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New multipled common fixed point theorems in Menger PMT-spaces

Cuiru Ji, Chuanxi Zhu*, Zhaoqi Wu

Department of Mathematics, Nanchang University, Nanchang, 330031, P. R. China.

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Abstract

In this work, we introduce the notion of Menger probabilistic metric type space, on the other hand, we introduce a more general class of auxiliary functions in contractivity condition, following that, we obtain some multipled common fixed point theorems for a pair of mappings $T : \underbrace{X \times X \cdots \times X}_{m\text{-times}} \to X$ and $A : X \to X$. As an application, we give out an example to demonstrate the validity of the obtained results.

Keywords: Multipled common fixed point, Menger PMT-spaces, ψ-contractive mapping.
2010 MSC: 47H10, 46S10.

1. Introduction

In 1942, Menger [12] initiated the study of PM-spaces and then Sehgal and Bharucha-Reid [16] followed Menger's line of research by using the notion of probabilistic q-contraction. They proved a unique fixed point result, which is an extension of the celebrated Banach's contraction principles [2]. Since then, many scholars have studied the existence of coupled fixed points in Menger spaces [3, 4, 8, 11, 15, 17, 18]. Recently, Choudhury and Das [5] gave a generalized unique fixed point theorem by using an altering distance function which was originally introduced by Khan et al. [9]. This extension of altering distance function is called ϕ-function, and has been further used in many related literatures [6, 13, 19]. Dutta et al. [7] defined nonlinear generalized contractive type mapping involving ψ-contractive mapping and proved their theorems for such kind of mapping in the setting of G-complete Menger PM-spaces. Then Kutbi et al. [10] weakened the notion of ψ-contractive mapping and established some fixed point theorems in G-complete Menger PM-spaces. After then, many fixed point results have been obtained by many authors. In 2015, Abdou et al. [1] introduced Menger PMT-spaces and established corresponding fixed point theorems. Moreover, Hierro and Sen [14] introduced a new auxiliary function and established corresponding fixed point theorems.

In this paper, motivated by the idea of Menger PMT-spaces and ψ-contractive mapping, we establish some multipled common fixed point theorems for a pair of mappings $T : \underbrace{X \times X \cdots \times X}_{m\text{-times}} \to X$ and $A : X \to X$ in complete PMT-spaces. Finally, an example is given to support our main results.

*Corresponding author
Email address: chuanxizhu@126.com (Chuanxi Zhu)

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2. Preliminaries

Let $\mathbb{R}$ denote the set of reals, $\mathbb{R}^+$ the nonnegative reals and $\mathbb{Z}^+$ be the set of all positive integers. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing and left continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$. We will denote by $\mathscr{D}$ the set of all distribution functions, while H will always denote the special distribution function defined by

$$H(t) = \begin{cases} 0, & t \leqslant 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 ([15]). A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if the following conditions are satisfied:

(1) $T(a, b) = T(b, a)$ and $T(a, T(b, c)) = T(T(a, b), c)$, for all $a, b, c \in [0, 1]$;
(2) T is continuous;
(3) $T(a, 1) = a$ for all $a \in [0, 1]$;
(4) $T(a, b) \geqslant T(c, d)$, whenever $a \geqslant c$ and $b \geqslant d$, for $a, b, c, d \in [0, 1]$.

Form the definition of T, it follows that $T(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. The following are three basic continuous t-norms:

(1) the minimum t-norm, defined by $T_M(a, b) = \min\{a, b\}$;
(2) the product t-norm, defined by $T_P(a, b) = ab$;
(3) the Lukasiewicz t-norm, defined by $T_L(a, b) = \max\{a + b - 1, 0\}$.

These t-norms are related in that way: $T_L \leqslant T_P \leqslant T_M$.

Definition 2.2 ([15]). A Menger probabilistic metric space (briefly, Menger PM-space) is a triplet (X, F, Δ) where X is a nonempty set, Δ is a continuous t-norm and F is a mapping from $X \times X$ into $\mathscr{D}^+$ such that, if $F_{x,y}$ denotes the value of F at the pair (x, y), the following conditions hold:

(PM-1) $F_{x,y}(t) = H(t)$ if and only if $x = y$, $t > 0$;

(PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$;

(PM-3) $F_{x,y}(t + s) \geqslant \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $t, s \geqslant 0$.

Definition 2.3 ([1]). A Menger probabilistic metric type space (briefly, Menger PMT-space) is a triplet (X, F, Δ) where X is a nonempty set, Δ is a continuous t-norm and F is a mapping from $X \times X$ into $\mathscr{D}^+$ such that, if $F_{x,y}$ denotes the value of F at the pair (x, y), the following conditions hold:

(PM-1) $F_{x,y}(t) = H(t)$ if and only if $x = y$, $t > 0$;

(PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$;

(PM-3) $F_{x,y}(K(t + s)) \geqslant \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $t, s \geqslant 0$ for some constant $K \geqslant 1$.

Clearly, every Menger PM-space is a Menger PMT-space, but the converse is false, as we can see in the following example.

Definition 2.4 ([1]). Let (X, F, Δ) be a PMT-space. For each $x \in X$ and $\lambda > 0$, the strong λ-neighborhood of x is the set

$$N_x(\lambda) = \{y \in X : F_{x,y}(\lambda) > 1 - \lambda\}$$

and strong neighborhood system for X is the union $\bigcup_{x\in V} N_x$, where

$$N_x = \{N_x(\lambda) : \lambda > 0\}.$$

The strong neighborhood system for X determines a Hausdorff topology for X.

Definition 2.5 ([1]). Let (X, F, Δ) be a PMT-space. Then,

(1) a sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer Z^+ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geqslant Z^+$;
(2) a sequence $\{x_n\}$ in X is called a Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer Z^+ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $m, n \geqslant Z^+$;
(3) a Menger PMT-space is said to be complete, if every Cauchy sequence in X is convergent to a point in X.

Definition 2.6 ([5]). A function $\Phi : R^+ \to R^+$ is said to be a ϕ-function if it satisfies the following conditions:

(1) $\phi(t)$=0 if and only if $t = 0$;
(2) $\phi(t)$ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$;
(3) ϕ is left continuous in $(0,\infty)$;
(4) ϕ is continuous at 0.

Definition 2.7 ([5]). Let Ψ_0 be the class of all non-decreasing functions $\psi : R^+ \to R^+$ satisfying:

(1) ψ is nondecreasing;
(2) ψ is continuous at $t = 0$;
(3) $\psi(0) = 0$;
(4) if $\{a_n\} \subset [0, +\infty)$ is a sequence such that $\{a_n\} \to 0$, then $\psi^n(a_n) \to 0$ (where ψ^n denotes the nth-iterate of ψ).

First of all, we show that we do not need to assume that ψ is continuous at $t = 0$ for function in Ψ_0 under the rest of the assumption.

Proposition 2.8 ([14]). *Let $\psi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function such that $\psi(0) = 0$.*

(1) *If ψ is not continuous at $t = 0$, then there exists $\varepsilon_0 \geqslant 0$ for all $t > 0$.*
(2) *If ψ satisfies $\psi^n(a_n) \to 0$ whenever $\{a_n\} \to 0$ as $n \to \infty$, then ψ is continuous at $t = 0$.*

Definition 2.9 ([19]). Let X be a non-empty set. Let $T : \underbrace{X \times X \cdots \times X}_{m\text{-times}} \to X$ and $A : X \to X$ be two mappings. A is said to be commutative with T if $AT(x, y, \cdots, z) = T(Ax, Ay, \cdots, Az)$ for all $x, y, \cdots, z \in X$. A point $u \in X$ is called a multipled common fixed point of T and A if $u = Au = T(u, u, \cdots, u)$.

Definition 2.10 ([14]). We shall denote by $\mathscr{H}$ the family of function $h : (0, 1] \to [0, +\infty)$ satisfying:

($\mathscr{H}_1$) if $\{a_n\} \subset (0, 1]$, the $a_n \to 1$ if and only if, $h(a_n) \to 1$;
($\mathscr{H}_2$) if $\{a_n\} \subset (0, 1]$, the $a_n \to 0$ if and only if, $h(a_n) \to \infty$.

Proposition 2.11 ([14]). *If $f \in \mathscr{H}$, then $h(1) = 0$. Furthermore, $h(t) = 0$ if and only if, $t = 1$.*

3. Main results

Theorem 3.1 ([19]). *Let (X, F, Δ) be a Menger PMT-space and Δ be a continuous t-norm. Then the following statements are equivalent:*

(1) *the sequence $\{x_n\}$ is a Cauchy sequence;*
(2) *for all $\varepsilon > 0$, there exists $M \in N^+$ such than $\lim_{n\to\infty} F_{x_n,x_m}(\varepsilon) = 1$ for all $n, m > M$.*

Proof. (1)⇒(2). This can be easily seen from Definition 2.5.

(2)⇒(1). Since Δ be a continuous t-norm, for every $\varepsilon > 0$ and $0 < \lambda < 1$, there exists $\lambda_0 \in (0, \lambda]$, such that $\Delta(1 - \lambda_0, 1 - \frac{\lambda}{2}) > 1 - \lambda$. Let $\lambda_1 = \min\{\lambda_0, \frac{\lambda}{2}\}$. Then $\Delta(1 - \lambda_1, 1 - \lambda_1) > 1 - \lambda$. Hence, from (2), there exists $M \in N^+$ and $K \geqslant 1$, such that $F_{x_n,x_m}(\frac{\varepsilon}{2K}) > 1 - \lambda_1$ and $F_{x_l,x_m}(\frac{\varepsilon}{2K}) > 1 - \lambda_1$ for all $n, m, l \geqslant M$. Then we have $F_{x_n,x_l}(\varepsilon) \geqslant \Delta(F_{x_n,x_m}(\frac{\varepsilon}{2K}), F_{x_l,x_m}(\frac{\varepsilon}{2K})) \geqslant \Delta(1 - \lambda_1, 1 - \lambda_1) > 1 - \lambda$. Thus, $\{x_n\}$ is a Cauchy sequence. □

Theorem 3.2. *Let* (X, F, Δ) *be a complete Menger PMT-space with* Δ *as a continuous* t*-norm. Let*

$$T : \underbrace{X \times X \cdots \times X}_{m\text{-times}} \to X$$

and $A : X \to X$ *be two mappings satisfying the following inequality:*

$$h(F_{T(x,y,\cdots,z),T(p,q,\cdots,r)}(\phi(ct))) \leqslant \psi\{\frac{h(F_{Ax,Ap}(\phi(t))) + h(F_{Ay,Aq}(\phi(t))) + \cdots + h(F_{Az,Ar}(\phi(t)))}{m}\}, \tag{3.1}$$

for all $x, y, \cdots, z \in X$*,* $p, q, \cdots, r \in X$*,* $c \in (0, 1)$*,* $\phi \in \Phi$*,* $\psi \in \Psi_0$*,* $t > 0$*, such that* $F_{Aa,Ap}(\phi(t)) > 0, F_{Ay,Aq}(\phi(t)) > 0, \cdots, F_{Az,Ar}(\phi(t)) > 0$*, where* $T(X \times X \cdots \times X) \subset A(X)$*, and* A *is continuous and commutative with* T*. Then there exists a unique multipled common fixed point of* A *and* T*, i.e., there exists* $u \in X$ *such that* $u = Au = T(u, u, \cdots, u)$*.*

Proof. Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}, \cdots, \{z_n\}_{n=1}^{\infty}$ be m-times sequences in X such that $Ax_{n+1} = T(x_n, y_n, \cdots, z_n)$ and $Ay_{n+1} = T(y_n, \cdots, z_n, x_n)$, $Az_{n+1} = T(z_n, x_n, y_n, \cdots)$. From $\sup_{t\in\mathbb{R}} F_{Ax_0,Ax_1}(t) = 1$, $\sup_{t\in\mathbb{R}} F_{Ay_0,Ay_1}(t) = 1, \cdots, \sup_{t\in\mathbb{R}} F_{Az_0,Az_1}(t) = 1$ and the definition of ϕ, one can find $t > 0$ such that $F_{Ax_0,Ax_1}(\phi(\frac{t}{c})) > 0, F_{Ay_0,Ay_1}(\phi(\frac{t}{c})) > 0, \cdots, F_{Az_0,Az_1}(\phi(\frac{t}{c})) > 0$. From (3.1), we have

$$\begin{aligned} h(F_{Ax_1,Ax_2}(\phi(t))) &= h(F_{T(x_0,y_0,\cdots,z_0),T(x_1,y_1,\cdots,z_1)}(\phi(\frac{t}{c}))) \\ &\leqslant \psi\{\frac{h(F_{Ax_0,Ax_1}(\phi(\frac{t}{c}))) + h(F_{Ay_0,Ay_1}(\phi(\frac{t}{c}))) + \cdots + h(F_{Az_0,Az_1}(\phi(\frac{t}{c})))}{m}\}. \end{aligned} \tag{3.2}$$

Similarly, we have

$$h(F_{Ay_1,Ay_2}(\phi(t))) \leqslant \psi\{\frac{h(F_{Ay_0,Ay_1}(\phi(\frac{t}{c}))) + h(F_{Az_0,Az_1}(\phi(\frac{t}{c}))) + \cdots + h(F_{Ax_0,Ax_1}(\phi(\frac{t}{c})))}{m}\}, \tag{3.3}$$

$$\vdots$$

$$h(F_{Az_1,Az_2}(\phi(t))) \leqslant \psi\{\frac{h(F_{Az_0,Az_1}(\phi(\frac{t}{c}))) + h(F_{Ax_0,Ax_1}(\phi(\frac{t}{c}))) + h(F_{Ay_0,Ay_1}(\phi(\frac{t}{c}))) + \cdots}{m}\}. \tag{3.4}$$

Suppose that $P_0(t) = \frac{h(F_{Ax_0,Ax_1}(\phi(t))) + h(F_{Ay_0,Ay_1}(\phi(t))) + \cdots + h(F_{Az_0,Az_1}(\phi(t)))}{m}$, from (3.2), (3.3), and (3.4) we deduce that $F_{Ax_1,Ax_2}(\phi(t)) > 0, F_{Ay_1,Ay_2}(\phi(t)) > 0, \cdots, F_{Az_1,Az_2}(\phi(t)) > 0$, and so $F_{Ax_1,Ax_2}(\phi(\frac{t}{c})) > 0, F_{Ay_1,Ay_2}(\phi(\frac{t}{c})) > 0, \cdots, F_{Az_1,Az_2}(\phi(\frac{t}{c})) > 0$, then we have

$$\begin{aligned} h(F_{Ax_2,Ax_3}(\phi(t))) &= h(F_{T(x_1,y_1,\cdots,z_1),T(x_2,y_2,\cdots,z_2)}(\phi(t))) \\ &\leqslant \psi\{\frac{h(F_{Ax_1,Ax_2}(\phi(\frac{t}{c}))) + h(F_{Ay_1,Ay_2}(\phi(\frac{t}{c}))) + \cdots + h(F_{Az_1,Az_2}(\phi(\frac{t}{c})))}{m}\} \\ &\leqslant \psi\{\frac{\psi(P_0(\frac{t}{c^2})) + \psi(P_0(\frac{t}{c^2})) + \cdots + \psi(P_0(\frac{t}{c^2}))}{m}\} \\ &= \psi^2\{P_0(\frac{t}{c^2})\}. \end{aligned}$$

Similarly, we have

$$h(F_{Ay_2,Ay_3}(\phi(t))) \leqslant \psi^2\{P_0(\frac{t}{c^2})\},$$

$$\vdots$$

$$h(F_{Az_2,Az_3}(\phi(t))) \leqslant \psi^2\{P_0(\frac{t}{c^2})\}.$$

Reaping the above procedure, we get

$$h(F_{Ax_n,Ax_{n+1}}(\phi(t))) \leqslant \psi^n\{P_0(\frac{t}{c^n})\}. \tag{3.5}$$

If we change Ax_0 with Ax_r in (3.5), then for all $n > r$ we get

$$h(F_{Ax_n,Ax_{n+1}}(\phi(c^r t))) \leqslant \psi^{n-r}\{P_0(\frac{c^r t}{c^{n-r}})\}.$$

Since $\psi^n(a_n) \to 0$ whenever $a_n \to 0$ as $n \to \infty$, therefore the above inequality implies that

$$\lim_{n\to\infty} h(F_{Ax_n,Ax_{n+1}}(\phi(c^r t))) = 0.$$

In particular, as $h \in \mathscr{H}$, condition $(\mathscr{H}_1)$ implies that

$$\lim_{n\to\infty} F_{Ax_n,Ax_{n+1}}(\phi(c^r t)) = 1.$$

Now, let $\varepsilon > 0$ be given, using the properties of function ϕ we can find $r \in \mathbb{Z}^+$ such that $\phi(c^r t) < \varepsilon$. Then we have

$$\lim_{n\to\infty} F_{Ax_n,Ax_{n+1}}(\varepsilon) \geqslant \lim_{n\to\infty} F_{Ax_n,Ax_{n+1}}(\phi(c^r t)) = 1. \tag{3.6}$$

By using a triangle inequality, we obtain

$$F_{Ax_n,Ax_{n+p}}(\varepsilon) \geqslant \Delta\left(\underbrace{F_{Ax_n,Ax_{n+1}}(\frac{\varepsilon}{Kp}), \Delta(F_{Ax_{n+1},Ax_{n+2}}(\frac{\varepsilon}{Kp}), \cdots, F_{Ax_{n+p-1},Ax_{n+p}}(\frac{\varepsilon}{Kp}))}_{\text{p-times}}\right).$$

Letting $n \to \infty$ and making use of (3.6), for any integer p, we get

$$\lim_{n\to\infty} F_{Ax_n,Ax_{n+p}}(\varepsilon) = 1 \ \text{ for every } \ \varepsilon > 0.$$

Hence $\{Ax_n\}$ is a Cauchy sequence, similarly, we can obtain $\{Ay_n\}, \cdots, \{Az_n\}$ are Cauchy sequences. Since (X, F, Δ) is complete, therefore $\lim_{n\to\infty} Ax_n = u, \lim_{n\to\infty} Ay_n = v, \cdots, \lim_{n\to\infty} Az_n = w$ for some $u, v, \cdots, w \in X$.

Now we show that $Au = T(u, v, \cdots, w)$.

Since A is continuous, we have $\lim_{n\to\infty} AAx_n = Au, \lim_{n\to\infty} AAy_n = Av, \cdots, \lim_{n\to\infty} AAz_n = Aw$. Then the commutative of A with T implies that $AAx_{n+1} = T(Ax_n, Ay_n, \cdots, Az_n)$. From (3.1) we obtain

$$\begin{aligned} h(F_{AAx_{n+1},T(u,v,\cdots,w)}(\phi(t))) &= h(F_{T(Ax_n,Ay_n,\cdots,Az_n),T(u,v,\cdots,w)}(\phi(t))) \\ &\leqslant \psi\left\{\frac{h(F_{AAx_n,Au}(\phi(\frac{t}{c}))) + h(F_{AAy_n,Av}(\phi(\frac{t}{c}))) + \cdots + h(F_{AAz_n,Aw}(\phi(\frac{t}{c})))}{m}\right\}. \end{aligned}$$

Letting $n \to \infty$, since $\psi(0) = 0$, we have $\lim_{n\to\infty} AAx_n = T(u, v, \cdots, w)$, from the above inequality, we get $Au = T(u, v, \cdots, w)$. Similarly, we have $Av = T(u, v, \cdots, w), \cdots, Aw = T(u, v, \cdots, w)$.

Next we show $Au = u$. From (3.1), we have

$$\begin{aligned} h(F_{Ax_1,Au}(\phi(t))) &= h(F_{T(Ax_0,Ay_0,\cdots,Az_0),T(u,v,\cdots,w)}(\phi(t))) \\ &\leqslant \psi\left\{\frac{h(F_{Ax_0,Au}(\phi(\frac{t}{c}))) + h(F_{Ay_0,Av}(\phi(\frac{t}{c}))) + \cdots + h(F_{Az_0,Aw}(\phi(\frac{t}{c})))}{m}\right\}, \end{aligned} \tag{3.7}$$

$$h(F_{Ay_1,Av}(\phi(t))) \leqslant \psi\left\{\frac{h(F_{Ay_0,Av}(\phi(\frac{t}{c}))) + \cdots + h(F_{Az_0,Aw}(\phi(\frac{t}{c}))) + h(F_{Ax_0,Au}(\phi(\frac{t}{c})))}{m}\right\}, \tag{3.8}$$

$$\vdots$$

$$h(F_{Az_1,Aw}(\phi(t))) \leqslant \psi\left\{\frac{h(F_{Az_0,Aw}(\phi(\frac{t}{c}))) + h(F_{Ax_0,Au}(\phi(\frac{t}{c}))) + h(F_{Ay_0,Av}(\phi(\frac{t}{c}))) + \cdots}{m}\right\}. \tag{3.9}$$

Suppose that $Q_0(t) = \frac{h(F_{Ax_0,Au}(\phi(t)))+h(F_{Ay_0,Av}(\phi(t)))+\cdots+h(F_{Az_0,Aw}(\phi(t)))}{m}$. Combining (3.7), (3.8), and (3.9) we obtain

$$\begin{aligned} h(F_{Ax_2,u}(\phi(t))) &\leqslant \psi\left\{\frac{h(F_{Ax_1,Au}(\phi(\frac{t}{c}))) + h(F_{Ay_1,Av}(\phi(\frac{t}{c}))) + \cdots + h(F_{Az_1,Aw}(\phi(\frac{t}{c})))}{m}\right\} \\ &\leqslant \psi\left\{\frac{\psi(Q_0(\frac{t}{c^2})) + \psi(Q_0(\frac{t}{c^2})) + \cdots + \psi(Q_0(\frac{t}{c^2}))}{m}\right\} \\ &= \psi^2\{Q_0(\frac{t}{c^2})\}. \end{aligned}$$

Similarly, we have

$$h(F_{Ay_2,Av}(\phi(t))) \leqslant \psi^2\{Q_0(\frac{t}{c^2})\},$$

$$\vdots$$

$$h(F_{Az_2,Aw}(\phi(t))) \leqslant \psi^2\{Q_0(\frac{t}{c^2})\}.$$

Repeating the above procedure, we obtain

$$h(F_{Ax_n,Au}(\phi(t))) \leqslant \psi^n\{Q_0(\frac{t}{c^n})\}.$$

Since $\psi^n(a_n) \to 0$ whenever $a_n \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} Ax_n = Au$, which implies that $Au = u = T(u, v, \cdots, w)$, similarly, we have $Av = v = T(u, v, \cdots, w), \cdots, Aw = w = T(u, v, \cdots, w)$.

Finally, we show $u = v = \cdots = w$. Without loss of generality, we denote $u = e_1, v = e_2, \cdots, w = e_n$, then $Ae_1 = e_1 = T(e_1, e_2, e_3, \cdots, e_{m-1}, e_m), Ae_2 = e_2 = T(e_2, e_3, \cdots, e_{m-1}, e_m, e_1), \cdots, Ae_m = e_m = T(e_m, e_1, e_2, e_3, \cdots, e_{m-1})$.

First, we prove that $F_{e_1,e_2}(\phi(s)) > 0$ for all $s > 0$. By the definition of ϕ, we have $\phi(\frac{s}{c^n}) \to 0$ as $n \to \infty$. Since $\sup\limits_{n\in\mathbb{Z}^+} F_{e_1,e_2}(\phi(\frac{s}{c^n})) = 1$, $\sup\limits_{n\in\mathbb{Z}^+} F_{e_2,e_3}(\phi(\frac{s}{c^n})) = 1, \cdots, \sup\limits_{n\in\mathbb{Z}^+} F_{e_m,e_1}(\phi(\frac{s}{c^n})) = 1$, we deduce that there exists $n \in \mathbb{Z}^+$ such that $F_{e_1,e_2}(\phi(\frac{s}{c^n})) > 0, F_{e_2,e_3}(\phi(\frac{s}{c^n})) > 0, \cdots, F_{e_m,e_1}(\phi(\frac{s}{c^n})) > 0$. Using (3.1), we obtain

$$\begin{aligned} &h(F_{e_1,e_2}(\phi(\frac{s}{c^{n-1}}))) \\ &\quad= h(F_{T(e_1,e_2,\cdots,e_{m-1},e_m),T(e_2,\cdots,e_{m-1},e_m,e_1)}(\phi(\frac{s}{c^{n-1}}))) \\ &\quad\leqslant \psi\{\frac{h(F_{e_1,e_2}(\phi(\frac{s}{c^n}))) + h(F_{e_2,e_3}(\phi(\frac{s}{c^n}))) + \cdots + h(F_{e_{m-1},e_m}(\phi(\frac{s}{c^n}))) + h(F_{e_m,e_1}(\phi(\frac{s}{c^n})))}{m}\}, \end{aligned}$$

which implies that $F_{e_1,e_2}(\phi(\frac{s}{c^{n-1}})) > 0$, similarly, we have $F_{e_2,e_3}(\phi(\frac{s}{c^{n-1}})) > 0, \cdots, F_{e_m,e_1}(\phi(\frac{s}{c^{n-1}})) > 0$. By reaping a similar reasoning n times we deduce that $F_{e_1,e_2}(\phi(s)) > 0, F_{e_2,e_3}(\phi(s)) > 0, \cdots, F_{e_m,e_1}(\phi(s)) > 0$ for all $s > 0$.

Second, we show that $F_{e_1,e_2}(\phi(s)) = 1$. In fact, for every $s > 0$, we have $F_{e_1,e_2}(\phi(\frac{s}{c^i})) > 0$ for all $1 \leqslant i \leqslant n$ and for all $n \in \mathbb{Z}^+$. Then by using (3.1), we get

$$h(F_{e_1,e_2}(\phi(s))) = h(F_{T(e_1,e_2,\cdots,e_{m-1},e_m),T(e_2,\cdots,e_{m-1},e_m,e_1)}(\phi(s))$$

$$\begin{aligned}
&\leqslant \psi\{\frac{h(F_{e_1,e_2}(\phi(\frac{s}{c}))) + h(F_{e_2,e_3}(\phi(\frac{s}{c}))) + \cdots + h(F_{e_{m-1},e_m}(\phi(\frac{s}{c}))) + h(F_{e_m,e_1}(\phi(\frac{s}{c})))}{m}\},\\
h(F_{e_2,e_3}(\phi(s))) &= h(F_{T(e_2,e_3,\cdots,e_m,e_1),T(e_3,e_4,\cdots,e_1,e_2)}(\phi(s)))\\
&\leqslant \psi\{\frac{h(F_{e_2,e_3}(\phi(\frac{s}{c}))) + h(F_{e_3,e_4}(\phi(\frac{s}{c}))) + \cdots + h(F_{e_m,e_1}(\phi(\frac{s}{c}))) + h(F_{e_1,e_2}(\phi(\frac{s}{c})))}{m}\},\\
&\vdots\\
h(F_{e_n,e_1}(\phi(s))) &= h(F_{T(e_m,e_1,\cdots,e_{m-2},e_{m-1}),T(e_1,e_2,\cdots,e_{m-1},e_m)}(\phi(s)))\\
&\leqslant \psi\{\frac{h(F_{e_m,e_1}(\phi(\frac{s}{c}))) + h(F_{e_1,e_2}(\phi(\frac{s}{c}))) + \cdots + h(F_{e_{m-2},e_{m-1}}(\phi(\frac{s}{c}))) + h(F_{e_{m-1},e_m}(\phi(\frac{s}{c})))}{m}\}.
\end{aligned}$$

Suppose that $E(s) = \frac{h(F_{e_1,e_2}(\phi(s)))+h(F_{e_2,e_3}(\phi(s)))+\cdots+h(F_{e_{m-1},e_m}(\phi(s)))+h(F_{e_m,e_1}(\phi(s)))}{m}$, then $E(s) \leqslant \psi\{E(\frac{s}{c})\}$. By n-iterations we get

$$h(F_{e_1,e_2}(\phi(s)) \leqslant \psi\{E(\frac{s}{c})\} \leqslant \psi^2\{E(\frac{s}{c^2})\} \leqslant \cdots \leqslant \psi^n\{E(\frac{s}{c^n})\}.$$

Thus, since $\psi^n(a_n) \to 0$ whenever $a_n \to 0$ as $n \to \infty$, we get $F_{e_1,e_2}(\phi(s)) = 1$. It follows that $F_{e_1,e_2}(t) = H(t)$ for all $t > 0$. In fact, if t is not in range of ϕ, since ϕ is continuous at 0, there exists $s > 0$ such that $\phi(s) < t$. This implies that $F_{e_1,e_2}(t) \geqslant F_{e_1,e_2}(\phi(s)) = 1$, then $e_1 = e_2$. Similarly, we have $e_2 = e_3, \cdots, e_m = e_1$, i.e., $u = v = \cdots = w$. Thus, $u \in X$ is the unique multipled common fixed point of A and T.

Taking $m = 1$ in Theorem 3.2, then $T : X \to X, A : X \to X$, $Ax = x$ for all $x \in X$. It is obvious that $T(X) \subset A(X)$. A is continuous and commutative with T, which also satisfy the conditions in Theorem 3.1, then we have the following consequence. □

Corollary 3.3. *Let (X, F, Δ) be a complete Menger space with Δ as a continuous t-norm. Let $T : X \to X$ satisfy the following inequality:*

$$h(F_{Tx,Ty}(\phi(ct))) \leqslant \psi(h(F_{x,y}(\phi(t)))),$$

for all $x, y \in X$, $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi_0$, $t > 0$, such that $F_{x,y}(\phi(t)) > 0$. Then T has a unique fixed point such that $u = Au = Tu$.

Taking $A = I$ (I is the identity mapping) in Theorem 3.2, we obtain the following corollary.

Corollary 3.4. *Let (X, F, Δ) be a complete Menger PMT-space and Δ be a continuous t-norm. Let*

$$T : \underbrace{X \times X \cdots \times X}_{m\text{-times}} \to X$$

and $A : X \to X$ be two mappings satisfying the following inequality:

$$h(F_{T(x,y,\cdots,z),T(p,q,\cdots,r)}(\phi(ct))) \leqslant \psi\{\frac{h(F_{x,p}(\phi(t))) + h(F_{y,q}(\phi(t))) + \cdots + h(F_{z,r}(\phi(t)))}{m}\},$$

for all $x, y, \cdots, z, p, q, \cdots, r \in X$, $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi_0$, $t > 0$. Let T be continuous and commutative. Then there exists a unique multipled common fixed point of T.

From the proof of Theorem 3.2, we can similarly prove the following result.

Theorem 3.5. *Let (X, F, Δ) be a complete Menger PMT-space with Δ as a continuous t-norm. Let*

$$T : \underbrace{X \times X \cdots \times X}_{m\text{-times}} \to X$$

and $A : X \to X$ *be two mappings satisfying the following inequality:*

$$h(F_{T(x,y,\cdots,z),T(p,q,\cdots,r)}(\phi(ct))) \leqslant \psi\{\min\{h(F_{Ax,Ap}(\phi(t))), h(F_{Ay,Aq}(\phi(t))), \cdots, h(F_{Az,Ar}(\phi(t)))\}\}$$

for all $x, y, \cdots, z, p, q, \cdots, r \in X$, $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi_0$, $t > 0$, *such that* $F_{Aa,Ap}(\phi(t)) > 0, F_{Ay,Aq}(\phi(t)) > 0, F_{Az,Ar}(\phi(t)) > 0$, *where* $T(X \times X \cdots \times X) \subset A(X)$, *and* A *is continuous and commutative with* T. *Then there exists a unique multipled common fixed point of* A *and* T, *i.e., there exists* $u \in X$ *such that* $u = Au = T(u, u, \cdots, u)$.

Taking $A = I$ (I is the identity mapping) in Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *Let* (X, F, Δ) *be a complete PMT-space with* Δ *as a continuous* t*-norm. Let*

$$T : \underbrace{X \times X \cdots \times X}_{m\text{-times}} \to X$$

and $A : X \to X$ *be two mappings satisfying the following inequality:*

$$h(F_{T(x,y,\cdots,z),T(p,q,\cdots,r)}(\phi(ct))) \leqslant \psi\{\min\{h(F_{x,p}(\phi(t))), h(F_{y,q}(\phi(t))), \cdots, h(F_{z,r}(\phi(t)))\}\},$$

for all $x, y, \cdots, z, p, q, \cdots, r \in X$, $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi_0$, $t > 0$, *and* T *is continuous and commutative. Then there exists a unique multipled common fixed point of* T.

Theorem 3.7. *Let* (X, F, Δ) *be a complete Menger PMT-space with* Δ *as a continuous* t*-norm and* $\Delta \leqslant \Delta_p$. *Let* $T : \underbrace{X \times X \cdots \times X}_{m\text{-times}} \to X$ *and* $A : X \to X$ *be two mappings satisfying the following inequality:*

$$h(F_{T(x,y,\cdots,z),T(p,q,\cdots,r)}(\phi(ct))) \leqslant \psi\left\{\sqrt[m]{\Delta(h(F_{Ax,Ap}(\phi(t))), \Delta(h(F_{Ay,Aq}(\phi(t))), \cdots, h(F_{Az,Ar}(\phi(t)))))}\right\}$$

for all $x, y, \cdots, z, p, q, \cdots, r \in X$, $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi_0$, $t > 0$, *such that* $F_{Aa,Ap}(\phi(t)) > 0, F_{Ay,Aq}(\phi(t)) > 0, \cdots, F_{Az,Ar}(\phi(t)) > 0$, *where* $T(X \times X \cdots \times X) \subset A(X)$, *and* A *is continuous and commutative with* T. *Then there exists a unique multipled common fixed point of* A *and* T, *i.e.,* $u \in X$ *such that* $u = Au = T(u, u, \cdots, u)$.

Proof. Since $\Delta \leqslant \Delta_p$, we get

$$\begin{aligned} h(F_{T(x,y,\cdots,z),T(p,q,\cdots,r)}(\phi(ct))) &\leqslant \psi\left\{\sqrt[m]{\Delta(h(F_{Ax,Ap}(\phi(t))), \Delta(h(F_{Ay,Aq}(\phi(t))), \cdots, h(F_{Az,Ar}(\phi(t)))))}\right\} \\ &\leqslant \psi\left\{\sqrt[m]{h(F_{Ax,Ap}(\phi(t)))h(F_{Ay,Aq}(\phi(t))), \cdots, h(F_{Az,Ar}(\phi(t)))}\right\} \\ &\leqslant \psi\left\{\frac{h(F_{Ax,Ap}(\phi(t))) + h(F_{Ay,Aq}(\phi(t))) + \cdots + h(F_{Az,Ar}(\phi(t)))}{m}\right\}. \end{aligned}$$

Then we can complete the proof by Theorem 3.2. □

4. An application

Example 4.1. Let $X = [0, 1]$, $h(x) = \frac{1}{x} - 1$, and d be the usual metric on X. Define $T : \underbrace{X \times X \cdots \times X}_{m\text{-times}} \to X$ as $T(x_1, x_2, \cdots, x_m) = \frac{x_1+x_2+\cdots+x_m}{5m}$. $A : X \to X$ as $Ax = \frac{x}{2}$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+d(x,y)}, & t > 0, \\ 0, & t = 0, \end{cases}$$

for all $x_1, x_2, \cdots, x_m, x, y \in X$ where $T(X \times X \cdots \times X) \subset A(X)$. Then (X, F, Δ) is a complete Menger PMT-space with Δ is a continuous t-norm. Define $\phi \in \Phi$, $\psi \in \Psi_0$ by $\phi(t) = \frac{t}{5}$ and $\psi(t) = \frac{9t}{10}$ for all $t > 0$. And $c = \frac{5}{6}$. We obtain

$$\begin{aligned} h(F_{T(x,y,\cdots,z),T(p,q,\cdots,r)}(\phi(ct))) &= \frac{1}{F_{T(x_1,x_2,\cdots,x_m),T(y_1,y_2,\cdots,y_m)}(\phi(ct))} - 1 \\ &= \frac{|T(x_1,x_2,\cdots,x_m) - T(y_1,y_2,\cdots,y_m)|}{\phi(ct)} \\ &= \frac{6|(x_1 + x_2 + \cdots + x_m) - (y_1 + y_2 + \cdots + y_m)|}{5mt} \end{aligned}$$

and

$$\begin{aligned} &\psi\left\{\frac{h(F_{Ax,Ap}(\phi(t))) + h(F_{Ay,Aq}(\phi(t))) + \cdots + h(F_{Az,Ar}(\phi(t)))}{m}\right\} \\ &= \psi\left\{\frac{(\frac{1}{F_{Ax_1,Ay_1}(\phi(t))} - 1) + (\frac{1}{F_{Ax_2,Ay_2}(\phi(t))} - 1) + \cdots + (\frac{1}{F_{Ax_m,Ay_m}(\phi(t))} - 1)}{m}\right\} \\ &= \psi\left\{\frac{|Ax_1 - Ay_1| + |Ax_2 - Ay_2| + \cdots + |Ax_m - Ay_m|}{m\phi(t)}\right\} \\ &= \frac{9(|x_1 - y_1| + |x_2 - y_2| + \cdots + |x_m - y_m|)}{4mt}. \end{aligned}$$

It is obvious that

$$h(F_{T(x,y,\cdots,z),T(p,q,\cdots,r)}(\phi(ct))) \leqslant \psi\left\{\frac{h(F_{Ax,Ap}(\phi(t))) + h(F_{Ay,Aq}(\phi(t))) + \cdots + h(F_{Az,Ar}(\phi(t)))}{m}\right\}.$$

Thus all the conditions of Theorem 3.5 are satisfied. Therefore, 0 is the unique multipled common fixed point of A and T.

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References

[1] A. A. N. Abdou, Y. J. Cho, R. Saadati, *Distance type and common fixed point theorems in Menger probabilistic metric type spaces*, Appl. Math. Comput., **265** (2015), 1145–1154. 1, 2.3, 2.4, 2.5

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181. 1

[3] S.-S. Chang, Y. J. Cho, S. M. Kang, *Nonlinear operator theory in probabilistic metric spaces*, Nova Science Publishers, Inc., Huntington, NY, (2001). 1

[4] Y. J. Cho, M. Grabiec, V. Radu, *On nonsymmetric topological and probabilistic structures*, Nova Science Publishers, Inc., New York, (2006). 1

[5] B. S. Choudhury, K. Das, *A new contraction principle in Menger spaces*, Acta Math. Sin. (Engl. Ser.), **24** (2008), 1379–1386. 1, 2.6, 2.7

[6] B. S. Choudhury, K. Das, *A coincidence point result in Menger spaces using a control function*, Chaos Solitons Fractals, **42** (2009), 3058–3063. 1

[7] P. N. Dutta, B. S. Choudhury, K. Das, *Some fixed point results in Menger spaces using a control function*, Surv. Math. Appl., **4** (2009), 41–52. 1

[8] O. Hadžić, E. Pap, *Fixed point theory in probabilistic metric spaces*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, (2001). 1

[9] M. S. Khan, M. Swalen, S. Sessa, *Fixed points theorems by altering distances between the points*, Bull. Austral. Math. Soc., **30** (1984), 1–9. 1

[10] M. A. Kutbi, D. Gopal, C. Vetro, W. Sintunavarat, *Further generalization of fixed point theorems in Menger PM-spaces*, Fixed Point Theory Appl., **2015** (2015), 10 pages. 1

[11] T. Luo, C.-X. Zhu, Z.-Q. Wu, *Tripled common fixed point theorems under probabilistic ϕ-contractive conditions in generalized Menger probabilistic metric spaces*, Fixed Point Theory Appl., **2014** (2014), 17 pages. 1

[12] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. U. S. A., **28** (1942), 535–537. 1

[13] D. Miheţ, *Altering distances in probabilistic Menger spaces*, Nonlinear Anal., **71** (2009), 2734–2738. 1

[14] A. F. Roldán López de Hierro, M. de la Sen, *Some fixed point theorems in Menger probabilistic metric-like spaces*, Fixed Point Theory Appl., **2015** (2015), 16 pages. 1, 2.8, 2.10, 2.11

[15] B. Schweizer, A. Sklat, *Probabilistic metric spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing Co., New York, (1983). 1, 2.1, 2.2

[16] V. M. Sehgal, A. T. Bharucha-Reid, *Fixed points of contraction mappings on probabilistic metric spaces*, Math. Systems Theory, **6** (1972), 72–102. 1

[17] C.-X. Zhu, *Several nonlinear operator problems in the Menger PN space*, Nonlinear Anal., **65** (2006), 1281–1284. 1

[18] C.-X. Zhu, *Research on some problems for nonlinear operators*, Nonlinear Anal., **71** (2009), 4568–4571. 1

[19] C.-X. Zhu, Z. Wei, Z.-Q. Wu, W.-Q. Xu, *Multidimensional common fixed point theorems under probabilistic ϕ-contractive conditions in multidimensional Menger probabilistic metric spaces*, Fixed Point Thenry Appl., **2015** (2015), 15 pages. 1, 2.9, 3.1