$ in Theorem 2.7, then

$$\left| f\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)dx \right| \leqslant \frac{2^{\frac{s-1}{q}}(a_2-a_1)^2}{16}\left[\frac{1}{\Gamma(2p_1+1)}\right]^{\frac{1}{p_1}}\left[\left|f''\left(\frac{3a_1+a_2}{4}\right)\right| + \left|f''\left(\frac{a_1+3a_2}{4}\right)\right|\right].$$

2. If $s = 1$ and $\frac{1}{3} < \left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}} < 1, p_1 > 1$ in Theorem 2.7, then

$$\left| \frac{\Gamma(1+2\alpha)}{2^\alpha} f\left(\frac{a_1+a_2}{2}\right) - \frac{\Gamma(1+2\alpha)\left[\Gamma(1+\alpha)\right]^2}{2^\alpha(a_2-a_1)^\alpha}\ {}_{a_1}I_{a_2}^{(\alpha)} f(x) \right| \leqslant \frac{(a_2-a_1)^{2\alpha}}{16^\alpha\left(\Gamma(1+\alpha)\right)^{\frac{1}{p_2}}}\left[\left|f^{(2\alpha)}\left(\frac{3a_1+a_2}{4}\right)\right| + \left|f^{(2\alpha)}\left(\frac{a_1+3a_2}{4}\right)\right|\right].$$

3. Applications to special means

As in [16], some generalized means are considered such as:

$$A(a_1,a_2) = \frac{a_1^\alpha + a_2^\alpha}{2^\alpha}, a_1, a_2 \geqslant 0,$$
$$L_n(a_1,a_2) = \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\left(a_2^{(n+1)\alpha} - a_1^{(n+1)\alpha}\right)\right]^{\frac{1}{n}}, n \in \mathbb{Z}\{-1,0\}, a_1, a_2 \in \mathbb{R}, a_1 \neq a_2.$$

In [12], the following example was given:
let $0 < s < 1$ and $a_1^\alpha, a_2^\alpha, a_3^\alpha \in \mathbb{R}^\alpha$. Define for $x \in \mathbb{R}_+$,

$$f(n) = \begin{cases} a_1^\alpha, & n = 0, \\ a_2^\alpha n^{s\alpha} + a_3^\alpha, & n > 0. \end{cases}$$

If $a_2^\alpha \geqslant 0^\alpha$ and $0^\alpha \leqslant a_3^\alpha \leqslant a_1^\alpha$, then $f \in GK_s^2$.

Proposition 3.1. *Let* $0 < a_1 < a_2$ *and* $s \in (0,1)$. *Then*

$$\left| \frac{\Gamma(1+2\alpha)}{2^\alpha} A^s(a_1,a_2) - \frac{\Gamma(1+2\alpha)\left[\Gamma(1+\alpha)\right]^2}{2^\alpha(a_2-a_1)^\alpha} L_s^s(a_1,a_2) \right| \leqslant \frac{(a_2-a_1)^{2\alpha}}{16^\alpha}\left|\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-2)\alpha)}\right|\left\{\frac{2^\alpha\Gamma(1+3\alpha)\left[\Gamma(1+\alpha)\right]^2}{\Gamma(1+4\alpha)\Gamma(1+2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{2^\alpha\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} + \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right\}\left[|a_1|^{(s-2)\alpha} + |a_2|^{(s-2)\alpha}\right].$$

Proof. The result follows from Remark 2.3 (2) with $f : [0,1] \longrightarrow [0^\alpha, 1^\alpha]$, $f(x) = x^{s\alpha}$. Also, when $\alpha = 1$, we have the following inequalitly:

$$\left| A^s(a_1,a_2) - \frac{1}{a_2-a_1} L_s^s(a_1,a_2) \right| \leqslant \frac{(a_2-a_1)^2|s(s-1)|}{48}\left\{|a_1|^{s-2} + |a_2|^{s-2}\right\}. \tag{3.1}$$

□

Proposition 3.2. *Let* $0 < a_1 < a_2$ *and* $s \in (0,1)$. *Then*

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}}A^{s}(a_1,a_2)-\frac{\Gamma(1+2\alpha)\,[\Gamma(1+\alpha)]^{2}}{2^{\alpha}(a_2-a_1)^{\alpha}}L_s^{s}(a_1,a_2)\right| \\
&\leqslant \frac{(a_2-a_1)^{2\alpha}}{16^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]^{\frac{1}{p_2}}\left|\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-2)\alpha)}\right|\left[\frac{\Gamma(1+2p_1\alpha)}{\Gamma(1+(2p_1+1)\alpha)}\right]^{\frac{1}{p_1}} \\
&\quad\times\left[\left(\left|\frac{a_1+a_2}{2}\right|^{(s-2)p_2\alpha}+|a_1|^{(s-2)p_2\alpha}\right)^{\frac{1}{p_2}}+\left(\left|\frac{a_1+a_2}{2}\right|^{(s-2)p_2\alpha}+|a_2|^{(s-2)p_2\alpha}\right)^{\frac{1}{p_2}}\right],
\end{aligned}$$

where $p_2 > 1$ *and* $\frac{1}{p_1}+\frac{1}{p_2}=1$.

Proof. The result follows (2.4) with $f:[0,1]\longrightarrow[0^{\alpha},1^{\alpha}]$, $f(x)=x^{s\alpha}$ and when $\alpha=1$, we have the following inequalitly:

$$\begin{aligned}
\left|A^{s}(a_1,a_2)-\frac{1}{a_2-a_1}L_s^{s}(a_1,a_2)\right| \leqslant \frac{(a_2-a_1)^{2}|s(s-1)|}{2^{\frac{1}{p_2}}16(2p_1+1)^{\frac{1}{p_1}}}&\left\{\left(\left|\frac{a_1+a_2}{2}\right|^{(s-2)p_2}+|a_1|^{(s-2)p_2}\right)^{\frac{1}{p_2}}\right. \\
&\left.+\left(\left|\frac{a_1+a_2}{2}\right|^{(s-2)p_2}+|a_2|^{(s-2)p_2}\right)^{\frac{1}{p_2}}\right\}.
\end{aligned} \tag{3.2}$$

Where $A(a_1,a_2)$ and $L_n(a_1,a_2)$ in (3.1) and (3.2) are known as

1. arithmetic mean: $A(a_1,a_2)=\frac{a_1+a_2}{2}, a_1,a_2\in\mathbb{R}^{+}$;
2. logarithmic mean: $L(a_1,a_2)=\frac{a_1-a_2}{\ln|a_1|-\ln|a_2|}, |a_1|\neq a_2, a_1,a_2\neq 0, a_1,a_2\in\mathbb{R}^{+}$;
3. generalized Log-mean: $L_n(a_1,a_2)=\left[\frac{a_2^{n+1}-a_1^{n+1}}{(n+1)(a_2-a_1)}\right]^{\frac{1}{n}}, n\in\mathbb{Z}\setminus\{-1,0\}, a_1,a_2\in\mathbb{R}^{+}$.

□

Now, we give application to wave equation on Cantor sets:

Not only the fractional calculus which deals with fractional differential equations [11] has various kinds of analytical methods for solving these equations [1, 2], but also local fractional calculus which deals with problems for nondifferentiable function [19] has analytical methods for solving the local fractional differential equations such as local fractional Fourier series method was applied to process the wave equation on Cantor sets (local fractional wave equation) with local fractional derivatives [20].

The wave equation on Cantor sets was given as

$$\frac{\partial^{2\alpha}f(x,t)}{\partial t^{2\alpha}}=A^{2\alpha}\frac{\partial^{2\alpha}f(x,t)}{\partial x^{2\alpha}}. \tag{3.3}$$

Following (3.3), a wave equation on Cantor sets was proposed as follows [22]:

$$\frac{\partial^{2\alpha}f(x,t)}{\partial t^{2\alpha}}=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}f(x,t)}{\partial x^{2\alpha}},\quad 0\leqslant\alpha\leqslant 1, \tag{3.4}$$

where $f(x,t)$ is a fractal wave function and the initial value is given by $f(x,0)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}$. The solution of (3.4) is given as $f(x,t)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$.

By using Lemma 2.1, we have

$$\frac{\Gamma(1+2\alpha)\Gamma(1+\alpha)}{2^{\alpha}(a_2-a_1)^{\alpha}}\int_{a_1}^{a_2}f(x,t)(dt)^{\alpha}-\frac{\Gamma(1+2\alpha)}{2^{\alpha}}f\left(x,\frac{a_1+a_2}{2}\right)$$

$$= \frac{(a_2-a_1)^\alpha}{8^\alpha\Gamma(1+2\alpha)}\left[\left(\frac{2}{a_2-a_1}\right)^{2\alpha} {}_{a_1}I^{(\alpha)}_{\frac{a_2+a_1}{2}}(t-a_1)^{2\alpha}x^{2\alpha}\frac{\partial^{2\alpha}f(x,t)}{\partial x^\alpha}\right.$$
$$\left.+\ {}_{a_1}I^{(\alpha)}_{\frac{a_2+a_1}{2}}\left(\frac{2(t-a_1)}{a_2-a_1}-1\right)^{2\alpha}x^{2\alpha}\frac{\partial^{2\alpha}f(x,t)}{\partial x^\alpha}\right].$$

4. Conclusion

In this article, we have studied some generalized Hermite-Hadamard type inequalities for the class of functions whose second local fractional derivatives in absolute value at certain powers are generalized s-convex functions in the second sense on fractal sets. In particular, our results extend some important inequalities in classical situation and, when $\alpha = 1$, some relationships between these inequalities and the classical inequalities have been established. Finally, we have given some applications for these inequalities on fractal sets.

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