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Multiple periodic solutions for second-order discrete Hamiltonian systems

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Abstract

By applying critical point theory, the multiplicity of periodic solutions to second-order discrete Hamiltonian systems with partially periodic potentials was considered. It is noticed that, in this paper, the nonlinear term is growing linearly and main results extend some present results.

Keywords: Discrete Hamiltonian systems, periodic solutions, the generalized saddle point theorem.
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1. Introduction

Consider the following systems

$$\Delta^2 x(t-1) + \nabla G(t, x(t)) = 0, \quad t \in \mathbb{Z}, \tag{1.1}$$

where $\Delta x(t) = x(t+1) - x(t)$, $\Delta^2 x(t) = \Delta(\Delta x(t))$. For any $t \in \mathbb{Z}$, $G : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$ is C^1 in x, and $G(t+T, x) = G(t, x)$ for any $x \in \mathbb{R}^N$, where $T \in \mathbb{Z}$ and $T > 0$.

As far as we known, Guo and Yu [6] obtained the first variational result about T-periodic solutions for system (1.1). Soon afterwards, applying variational methods, there have been many studies in the literature consider about periodic solutions to discrete systems [2–6, 9, 11, 14–18].

It is noticed that the existence of one periodic solution to system (1.1) were obtained in [6] in case $\nabla G(t, x)$ is bounded. Afterward, in [14, 15], Xue and Tang studied the system (1.1), in which ∇G is growing sublinearly: there exist $\delta > 0, \eta > 0$ satisfying

$$|\nabla G(t, u)| \leqslant \delta |u|^\alpha + \eta, \quad \forall (t, u) \in [1, T] \cap \mathbb{Z} \times \mathbb{R}^N, \tag{1.2}$$

where $\alpha \in [0, 1)$. Furthermore, G satisfies:

$$\lim_{|u| \to \infty} |u|^{-2\alpha} \sum_{t=1}^{T} G(t, u) = +\infty,$$

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or

$$\lim_{|u|\to\infty} |u|^{-2\alpha} \sum_{t=1}^{T} G(t,u) = -\infty.$$

In case ∇G is growing sublinearly or linearly (that is ∇G satisfies (1.2) with $\alpha = 1$), Tang and Zhang [11] extended the main results obtained in [6, 14, 15] under more weakened conditions on G:

$$\lim_{|u|\to\infty} |u|^{-2\alpha} \sum_{t=1}^{T} G(t,u) < +\infty,$$

or

$$\lim_{|u|\to\infty} |u|^{-2\alpha} \sum_{t=1}^{T} G(t,u) > -\infty.$$

Recently, in [16], under $\nabla G(t,u)$ satisfies (1.2), G is coercive or resonant and periodic only in a part of the variables, that is, there exists an integer $k \in [0,N]$ such that:

(i) $G(t,u)$ is T_j-periodic in $u_j, 1 \leqslant j \leqslant k$.

(ii)
$$|u|^{-2\alpha} \sum_{t=1}^{T} G(t,u) \to \pm\infty \text{ as } |u| \to \infty, u \in \{0\} \times \mathbb{R}^{N-k},$$

by using generalized saddle point theorem [8], Yan et al. considered the multiple periodic solutions for system (1.1) and got some interesting results.

Motivated by [6, 11, 14–16], especially by [11, 16], one natural question is: What will happen when ∇G is growing linearly and G is coercive or resonant and periodic only in a part of the variables? More concretely, can we obtain some results similar to that of in [16] with ∇G is growing linearly? It seems one interesting question. In this paper, we will state them.

Theorem 1.1. *Suppose that there exists an integer* $k \in [0,N]$ *such that*

(H_1) $G(t,u)$ *is* T_j*-periodic in* u_j, $1 \leqslant j \leqslant k$.

(H_2) *There exist constants* $0 < \delta < \frac{\lambda_1}{4}$, $\eta > 0$ *satisfying*

$$|\nabla G(t,u)| \leqslant \delta|u| + \eta, \ \forall (t,u) \in [1,T] \cap \mathbb{Z} \times \mathbb{R}^N,$$

here $\lambda_l = 2 - 2\cos l\omega$, $\omega = \frac{2}{T}$, $l \in [0,[T/2]] \cap \mathbb{Z}$, $[\cdot]$ *is the integral function.*

(H_3)
$$\liminf_{|u|\to\infty} |u|^{-2} \sum_{t=1}^{T} G(t,u) > \frac{\lambda_{[T/2]}T}{4} + \frac{\delta^2 T}{\lambda_{[T/2]}} + \delta T, \quad u \in \{0\} \times \mathbb{R}^{N-k}.$$

Then the Hamiltonian system (1.1) *possesses* $k+1$ *periodic solutions.*

Theorem 1.2. *Suppose that* G *satisfies* (H_1), (H_2), *and*

(H_4)
$$\limsup_{|u|\to\infty} |u|^{-2} \sum_{t=1}^{T} G(t,u) < -\frac{\delta T}{2}, \quad u \in \{0\} \times \mathbb{R}^{N-k}.$$

Then the Hamiltonian system (1.1) *possesses* $k+1$ *periodic solutions.*

2. Definitions and lemmas

Let

$$H_T = \{x : \mathbb{Z} \to \mathbb{R}^N | x(t+T) = x(t), \ \forall t \in \mathbb{Z}\},$$

and

$$\langle x, y\rangle = \sum_{t=1}^{T}(x(t), y(t)),\ \|x\| = \left(\sum_{t=1}^{T}|x(t)|^2\right)^{1/2},\quad \forall x, y \in H_T.$$

Obviously, $(H_T, \langle\cdot,\cdot\rangle)$ is a Hilbert space (in fact, is finite-dimensional space).

For $x \in H_T$, denote

$$x_1 = \frac{1}{T}\sum_{t=1}^{T}|x(t)|,\quad x_2(t) = x(t) - x_1.$$

Let $\{e_j | 1 \leqslant j \leqslant N\}$ be the canonical basis of $\mathbb{R}^N$, and k_j be the unique integer such that

$$0 \leqslant (x_1, e_j) - k_j T_j < T_j \text{ for } 1 \leqslant j \leqslant k.$$

Set

$$\tilde{x}(t) = Px_1 + Qx_1 + x_2(t),$$

for all $x \in H_T$, where

$$Px_1 = \sum_{j=k+1}^{N}(x_1, e_j)e_j,\quad Qx_1 = \sum_{j=1}^{k}((x_1, e_j) - k_j T_j)e_j.$$

It is easy to see that there exists $\mu > 0$ such that

$$|Qx_1| < \mu. \tag{2.1}$$

Let I be defined on H_T by

$$I(x) = -\frac{1}{2}\sum_{t=1}^{T}|\Delta x(t)|^2 + \sum_{t=1}^{T}G(t, x(t)).$$

Then

$$\langle I'(x), y\rangle = -\sum_{t=1}^{T}(\Delta x(t), \Delta y(t)) + \sum_{t=1}^{T}(\nabla G(t, x(t)), y(t)),$$

for any $x, y \in H_T$. According to fact that of in [15], the periodic solutions for system (1.1) are critical points for the functional I.

Denote

$$F = \left\{\sum_{j=1}^{k} l_j T_j e_j | l_j \in \mathbb{Z}, 1 \leqslant j \leqslant k\right\}.$$

Let $\pi : H_T \to H_T/F$ be the canonical surjection (in fact, F is a discrete subgroup of H_T).

In fact, $H_T/F = U \times V$, here

$$U = E + W,$$
$$W = H_T^1 = \{x \in H_T |\ x_1 = 0\},$$
$$E = \operatorname{span}\{e_{k+1}, \dots, e_N\},$$

and

$$V = \operatorname{span}\{e_1, \dots, e_k\}/F.$$

It is easy to see that V and the torus T^k are isomorphic. Let $f(\pi(x)) = I(x)$, in fact $f : U \times V \to \mathbb{R}$.

Lemma 2.1 (The generalized saddle point theorem [8]). *Let U be a Banach space with a decomposition $U = E + W$, where E and W are two subspaces of U with $\dim W < +\infty$. Let V be a finite-dimensional, and compact*

C^2-manifold without boundary. Let $f : U \times V \to \mathbb{R}$ be a C^1-function and satisfy the (P.S.) condition. If there exist $\rho > 0$, $\gamma < \beta$ satisfy

(a) $\inf_{x\in E\times V} f(x) \geqslant \beta$;

(b) $\sup_{x\in S\times V} f(x) \leqslant \gamma$,

where $S = \partial D$, $D = \{z \in W \mid |z| \leqslant \rho\}$. Then the functional φ has at least cuplength$(V) + 1$ critical points.

Lemma 2.2 (Theorem 4.12 in [10]). *Let $\varphi \in C^1(H_T, \mathbb{R})$ be a G-invariant functional ($\varphi(u + g) = \varphi(u)$ for every $u \in H_T$ and $g \in F$) satisfying the $(P.S.)_G$ condition (that is every sequence $\{x_n\}$ of H_T such that $\varphi(x_n)$ is bounded, $\varphi'(x_n) \to 0$ as $n \to \infty$, $\pi(x_n)$ has a convergent subsequence). If φ is bounded from below and if the dimension k of the space generated by F is finite, then φ has at least $k + 1$ critical orbits.*

Lemma 2.3. *Let*

$$M_r := \{x \in H_T : -\Delta^2 x(t-1) = \lambda_r x(t)\},$$

here λ_r are defined as in condition (H_2). Then M_r is a subspace of H_T and

(i) $M_r \perp M_i, r \neq i, r, i \in [0, [T/2]] \cap \mathbb{Z}$;

(ii) $H_T = \oplus_{r=0}^{[T/2]} M_r$.

Lemma 2.4 ([15]). *Let*

$$H_r = \oplus_{i=0}^{r} M_i, \quad H_r^{\perp} = \oplus_{i=r+1}^{[T/2]} M_i,$$

where $r \in [0, [T/2]] \cap \mathbb{Z}$. Then

$$\sum_{t=1}^{T} |\Delta x(t)|^2 \leqslant \lambda_r \|x\|^2, \quad \forall x \in H_r, \tag{2.2}$$

and

$$\sum_{t=1}^{T} |\Delta x(t)|^2 \geqslant \lambda_{r+1} \|x\|^2, \quad \forall x \in H_r^{\perp}. \tag{2.3}$$

3. Proof of theorems

Proof of Theorem 1.1.

Step 1. We will assert the (P.S.) condition holds.

Assume that $\{\pi(x_n)\}$ is a (P.S.) sequence of f, i.e., $I(x_n)$ is bounded, $I'(x_n) \to 0$. Combing (H_3) with $\lambda_1 > 4\delta$, we have that

$$\liminf_{|u|\to\infty} |u|^{-2} \sum_{t=1}^{T} G(t, u) > \frac{4\delta^2 T \lambda_{[T/2]}}{\lambda_1^2} + \frac{\delta^2 T}{\lambda_{[T/2]}} + \frac{16\delta^3 T}{\lambda_1^2}. \tag{3.1}$$

By (H_2) and (3.1), one has that

$$\begin{aligned}
\left|\sum_{t=1}^{T} (G(t, \tilde{x}(t)) - G(t, Px_1))\right| &\leqslant \sum_{t=1}^{T} |G(t, \tilde{x}(t)) - G(t, Px_1)| \\
&\leqslant \sum_{t=1}^{T} \left|\int_0^1 (\nabla G(t, Px_1 + s(Qx_1 + x_2(t))), Qx_1 + x_2(t)) ds\right| \\
&\leqslant \sum_{t=1}^{T} \int_0^1 |(\nabla G(t, Px_1 + s(Qx_1 + x_2(t))), Qx_1 + x_2(t))| ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{t=1}^{T} \int_0^1 |\nabla G(t, Px_1 + s(Qx_1 + x_2(t)))||Qx_1 + x_2(t)|ds \\
&\leqslant \sum_{t=1}^{T} \int_0^1 (\delta|Px_1 + s(Qx_1 + x_2(t))| + \eta)|Qx_1 + x_2(t)|ds \\
&\leqslant \sum_{t=1}^{T} \delta(|Px_1| + |Qx_1 + x_2(t)|)|Qx_1 + x_2(t)| + \sum_{t=1}^{T} \eta|Qx_1 + x_2(t)| \\
&\leqslant \delta\mu T|Px_1| + \sum_{t=1}^{T} \delta|Px_1||x_2(t)| + \sum_{t=1}^{T} 4\delta(|Qx_1|^2 + |x_2(t)|^2) \\
&\quad + \sum_{t=1}^{T} \eta(|Qx_1| + |x_2(t)|) \qquad (3.2)\\
&\leqslant \delta\mu T|Px_1| + \sum_{t=1}^{T} \delta|Px_1||x_2(t)| + 4\delta\mu^2 T + 4\delta \sum_{t=1}^{T} |x_2(t)|^2 \\
&\quad + \eta\mu T + \sum_{t=1}^{T} \eta|x_2(t)| \\
&\leqslant \delta\mu T|Px_1| + \frac{\delta^2 T}{\lambda_{[T/2]}}|Px_1|^2 + \frac{\lambda_{[T/2]}}{2}\|x_2\|^2 + 4\delta\mu^2 T + 4\delta\|x_2\|^2 \\
&\quad + \eta\mu T + \eta\sqrt{T}\|x_2\|.
\end{aligned}
$$

In the same way, we get

$$
\begin{aligned}
\left|\sum_{t=1}^{T} (\nabla G(t, \tilde{x}(t)), x_2(t))\right| &= \left|\sum_{t=1}^{T} (\nabla G(t, Px_1 + Qx_1 + x_2(t)), x_2(t))\right| \\
&\leqslant \sum_{t=1}^{T} |\nabla G(t, Px_1 + Qx_1 + x_2(t))||x_2(t)| \\
&\leqslant \sum_{t=1}^{T} (\delta|Px_1 + Qx_1 + x_2(t)| + \eta)|x_2(t)| \qquad (3.3)\\
&\leqslant \sum_{t=1}^{T} \delta|Px_1||x_2(t)| + \delta\mu\sqrt{T}\|x_2\| + \delta \sum_{t=1}^{T} |x_2(t)|^2 + \eta\sqrt{T}\|x_2\| \\
&\leqslant \frac{\lambda_1}{2}\|x_2\|^2 + \frac{\delta^2 T}{\lambda_1}|Px_1|^2 + \delta\|x_2\|^2 + (\delta\mu + \eta)\sqrt{T}\|x_2\|.
\end{aligned}
$$

Since the fact $x_{1n} \in H_0$ and $x_{2n} \in H_0^{\perp}$, by (2.3), one has that

$$
\sum_{t=1}^{T} (\Delta x_n(t), \Delta x_{2n}(t)) = \sum_{t=1}^{T} (\Delta x_{2n}(t), \Delta x_{2n}(t)) = \sum_{t=1}^{T} |\Delta x_{2n}(t)|^2 \geqslant \lambda_1\|x_{2n}\|^2. \qquad (3.4)
$$

By $I(x_n)$ is bounded, $I'(x_n) \to 0$ and (3.3), for n large enough, one gets

$$
\begin{aligned}
\sum_{t=1}^{T} (\Delta x_n(t), \Delta x_{2n}(t)) &= -\langle I'(x_n), x_{2n}\rangle + \sum_{t=1}^{T} (\nabla G(t, x_n(t)), x_{2n}(t)) \\
&= -\langle I'(x_n), x_{2n}\rangle + \sum_{t=1}^{T} (\nabla G(t, \tilde{x}_n(t)), x_{2n}(t)) \qquad (3.5)\\
&\leqslant \|x_{2n}\| + \frac{\lambda_1}{2}\|x_{2n}\|^2 + \frac{\delta^2 T}{\lambda_1}|Px_{1n}|^2 + \delta\|x_{2n}\|^2 + (\delta\mu + \eta)\sqrt{T}\|x_{2n}\|.
\end{aligned}
$$

Combing (3.4) with (3.5), one has that

$$(\frac{\lambda_1}{2}-\delta)\|x_{2n}\|^2-((\delta\mu+\eta)\sqrt{T}+1)\|x_{2n}\| \leqslant \frac{\delta^2 T}{\lambda_1}|Px_{1n}|^2.$$

Therefore, there exists C_1 such that

$$\frac{\delta^2 T}{\lambda_1}|Px_{1n}|^2 \geqslant \frac{\lambda_1}{4}\|x_{2n}\|^2+C_1, \tag{3.6}$$

for all large n, where

$$C_1=\min_{s\in[0,+\infty)}\left\{(\frac{\lambda_1}{4}-\delta)s^2-((\delta\mu+\eta)\sqrt{T}+1)s\right\}.$$

Since $\lambda_1>4\delta$, $-\infty<C_1<0$, it follows from (3.6) that

$$\|x_{2n}\|^2 \leqslant \frac{4\delta^2 T}{\lambda_1^2}|Px_{1n}|^2-\frac{4C_1}{\lambda_1}. \tag{3.7}$$

Then

$$\|x_{2n}\| \leqslant \frac{2\delta\sqrt{T}}{\lambda_1}|Px_{1n}|+C_2, \tag{3.8}$$

where $0<C_2<+\infty$.

So by the boundedness of $I(x_n)$, (2.2), (3.2), (3.7), and (3.8), we have that

$$\begin{aligned}
C_3 \geqslant I(x_n)=I(\tilde{x}_n) &= -\frac{1}{2}\sum_{t=1}^{T}|\Delta\tilde{x}_n(t)|^2+\sum_{t=1}^{T}G(t,\tilde{x}_n(t))\\
&= -\frac{1}{2}\sum_{t=1}^{T}|\Delta\tilde{x}_n(t)|^2+\sum_{t=1}^{T}[G(t,\tilde{x}_n(t))-G(t,Px_{1n})]+\sum_{t=1}^{T}G(t,Px_{1n})\\
&\geqslant -\frac{1}{2}\lambda_{[T/2]}\|x_{2n}\|^2-\delta\mu T|Px_{1n}|-\frac{\delta^2 T}{\lambda_{[T/2]}}|Px_{1n}|^2-\frac{\lambda_{[T/2]}}{2}\|x_{2n}\|^2\\
&\quad -4\delta\mu^2 T-4\delta\|x_{2n}\|^2-\eta\mu T-\eta\sqrt{T}\|x_{2n}\|+\sum_{t=1}^{T}F(t,Px_{1n})\\
&\geqslant -\lambda_{[T/2]}\left(\frac{4\delta^2 T}{\lambda_1^2}|Px_{1n}|^2-\frac{4C_1}{\lambda_1}\right)-\frac{\delta^2 T}{\lambda_{[T/2]}}|Px_{1n}|^2-\delta\mu T|Px_{1n}|-4\delta\mu^2 T-\eta\mu T\\
&\quad -4\delta\left(\frac{4\delta^2 T}{\lambda_1^2}|Px_{1n}|^2-\frac{4C_1}{\lambda_1}\right)-\eta\sqrt{T}\left(\frac{2\delta\sqrt{T}}{\lambda_1}|Px_{1n}|+C_2\right)+\sum_{t=1}^{T}F(t,Px_{1n})\\
&\geqslant |Px_{1n}|^2\left(|Px_{1n}|^{-2}\sum_{t=1}^{T}F(t,Px_{1n})-\frac{4\delta^2 T\lambda_{[T/2]}}{\lambda_1^2}-\frac{\delta^2 T}{\lambda_{[T/2]}}-\frac{16\delta^3 T}{\lambda_1^2}\right)\\
&\quad -C_4|Px_{1n}|-C_5,
\end{aligned} \tag{3.9}$$

for n sufficient large, where $C_4>0, C_5>0$ are constants.

It follows from (3.9) and (3.1) that $|Px_{1n}|$ is bounded. Then by (3.8), we obtain $\|x_{2n}\|$ is bounded, so $\{\tilde{x}_n\}$ is also bounded. Observe that $\pi(x_n)=\pi(\tilde{x}_n)$ and H_T is a finite-dimensional space, then we obtain the result that f satisfies the (P.S.) condition.

Step 2. To prove that (a) and (b) of Lemma 2.1 are satisfied.

Since $\pi(x) \in E \times V$, $x = Px_1 + Qx_1$, from (H_3), we get

$$f(\pi(x)) = \sum_{t=1}^{T} G(t, Px_1 + Qx_1) \to \infty$$

uniformly for $\pi(Qx_1) \in V$ as $|Px_1| \to \infty$. So, there exists a constant β satisfying $\inf_{\pi(x)\in E\times V} f(\pi(x)) \geqslant \beta$.

By (H_2), $\exists\, C_6 > 0$ satisfying

$$|G(t,u)| \leqslant |\int_0^1 (\nabla G(t,su),u)ds| + G(t,0) \leqslant \int_0^1 |\nabla G(t,su)||u|ds + G(t,0) \leqslant \frac{\delta}{2}|u|^2 + \eta|u| + C_6, \tag{3.10}$$

for all $t \in [1,T] \cap \mathbb{Z}, u \in \mathbb{R}^N$.

Since $\pi(x) \in W \times V$, $x = Qx_1 + x_2$. Hence, by (2.3), (2.1), and (3.10), one has that

$$\begin{aligned}
f(\pi(x)) = I(x) = I(Qx_1 + x_2) &= -\frac{1}{2}\sum_{t=1}^{T} |\Delta x_2(t)|^2 + \sum_{t=1}^{T} G(t, Qx_1 + x_2(t)) \\
&\leqslant -\frac{1}{2}\lambda_1 \|x_2\|^2 + \sum_{t=1}^{T} \frac{\delta}{2}|Qx_1 + x_2(t)|^2 + \sum_{t=1}^{T} \eta|Qx_1 + x_2(t)| + C_6 T \\
&\leqslant -\frac{1}{2}\lambda_1 \|x_2\|^2 + 2\delta \sum_{t=1}^{T} (|Qx_1|^2 + |x_2(t)|^2) + \sum_{t=1}^{T} \eta|Qx_1| + \sum_{t=1}^{T} \eta|x_2(t)| + C_6 T \\
&\leqslant -\frac{1}{2}\lambda_1 \|x_2\|^2 + 2\delta\mu^2 T + 2\delta\|x_2\|^2 + \eta\mu T + \eta\sqrt{T}\|x_2\| + C_6 T \\
&\leqslant (-\frac{1}{2}\lambda_1 + 2\delta)\|x_2\|^2 + C_7\|x_2\| + C_8,
\end{aligned}$$

where $C_7 > 0, C_8 > 0$ are constants. Note that $\lambda_1 > 4\delta$, take $\|x_2\|$ sufficient large such that

$$\sup_{\pi(x)\in S\times V} f(\pi(x)) \leqslant \gamma < \beta.$$

By all above, the linking conditions (a) and (b) are satisfied. By Lemma 2.1, the system (1.1) possesses $k+1$ periodic solutions. □

Proof of Theorem 1.2. Assume that $\{\pi(x_n)\}$ is a (P.S.) sequence of f, i.e., $I(x_n)$ is bounded, $I'(x_n) \to 0$.

For $\lambda_1 > 4\delta$, we have that

$$\limsup_{|u|\to\infty} |u|^{-2} \sum_{t=1}^{T} G(t,u) < -\frac{2\delta^2 T}{\lambda_1}.$$

Same as (3.2), we get

$$\left|\sum_{t=1}^{T} (G(t,\tilde{x}(t)) - G(t,Px_1))\right| \leqslant \delta\mu T|Px_1| + \frac{2\delta^2 T}{\lambda_1}|Px_1|^2 + \frac{\lambda_1}{4}\|x_2\|^2 + 4\delta\mu^2 T + 4\delta\|x_2\|^2 + \eta\mu T + \eta\sqrt{T}\|x_2\|. \tag{3.11}$$

For $x \in H_T$, we set $\psi(x) = -I(x)$. It is clear that $\psi(x)$ is a G-invariant functional, that is $\psi(x+g) = \psi(x)$, for all $g \in F$, $x \in H_T$. For all $x \in H_T$, by (3.11), one has that

$$\psi(x) = \psi(\tilde{x}) = \frac{1}{2}\sum_{t=1}^{T} |\Delta x_2(t)|^2 - \sum_{t=1}^{T} G(t,\tilde{x}(t))$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{t=1}^{T}|\Delta x_2(t)|^2 - \sum_{t=1}^{T}[G(t,\tilde{x}(t)) - G(t,Px_1)] - \sum_{t=1}^{T}G(t,Px_1)\\
&\geqslant \frac{1}{2}\lambda_1\|x_2\|^2 - \delta\mu T|Px_1| - \frac{2\delta^2 T}{\lambda_1}|Px_1|^2 - \frac{\lambda_1}{4}\|x_2\|^2\\
&\quad - 4\delta\mu^2 T - 4\delta\|x_2\|^2 - \eta\mu T - \eta\sqrt{T}\|x_2\| - \sum_{t=1}^{T}G(t,Px_1) \qquad (3.12)\\
&= (\frac{1}{4}\lambda_1 - 4\delta)\|x_2\|^2 - \eta\sqrt{T}\|x_2\| - \delta\mu T|Px_1|\\
&\quad - 4\delta\mu^2 T - \eta\mu T - |Px_1|^2\left(|Px_1|^{-2}\sum_{t=1}^{T}G(t,Px_1) + \frac{2\delta^2 T}{\lambda_1}\right).
\end{aligned}
$$

So, ψ is bounded from below.

For $I(x_n)$ is bounded, $I'(x_n) \to 0$, there exists C_9 satisfies $\psi(x_n) \leqslant C_9$. By (3.12), we have that

$$
\begin{aligned}
C_9 \geqslant \psi(x_n) = \psi(\tilde{x}_n) &\geqslant (\frac{1}{4}\lambda_1 - 4\delta)\|x_{2n}\|^2 - \eta\sqrt{T}\|x_{2n}\|\\
&\quad - |Px_{1n}|^2\left(|Px_{1n}|^{-2}\sum_{t=1}^{T}G(t,Px_{1n}) + \frac{2\delta^2 T}{\lambda_1}\right) - C_{10}|Px_{1n}| - C_{11},
\end{aligned}
\tag{3.13}
$$

where C_{10}, C_{11} are some positive constants.

Combing (H_4) with (3.13), we conclude that $|Px_{1n}|$ and $\|x_{2n}\|$ are bounded, so $\{\tilde{x}_n\}$ is also bounded. Since H_T is finite-dimensional and $\{\tilde{x}_n\} \in H_T$, so $\{\tilde{x}_n\}$ contains a convergent subsequence. By $\pi(x_n) = \pi(\tilde{x}_n)$, then $\pi(x_n)$ has a convergent subsequence, that is the functional ψ satisfies the (P.S.) condition.

Hence, all assumptions of Lemma 2.2 are held. Then, by Lemma 2.2 we have that the system (1.1) possesses $k+1$ geometrically distinct periodic solutions in H_T. □

4. Conclusion

From the main conclusion, that is, Theorem 1.1 and Theorem 1.2, our results complete and extend some results that of in [11, 16]. In the last, we would like to point out that based on the results reported in [1, 7, 12, 13] on fractional calculus and time scales we will study some interesting problems, for example, the fractional Hamiltonian system on time scales.

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