Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 483–491
Research Article

ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

A projected fixed point algorithm with Meir-Keeler contraction for pseudocontractive mappings

Yonghong Yao[a], Naseer Shahzad[b,*], Yeong-Cheng Liou[c], Li-Jun Zhu[d]

[a] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China.*

[b] *Department of Mathematics, King Abdulaziz University, P. O. B. 80203, Jeddah 21589, Saudi Arabia.*

[c] *Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan and Center for General Education, Kaohsiung Medical University, Kaohsiung 807, Taiwan.*

[d] *School of Mathematics and Information Science, Beifang University of Nationalities, Yinchuan 750021, China.*

Communicated by Y. J. Cho

Abstract

In this paper, we introduce a projected algorithm with Meir-Keeler contraction for finding the fixed points of the pseudocontractive mappings. We prove that the presented algorithm converges strongly to the fixed point of the pseudocontractive mapping in Hilbert spaces.

Keywords: Projected algorithm, pseudocontractive mapping, fixed point.
2010 MSC: 47H05, 47H10, 47H17.

1. Introduction

In this paper, we assume that H is a real Hilbert space with inner $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and $C \subset H$ is a nonempty closed convex set.

Recall that a mapping $T : C \to C$ is said to be pseudocontractive, if

$$\langle Tu - Tu^{\dagger}, u - u^{\dagger} \rangle \leqslant \|u - u^{\dagger}\|^2, \quad \forall u, u^{\dagger} \in C. \tag{1.1}$$

It is clear that (1.1) is equivalent to

$$\|Tu - Tu^{\dagger}\|^2 \leqslant \|u - u^{\dagger}\|^2 + \|(I - T)u - (I - T)u^{\dagger}\|^2, \quad \forall u, u^{\dagger} \in C. \tag{1.2}$$

We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T. Recall also that a mapping $T : C \to C$ is said to be L-Lipschitzian, if

$$\|Tu - Tu^{\dagger}\| \leqslant L\|u - u^{\dagger}\|, \quad \forall u, u^{\dagger} \in C,$$

*Corresponding author
Email addresses: yaoyonghong@aliyun.com (Yonghong Yao), nshahzad@kau.edu.sa (Naseer Shahzad), simplex_liou@hotmail.com (Yeong-Cheng Liou), zhulijun1995@sohu.com (Li-Jun Zhu)

doi:10.22436/jnsa.010.02.13

Received 2016-04-06

where $L > 0$ is a constant. If $L = 1$, T is called nonexpansive.

The interest of pseudocontractions lies in their connection with monotone operators, namely, T is a pseudocontraction, if and only if the complement $I - T$ is a monotone operator. In the literature, there are a large number references associated with the fixed point algorithms for nonexpansive mappings and pseudocontractive mappings. See, for instance, [1–7, 11] and [9, 10, 12–31]. The first interesting result for finding the fixed points of the pseudocontractive mappings was presented by Ishikawa in 1974 as follows.

Theorem 1.1 (Ishikawa Algorithm, [7]). *Let H be a Hilbert space. Let $C \subset H$ be a convex compact set. Let $T : C \to C$ be an L-Lipschitzian pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. For any $x_0 \in C$, define the sequence $\{x_n\}$ iteratively by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \end{cases} \tag{1.3}$$

for all $n \in \mathbb{N}$, where $\{\beta_n\} \subset [0,1]$, $\{\alpha_n\} \subset [0,1]$ satisfy the conditions: $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \beta_n \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by (1.3) converges strongly to a fixed point of T.

Remark 1.2. The iteration (1.3) is now referred as the Ishikawa iterative sequence. We observe that C is compact subset. We know that strong convergence has not been achieved without compactness assumption (a counter example can be found in [3]).

In order to obtain strong convergence for pseudocontractive mappings without the compactness assumption, Zhou [30] coupled the Ishikawa algorithm with the hybrid technique and proved the following theorem for Lipschitz pseudocontractive mappings.

Theorem 1.3 (Hybrid Ishikawa Algorithm, [30]). *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a Lipschitz pseudocontraction such that $\mathrm{Fix}(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$ satisfying the conditions:*

(i) $\alpha_n \leqslant \beta_n$, *for all* $n \in \mathbb{N}$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n \leqslant \beta < \frac{1}{\sqrt{1+L^2}+1}$.

Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ z_n = (1-\alpha_n)x_n + \alpha_n T y_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leqslant x_n - z\|^2 - \beta_n \alpha_n (1 - 2\beta_n - \beta_n^2 L^2)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geqslant 0\}, \\ x_{n+1} = \mathrm{proj}_{C_n \bigcap Q_n}(x_0), \quad n \in \mathbb{N}. \end{cases} \tag{1.4}$$

Then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $\mathrm{proj}_{\mathrm{Fix}(T)}(x_0)$.

Further, Yao et al. [16] introduced the hybrid Mann algorithm and obtained the strong convergence theorem.

Theorem 1.4 (Hybrid Mann Algorithm, [16]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an L-Lipschitz pseudocontractive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = \mathrm{proj}_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ C_{n+1} = \{z \in C_n : \|\alpha_n (I-T) y_n\|^2 \leqslant 2\alpha_n \langle x_n - z, (I-T)y_n \rangle\}, \\ x_{n+1} = \mathrm{proj}_{C_{n+1}}(x_0), \quad n \in \mathbb{N}. \end{cases} \tag{1.5}$$

Assume the sequence $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{L+1})$. Then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $\mathrm{proj}_{\mathrm{Fix}(T)}(x_0)$.

Motivated and inspired by the above results, in this paper we introduce a projected algorithm with Meir-Keeler contraction for finding the fixed points of the pseudocontractive mappings. We prove that the presented algorithm converges strongly to the fixed point of the pseudocontractive mapping in Hilbert spaces.

2. Preliminaries

Recall that the metric projection $\mathrm{proj}_C : H \to C$ satisfies

$$\|u - \mathrm{proj}_C(u)\| = \inf\{\|u - u^\dagger\| : u^\dagger \in C\}.$$

The metric projection proj is a typical firmly nonexpansive mapping. The characteristic inequality of the projection is

$$\langle u - \mathrm{proj}_C(u), u^\dagger - \mathrm{proj}_C(u)\rangle \leqslant 0,$$

for all $u \in H$, $u^\dagger \in C$.

Recall that a mapping T is said to be demiclosed, if for any sequence $\{x_n\}$ which weakly converges to $\tilde{x}$, and if the sequence $\{T(x_n)\}$ strongly converges to $x^\dagger$, then $T(\tilde{x}) = x^\dagger$.

It is well-known that in a real Hilbert space H, the following equality holds:

$$\|\xi u + (1-\xi)u^\dagger\|^2 = \xi\|u\|^2 + (1-\xi)\|u^\dagger\|^2 - \xi(1-\xi)\|u - u^\dagger\|^2, \tag{2.1}$$

for all $u, u^\dagger \in H$ and $\xi \in [0, 1]$.

Lemma 2.1 ([30])**.** *Let H be a real Hilbert space, C a closed convex subset of H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Then*

(i) $\mathrm{Fix}(T)$ *is a closed convex subset of* C*;*

(ii) $(I - T)$ *is demiclosed at zero.*

For convenient, in the sequel we shall use the following expressions:

- $x_n \rightharpoonup x^\dagger$ denotes the weak convergence of x_n to $x^\dagger$;
- $x_n \to x^\dagger$ denotes the strong convergence of x_n to $x^\dagger$.

Let the sequence $\{C_n\}$ be a nonempty closed convex subset of a Hilbert space H. We define $s - \mathrm{Li}_n C_n$ and $w - \mathrm{Ls}_n C_n$ as follows.

- $x \in s - \mathrm{Li}_n C_n$, if and only if there exists $\{x_n\} \subset C_n$ such that $x_n \to x$.
- $x \in w - \mathrm{Ls}_n C_n$, if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset C_{n_i}$ such that $y_i \rightharpoonup y$.

If C_0 satisfies

$$C_0 = s - \mathrm{Li}_n C_n = w - \mathrm{Ls}_n C_n,$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [10] and we write $C_0 = M - \lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. Tsukada [14] proved the following theorem for the metric projection.

Lemma 2.2 ([14])**.** *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 = M - \lim_{n\to\infty} C_n$ exists and is nonempty, then for each $x \in H$, $\{\mathrm{proj}_{C_n}(x)\}$ converges strongly to $\mathrm{proj}_{C_0}(x)$, where proj_{C_n} and proj_{C_0} are the metric projections of H onto C_n and C_0, respectively.*

Let (X, d) be a complete metric space. A mapping $f : X \to X$ is called a Meir-Keeler contraction [8], if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \epsilon + \delta \ \text{ implies } \ d(f(x), f(y)) < \epsilon,$$

for all $x, y \in X$. It is well-known that the Meir-Keeler contraction is a generalization of the contraction.

Lemma 2.3 ([8])**.** *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

Lemma 2.4 ([13])**.** *Let f be a Meir-Keeler contraction on a convex subset C of a Banach space E. Then, for every $\epsilon > 0$, there exists $r \in (0, 1)$ such that*

$$\|x - y\| \geqslant \epsilon \ \textit{implies} \ \|f(x) - f(y)\| \leqslant r\|x - y\|,$$

for all $x, y \in C$.

Lemma 2.5 ([13])**.** *Let C be a convex subset of a Banach space E. Let T be a nonexpansive mapping on C, and let f be a Meir-Keeler contraction on C. Then the following hold:*

(i) *Tf is a Meir-Keeler contraction on C;*

(ii) *for each $\alpha \in (0, 1)$, $(1 - \alpha)T + \alpha f$ is a Meir-Keeler contraction on C.*

3. Main results

In this section, we firstly introduce a projected fixed point algorithm with Meir-Keeler contraction for pseudocontractive mappings in Hilbert spaces. Consequently, we show the strong convergence of our presented algorithm.

In the sequel, we assume that H is a real Hilbert space and $C \subset H$ is a nonempty closed convex set. Let $T : C \to C$ be an $L(> 1)$-Lipschitzian pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $f : C \to C$ be a Meir-Keeler contractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$.

Algorithm 3.1. For $x_0 \in C_0 = C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ C_{n+1} = \{z \in C_n : \|(1 - \alpha_n)x_n + \alpha_n T y_n - z\| \leqslant \|x_n - z\|\}, \\ x_{n+1} = \mathrm{proj}_{C_{n+1}} f(x_n), \quad \forall n \geqslant 0, \end{cases} \tag{3.1}$$

where proj is the metric projection.

Theorem 3.2. *If $0 < a < \alpha_n \leqslant \beta_n < b < \frac{1}{\sqrt{1+L^2}+1}$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $x^\dagger = \mathrm{proj}_{\mathrm{Fix}(T)} f(x^\dagger)$.*

Remark 3.3. By Lemma 2.1, $\mathrm{Fix}(T)$ is a closed convex subset of C. Thus $\mathrm{proj}_{\mathrm{Fix}(T)}$ is well-defined. Since f is a Meir-Keeler contraction of C, we get $\mathrm{proj}_{\mathrm{Fix}(T)} f$ is a Meir-Keeler contraction of C by Lemma 2.5. According to Lemma 2.3, there exists a unique fixed point $x^\dagger \in C$ such that $x^\dagger = \mathrm{proj}_{\mathrm{Fix}(T)} f(x^\dagger)$.

Proof. We first show by induction that $\mathrm{Fix}(T) \subset C_n$ for all $n \geqslant 0$.

(i) $\mathrm{Fix}(T) \subset C_0$ is obvious.

(ii) Suppose that $\mathrm{Fix}(T) \subset C_k$ for some $k \in \mathbb{N}$. Then for $x^* \in \mathrm{Fix}(T) \subset C_k$, we have from (1.2) that

$$\|Tx_n - x^*\|^2 \leqslant \|x_n - x^*\|^2 + \|Tx_n - x_n\|^2, \tag{3.2}$$

and

$$\begin{aligned} \|Ty_n - x^*\|^2 &= \|T((1 - \beta_n)I + \beta_n T)x_n - x^*\|^2 \\ &\leqslant \|(1 - \beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\|^2 + \|(1 - \beta_n)x_n + \beta_n Tx_n - Ty_n\|^2. \end{aligned} \tag{3.3}$$

From (2.1) we have that

$$\begin{aligned}\|(1-\beta_n)x_n+\beta_n Tx_n-Ty_n\|^2 &= \|(1-\beta_n)(x_n-Ty_n)+\beta_n(Tx_n-Ty_n)\|^2\\ &= (1-\beta_n)\|x_n-Ty_n\|^2+\beta_n\|Tx_n-Ty_n\|^2\\ &\quad -\beta_n(1-\beta_n)\|x_n-Tx_n\|^2.\end{aligned} \tag{3.4}$$

Since T is L-Lipschitzian and $x_n-y_n=\beta_n(x_n-Tx_n)$, by (3.4) we get that

$$\begin{aligned}\|(1-\beta_n)x_n+\beta_n Tx_n-Ty_n\|^2 &\leqslant (1-\beta_n)\|x_n-Ty_n\|^2+\beta_n^3L^2\|x_n-Tx_n\|^2\\ &\quad -\beta_n(1-\beta_n)\|x_n-Tx_n\|^2\\ &= (1-\beta_n)\|x_n-Ty_n\|^2+(\beta_n^3L^2+\beta_n^2-\beta_n)\|x_n-Tx_n\|^2.\end{aligned} \tag{3.5}$$

By (2.1) and (3.2) we have that

$$\begin{aligned}\|(1-\beta_n)(x_n-x^*)+\beta_n(Tx_n-x^*)\|^2 &= \|(1-\beta_n)(x_n-x^*)+\beta_n(Tx_n-x^*)\|^2\\ &= (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|Tx_n-x^*\|^2\\ &\quad -\beta_n(1-\beta_n)\|x_n-Tx_n\|^2\\ &\leqslant (1-\beta_n)\|x_n-x^*\|^2+\beta_n(\|x_n-x^*\|^2+\|x_n-Tx_n\|^2)\\ &\quad -\beta_n(1-\beta_n)\|x_n-Tx_n\|^2\\ &= \|x_n-x^*\|^2+\beta_n^2\|x_n-Tx_n\|^2.\end{aligned} \tag{3.6}$$

By (3.3), (3.5) and (3.6) we obtain that

$$\|Ty_n-x^*\|^2 \leqslant \|x-x^*\|^2+(1-\beta_n)\|x_n-Ty_n\|^2-\beta_n(1-2\beta_n-\beta_n^2L^2)\|x_n-Tx_n\|^2. \tag{3.7}$$

Since $\beta_n < b < \frac{1}{\sqrt{1+L^2}+1}$, we derive that

$$1-2\beta_n-\beta_n^2L^2>0,\quad \forall n\geqslant 0.$$

This together with (3.7) implies that

$$\|Ty_n-x^*\|^2 \leqslant \|x_n-x^*\|^2+(1-\beta_n)\|x_n-Ty_n\|^2. \tag{3.8}$$

By (2.1) and (3.8) and noting that $\alpha_n \leqslant \beta_n$, we have that

$$\begin{aligned}\|(1-\alpha_n)x_n+\alpha_n Ty_n-x^*\|^2 &= (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|Ty_n-x^*\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-Ty_n\|^2\\ &\leqslant \|x_n-x^*\|^2-\alpha_n(\beta_n-\alpha_n)\|Ty_n-x^*\|^2\\ &\leqslant \|x_n-x^*\|^2,\end{aligned}$$

and hence $x^*\in C_{k+1}$. This indicates that

$$\mathrm{Fix}(T)\subset C_n,$$

for all $n\geqslant 0$.

Next, we show that C_n is closed and convex for all $n\geqslant 0$.

(i) It is obvious from the assumption that $C_0=C$ is closed convex.

(ii) Suppose that C_k is closed and convex for some $k\in N$. For $z\in C_k$, we know that $\|(1-\alpha_k)x_k+\alpha_k Ty_k-z\|\leqslant\|x_k-z\|$ is equivalent to

$$\alpha_k\|Ty_k-x_k\|^2+2\langle Ty_k-x_k,x_k-z\rangle\leqslant 0.$$

So, C_{k+1} is closed and convex. By induction, we deduce that C_n is closed and convex for all $n \geqslant 0$. This implies that $\{x_n\}$ is well-defined.

Next, we prove that

$$\lim_{n\to\infty} \|x_n - u\| = 0,$$

for some $u \in \cap_{n=1}^{\infty} C_n$ and

$$\langle f(u) - u, u - y\rangle \geqslant 0,$$

for all $y \in Fix(T)$.

Since $\bigcap_{n=1}^{\infty} C_n$ is closed convex, we also have that $proj_{\bigcap_{n=1}^{\infty} C_n}$ is well-defined and so $proj_{\bigcap_{n=1}^{\infty} C_n} f$ is a Meir-Keeler contraction on C. By Lemma 2.3, there exists a unique fixed point $u \in \bigcap_{n=1}^{\infty} C_n$ of $proj_{\bigcap_{n=1}^{\infty} C_n} f$. Since C_n is a nonincreasing sequence of nonempty closed convex subset of H with respect to inclusion, it follows that

$$\emptyset \neq Fix(T) \subset \bigcap_{n=1}^{\infty} C_n = M - \lim_{n\to\infty} C_n.$$

Setting $u_n := proj_{C_n} f(u)$ and applying Lemma 2.2, we can conclude that

$$\lim_{n\to\infty} u_n = proj_{\bigcap_{n=1}^{\infty} C_n} f(u) = u.$$

Now we show that $\lim_{n\to\infty} \|x_n - u\| = 0$.

Assume $d = \overline{\lim}_n \|x_n - u\| > 0$, then for all $\epsilon \in (0, d)$, we can choose a $\delta_1 > 0$ such that

$$\overline{\lim_{n}} \|x_n - u\| > \epsilon + \delta_1. \tag{3.9}$$

Since f is a Meir-Keeler contraction, for above ϵ there exists another $\delta_2 > 0$ such that

$$\|x - y\| < \epsilon + \delta_2 \text{ implies } \|f(x) - f(y)\| < \epsilon, \tag{3.10}$$

for all $x, y \in C$.

In fact, we can choose a common $\delta > 0$ such that (3.9) and (3.10) hold. If $\delta_1 > \delta_2$, then

$$\overline{\lim_{n}} \|x_n - u\| > \epsilon + \delta_1 > \epsilon + \delta_2.$$

If $\delta_1 \leqslant \delta_2$, then from (3.10) we deduce that

$$\|x - y\| < \epsilon + \delta_1 \text{ implies } \|f(x) - f(y)\| < \epsilon,$$

for all $x, y \in C$.

Thus, we have that

$$\overline{\lim_{n}} \|x_n - u\| > \epsilon + \delta, \tag{3.11}$$

and

$$\|x - y\| < \epsilon + \delta \text{ implies } \|f(x) - f(y)\| < \epsilon, \quad \text{for all } x, y \in C. \tag{3.12}$$

Since $u_n \to u$, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n - u\| < \delta, \quad \forall n \geqslant n_0. \tag{3.13}$$

We now consider two possible cases.

Case 1. There exists $n_1 \geqslant n_0$ such that

$$\|x_{n_1} - u\| \leqslant \epsilon + \delta.$$

By (3.12) and (3.13), we get that

$$\begin{aligned}
\|x_{n_1+1} - u\| &\leqslant \|x_{n_1+1} - u_{n_1+1}\| + \|u_{n_1+1} - u\| \\
&= \|proj_{C_{n_1+1}} f(x_{n_1}) - proj_{C_{n_1+1}} f(u)\| + \|u_{n_1+1} - u\| \\
&\leqslant \|f(x_{n_1}) - f(u)\| + \|u_{n_1+1} - u\| \\
&\leqslant \epsilon + \delta.
\end{aligned}$$

By induction, we can obtain that

$$\|x_{n_1+m} - u\| \leqslant \epsilon + \delta,$$

for all $m \geqslant 1$, which implies that

$$\overline{\lim_n} \|x_n - u\| \leqslant \epsilon + \delta,$$

which contradicts with (3.11). Therefore, we conclude that $\|x_n - u\| \to 0$ as $n \to \infty$.

Case 2. $\|x_n - u\| > \epsilon + \delta$ for all $n \geqslant n_0$.

We shall prove that Case 2 is impossible. Suppose Case 2 holds true. By Lemma 2.4, there exists $r \in (0, 1)$ such that

$$\|f(x_n) - f(u)\| \leqslant r\|x_n - u\|, \quad \forall n \geqslant n_0.$$

Thus, we have that

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &= \|\mathrm{proj}_{C_{n+1}} f(x_n) - \mathrm{proj}_{C_{n+1}} f(u)\| \\ &\leqslant \|f(x_n) - f(u)\| \\ &\leqslant r\|x_n - u\|,\end{aligned}$$

for every $n \geqslant n_0$.

It follows that

$$\begin{aligned}\overline{\lim_n} \|x_{n+1} - u\| &= \overline{\lim_n} \|x_{n+1} - u_{n+1}\| \\ &\leqslant r\overline{\lim_n} \|x_n - u\| \\ &< \overline{\lim_n} \|x_n - u\|,\end{aligned}$$

which gives a contradiction.

Hence, we obtain that

$$\lim_{n\to\infty} \|x_n - u\| = 0,$$

and therefore, $\{x_n\}$ is bounded.

Finally, we prove that $u \in \mathrm{Fix}(T)$. Observe that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leqslant \|x_n - u\| + \|u - u_{n+1}\| + \|u_{n+1} - x_{n+1}\| \\ &= \|x_n - u\| + \|u - u_{n+1}\| + \|\mathrm{proj}_{C_{n+1}} f(x_n) - \mathrm{proj}_{C_{n+1}} f(u)\| \\ &\leqslant \|x_n - u\| + \|u - u_{n+1}\| + \|f(x_n) - f(u)\|.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

From $x_{n+1} \in C_{n+1}$, we have that

$$\|(1 - \alpha_n)x_n + \alpha_n Ty_n - x_{n+1}\| \leqslant \|x_n - x_{n+1}\|.$$

This together with (3.14) implies that

$$\lim_{n\to\infty} \|Ty_n - x_n\| = 0.$$

Note that

$$\begin{aligned}\|x_n - Tx_n\| &\leqslant \|x_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\leqslant \|x_n - Ty_n\| + L\|x_n - y_n\| \\ &\leqslant \|x_n - Ty_n\| + L(1 - \beta_n)\|x_n - Tx_n\|.\end{aligned}$$

It follows

$$\|x_n - Tx_n\| \leqslant \frac{1}{1 - (1 - \beta_n)L} \|x_n - Ty_n\| \leqslant \frac{1}{1 - (1 - a)L} \|x_n - Ty_n\| \to 0. \tag{3.15}$$

By Lemma 2.1 and (3.15), we have that $u \in \mathrm{Fix}(T)$.

Since $x_{n+1} = \mathrm{proj}_{C_{n+1}} f(x_n)$, we have that

$$\langle f(x_n) - x_{n+1}, x_{n+1} - y\rangle \geqslant 0, \quad \forall y \in C_{n+1}.$$

Since $\mathrm{Fix}(T) \subset C_{n+1}$, we get

$$\langle f(x_n) - x_{n+1}, x_{n+1} - y\rangle \geqslant 0, \quad \forall y \in \mathrm{Fix}(T).$$

We have from $x_n \to u \in \mathrm{Fix}(T)$ that

$$\langle f(u) - u, u - y\rangle \geqslant 0, \quad \forall y \in \mathrm{Fix}(T).$$

Thus, $u = \mathrm{proj}_{\mathrm{Fix}(T)} f(u) = x^\dagger$. This completes the proof. □

Remark 3.4. It is obvious that (3.1) is simpler than (1.4) and (1.5).

From Theorem 3.2, we can deduce several corollaries.

Corollary 3.5. *Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T : C \to C$ be an $L(> 1)$-Lipschitzian pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $f : C \to C$ be a ρ-contraction. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$. If $0 < a < \alpha_n \leqslant \beta_n < b < \frac{1}{\sqrt{1+L^2}+1}$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $x^\dagger = \mathrm{proj}_{\mathrm{Fix}(T)} f(x^\dagger)$.*

Corollary 3.6. *Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $f : C \to C$ be a Meir-Keeler contractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$. If $0 < a < \alpha_n \leqslant \beta_n < b < \frac{1}{1+\sqrt{2}}$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $x^\dagger = \mathrm{proj}_{\mathrm{Fix}(T)} f(x^\dagger)$.*

Corollary 3.7. *Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $f : C \to C$ be a ρ-contraction. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$. If $0 < a < \alpha_n \leqslant \beta_n < b < \frac{1}{1+\sqrt{2}}$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $x^\dagger = \mathrm{proj}_{\mathrm{Fix}(T)} f(x^\dagger)$.*

Algorithm 3.8. For $x_0 \in C_0 = C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ C_{n+1} = \{z \in C_n : \|(1-\alpha_n)x_n + \alpha_n T y_n - z\| \leqslant \|x_n - z\|\}, \\ x_{n+1} = \mathrm{proj}_{C_{n+1}}(x_0), \quad \forall n \geqslant 0, \end{cases} \tag{3.16}$$

where proj is the metric projection.

Corollary 3.9. *Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T : C \to C$ be an $L(> 1)$-Lipschitzian pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$. If $0 < a < \alpha_n \leqslant \beta_n < b < \frac{1}{\sqrt{1+L^2}+1}$, then the sequence $\{x_n\}$ defined by (3.16) converges strongly to $x^\dagger = \mathrm{proj}_{\mathrm{Fix}(T)}(x_0)$.*

Corollary 3.10. *Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$. If $0 < a < \alpha_n \leqslant \beta_n < b < \frac{1}{1+\sqrt{2}}$, then the sequence $\{x_n\}$ defined by (3.16) converges strongly to $x^\dagger = \mathrm{proj}_{\mathrm{Fix}(T)}(x_0)$.*

Acknowledgment

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. N. Shahzad acknowledges with thanks DSR for financial support. Yeong-Cheng Liou was supported in part by NSC 101-2628-E-230-001-MY3 and NSC 101-2622-E-230-005-CC3. Li-Jun Zhu was supported in part by NNSF of China (61362033).

References

[1] L.-C. Ceng, A. Petruşel, J.-C. Yao, *Strong convergence of modified implicit iterative algorithms with perturbed mappings for continuous pseudocontractive mappings*, Appl. Math. Comput., **209** (2009), 162–176. 1

[2] C. E. Chidume, M. Abbas, B. Ali, *Convergence of the Mann iteration algorithm for a class of pseudocontractive mappings*, Appl. Math. Comput., **194** (2007), 1–6.

[3] C. E. Chidume, S. A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontractions*, Proc. Amer. Math. Soc., **129** (2001), 2359–2363. 1.2

[4] Y. J. Cho, S. M. Kang, X.-L. Qin, *Strong convergence of an implicit iterative process for an infinite family of strict pseudocontractions*, Bull. Korean Math. Soc., **47** (2010), 1259–1268.

[5] L. Ćirić, A. Rafiq, N. Cakić, J. S. Ume, *Implicit Mann fixed point iterations for pseudo-contractive mappings*, Appl. Math. Lett., **22** (2009), 581–584.

[6] W.-P. Guo, M. S. Choi, Y. J. Cho, *Convergence theorems for continuous pseudocontractive mappings in Banach spaces*, J. Inequal. Appl., **2014** (2014), 10 pages.

[7] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44** (1974), 147–150. 1, 1.1

[8] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28** (1969), 326–329. 2, 2.3

[9] C. Moore, B. V. C. Nnoli, *Strong convergence of averaged approximants for Lipschitz pseudocontractive maps*, J. Math. Anal. Appl., **260** (2001), 269–278. 1

[10] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Advances in Math., **3** (1969), 510–585. 1, 2

[11] X.-L. Qin, Y. J. Cho, H.-Y. Zhou, *Strong convergence theorems of fixed point for quasi-pseudo-contractions by hybrid projection algorithms*, Fixed Point Theory, **11** (2010), 347–354. 1

[12] N. Shahzad, N. Zegeye, *On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces*, Nonlinear Anal., **71** (2009), 838–844. 1

[13] T. Suzuki, *Moudafi's viscosity approximations with Meir-Keeler contractions*, J. Math. Anal. Appl., **325** (2007), 342–352. 2.4, 2.5

[14] M. Tsukada, *Convergence of best approximations in a smooth Banach space*, J. Approx. Theory, **40** (1984), 301–309. 2, 2.2

[15] A. Udomene, *Path convergence, approximation of fixed points and variational solutions of Lipschitz pseudocontractions in Banach spaces*, Nonlinear Anal., **67** (2007), 2403–2414.

[16] Y.-H. Yao, Y.-C. Liou, G. Marino, *A hybrid algorithm for pseudo-contractive mappings*, Nonlinear Anal., **71** (2009), 997–5002. 1, 1.4

[17] Y.-H. Yao, Y.-C. Liou, J.-C. Yao, *Split common fixed point problem for two quasi-pseudo-contractive operators and its algorithm construction*, Fixed Point Theory Appl., **2015** (2015), 19 pages.

[18] Y.-H. Yao, M. A. Noor, S. Zainab, Y.-C. Liou, *Mixed equilibrium problems and optimization problems*, J. Math. Anal. Appl., **354** (2009), 319–329.

[19] Y.-H. Yao, M. Postolache, S. M. Kang, *Strong convergence of approximated iterations for asymptotically pseudocontractive mappings*, Fixed Point Theory Appl., **2014** (2014), 13 pages.

[20] Y.-H. Yao, M. Postolache, Y.-C. Liou, *Coupling Ishikawa algorithms with hybrid techniques for pseudocontractive mappings*, Fixed Point Theory Appl., **2013** (2013), 8 pages.

[21] Y.-H. Yao, M. Postolache, Y.-C. Liou, *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl., **2013** (2013), 12 pages.

[22] Y.-H. Yao, N. Shahzad, *New methods with perturbations for non-expansive mappings in Hilbert spaces*, Fixed Point Theory Appl., **2011** (2011), 9 pages.

[23] Y.-H. Yao, N, Shahzad, *Implicit and explicit methods for finding fixed points of strictly pseudo-contractive mappings in Banach spaces*, J. Nonlinear Convex Anal., **13** (2012), 183–194.

[24] H. Zegeye, E. U. Ofoedu, N. Shahzad, *Convergence theorems for equilibrium problem, variational inequality problem and countably infinite relatively quasi-nonexpansive mappings*, Appl. Math. Comput. **216** (2010), 3439–3449.

[25] H. Zegeye, N. Shahzad, *Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings*, Nonlinear Anal., **70** (2009), 2707–2716.

[26] H. Zegeye, N. Shahzad, *Approximating common solution of variational inequality problems for two monotone mappings in Banach spaces*, Optim. Lett., **5** (2011), 691–704.

[27] H. Zegeye, N. Shahzad, *An algorithm for a common fixed point of a family of pseudocontractive mappings*, Fixed Point Theory Appl., **2013** (2013), 14 pages.

[28] H. Zegeye, N. Shahzad, M. A. Alghamdi, *Convergence of Ishikawa's iteration method for pseudocontractive mappings*, Nonlinear Anal., **74** (2011), 7304–7311.

[29] H. Zegeye, N. Shahzad, T. Mekonen, *Viscosity approximation methods for pseudocontractive mappings in Banach spaces*, Appl. Math. Comput., **185** (2007), 538–546.

[30] H.-Y. Zhou, *Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl., **343** (2008), 546–556. 1, 1.3, 2.1

[31] H.-Y. Zhou, *Strong convergence of an explicit iterative algorithm for continuous pseudo-contractions in Banach spaces*, Nonlinear Anal., **70** (2009), 4039–4046. 1