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Levitin-Polyak well-posedness for lexicographic vector equilibrium problems

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Abstract

We introduce the notions of Levitin-Poljak (LP) well-posedness and LP well-posedness in the generalized sense for the lexicographic vector equilibrium problems. Then, we establish some sufficient conditions for lexicographic vector equilibrium problems to be LP well-posedness at the reference point. Numerous examples are provided to explain that all the assumptions we impose are very relaxed and cannot be dropped. The results in this paper unify, generalize and extend some known results in the literature. ©2017 All rights reserved.

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1. Introduction and preliminaries

Equilibrium problems first considered by Blum and Oettli [8] have been playing an important role in optimization theory with many striking applications particularly in transportation, mechanics, economics, etc. Equilibrium models incorporate many other important problems such as: optimization problems, variational inequalities, complementarity problems, saddlepoint/minimax problems, and fixed points. Equilibrium problems with scalar and vector objective functions have been widely studied. The crucial issue of solvability (the existence of solutions) has attracted the most considerable attention of researchers, see for example [14, 18, 21, 40].

On the other hand, well-posedness plays an important role in the stability analysis and numerical methods for optimization theory and applications. Since any algorithm can generate only an approximating solution sequence which is meaningful only if the problem is well-posed under consideration. The first and oldest well-posedness is Hadamard well-posedness [20], which means existence, uniqueness and continuous dependence of the optimal solution and optimal value from perturbed data. The second is Tikhonov well-posedness [41], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Well-posedness properties have been intensively studied and the two classical well-posedness notions have been extended and blended. For parametric problems, well-posedness is closely related to stability. Up to now, there have been many works dealing with well-posedness of optimization-related problems as mathematical programming [22, 39], constrained minimization [12, 16, 43, 44] variational inequalities [10, 12, 17, 30, 42], Nash equilibria [34, 42],

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and equilibrium problems [3, 16, 23]. A fundamental requirement in Tykhonov well-posedness is that every minimizing sequence is from within the feasible region. However, in several numerical methods such as exterior penalty methods and augmented Lagrangian methods, the minimizing sequence generated may not be feasible. Taking this into account, Levitin and Poljak [28] introduced another notion of well-posedness which does not necessarily require the feasibility of the minimizing sequence. However, it requires the distance of the minimizing sequence from the feasible set to approach to zero eventually. Since then, many authors investigated the well-posedness and well-posedness in the generalized sense for optimization, variational inequalities and equilibrium problems. The study of Levitin-Polyak type well-posedness for scalar convex optimization problems with functional constraints was initiated by Konsulova and Revalski [25]. In 1981, Lucchetti and Patrone [33] introduced and studied the well-posedness for variational inequalities, which is a generalization of the Tykhonov well-posedness of minimization problems. Long et al. [31] introduced and studied four types of Levitin-Polyak well-posedness of equilibrium problems with abstract set constraints and functional constraints. Li and Li [29] introduced and researched two types of Levitin-Polyak well-posedness of vector equilibrium problems with abstract set constraints. Peng et al. [36] introduced and studied four types of Levitin-Polyak well-posedness of vector equilibrium problems with abstract set constraints and functional constraints. Peng et al. [37] introduced several types of Levitin-Polyak well-posedness for a generalized vector quasi-equilibrium problem with functional constraints and abstract set constraints. Chen et al. [11] studied the Levitin-Polyak well-posedness by perturbations for a class of general systems of set-valued vector quasi-equilibrium problems in Hausdorff topological vector spaces. Very recently Lalitha and Bhatia [27] studied the LP well-posedness for a parametric quasi variational inequality problem of the Minty type.

With regard to vector equilibrium problems, most of existing results correspond to the case when the order is induced by a closed convex cone in a vector space. Thus, they cannot be applied to lexicographic cones, which are neither closed nor open. These cones have been extensively investigated in the framework of vector optimization, see for example [1, 6, 7, 9, 15, 19, 24, 26].

For instance, Konnov and Ali [24] studied sequential problems, especially exploiting its relation with regularization methods. Bianchi et al. in [6] analyzed lexicographic equilibrium problems on a topological Hausdorff vector space, and their relationship with some other vector equilibrium problems. They obtained the existence results for the tangled lexicographic problem via the study of a related sequential problem. However, for equilibrium problems, the main emphasis has been on the issue of solvability/existence. To the best of the knowledge, very recently, Anh et al. [1] studied the Tikhonov well-posedness for lexicographic vector equilibrium problems in metric spaces and gave the sufficient conditions for a family of such problems to be well-posed and uniquely well-posed at the considered point. Furthermore, they derived several results on well-posedness for a class of variational inequalities.

In this paper, we first introduce the new notions of Levitin-Polyak (LP) well-posedness and LP well-posedness in the generalized sense for the lexicographic vector equilibrium problems. Then, we establish some sufficient conditions for this problems to be LP well-posedness at the reference point. Furthermore, we give numerous examples to explain that all the imposed assumptions are very relaxed and cannot be dropped.

The layout of the paper is as follows. In Section 2, we give some necessary and/or sufficient conditions for (LEP) to be LP well-posed at the reference point. In Section 3 is devoted to results for a generalization of LP well-posedness for (LEP).

We first recall the concept of lexicographic cone in finite dimensional spaces and models of equilibrium problems with the order induced by such a cone. The lexicographic cone of $\mathbb{R}^n$, denoted C_l, is the collection of zero and all vectors in $\mathbb{R}^n$ with the first nonzero coordinate being positive, i.e.,

$$C_l := \{0\} \cup \{x \in \mathbb{R}^n | \exists i \in \{1, 2, \dots, n\} : x_i > 0 \text{ and } x_j = 0, \quad \forall j < i\}.$$

This cone is convex and pointed, and induces the total order as follows

$$x \geqslant_l y \Leftrightarrow x - y \in C_l.$$

We also observe that it is neither closed nor open. Indeed, when comparing with the cone $C_1 := \{x \in \mathbb{R}^n | x_1 \geqslant 0\}$, we see that $\operatorname{int} C_1 \subsetneq C_l \subsetneq C_1$, while

$$\operatorname{int} C_l = \operatorname{int} C_1, \quad \text{and} \quad \operatorname{cl} C_l = C_1.$$

Throughout this paper, if not other specified, X is a metric space and Λ denotes the metric space. Let $X_0 \subset X$ be nonempty and closed sets . Let $f := (f_1, f_2, \ldots, f_n) : X \times X \times \Lambda \to \mathbb{R}^n$ be vector-valued function and $K : \Lambda \to 2^X$ being a closed valued map. The lexicographic vector quasiequilibrium problem consists of, for each $\lambda \in \Lambda$,

$$(\text{LEP}_\lambda) \quad \text{find } \bar{x} \in K(\lambda) \quad \text{such that} \quad f(\bar{x}, y, \lambda) \geqslant_l 0, \quad \forall y \in K(\lambda).$$

Instead of writing $\{(\text{LEP}_\lambda) | \lambda \in \Lambda\}$ for the family of lexicographic vector equilibrium problem, i.e., the lexicographic parametric problem, we will simply write (LEP) in the sequel.

Let $S : \Lambda \to 2^X$ be the solution map of (LEP); that is, for each $\lambda \in \Lambda$,

$$S(\bar{\lambda}) := \{x \in K(\bar{\lambda}) | f(x, y, \bar{\lambda}) \geqslant_l 0, \ \forall y \in K(\bar{\lambda})\}.$$

By following the lines of investigating ε-solutions to vector optimization problems initiated by Loridan [32], we consider, for each $\lambda \in \Lambda$ and each $\varepsilon \in [0, \infty)$, the following approximate problem:

$$(\text{LEP}_{\lambda, \varepsilon}) \ \text{find } \bar{x} \in K(\lambda) \ \text{such that} \ d(\bar{x}, K(\lambda)) \leqslant \varepsilon, \quad \text{and} \quad f(\bar{x}, y, \lambda) + \varepsilon e \geqslant_l 0, \quad \forall y \in K(\lambda),$$

where $e := (\underbrace{0, 0, \ldots, 0}_{n-1}, 1) \in \mathbb{R}^n$. The solution set of $(\text{LEP}_{\lambda, \varepsilon})$ is denoted by $\tilde{S}(\lambda, \varepsilon)$; that is, the set-valued map $\tilde{S} : \Lambda \times \mathbb{R} \to 2^X$ is defined by

$$\tilde{S}(\lambda, \varepsilon) = \{x \in X | d(x, K(\lambda)) \leqslant \varepsilon \ \text{and} \ f(x, y, \lambda) + \varepsilon e \geqslant_l 0, \ \forall y \in K(\lambda)\}, \tag{1.1}$$

for all $(\lambda, \varepsilon) \in \Lambda \times \mathbb{R}$.

Now we introduce the concept of LP well-posedness for LEP. For this purpose, we require the following notions of an LP approximating sequence.

Definition 1.1. Let $\{\lambda_n\}$ be a sequence in Λ such that $\lambda_n \to \bar{\lambda}$. A sequence $\{x_n\}$ is said to be an LP approximating sequence for LEP with respect to $\{\lambda_n\}$, if there is a sequence $\{\epsilon_n\}$ in $(0, \infty)$ satisfying $\epsilon_n \to 0$ as $n \to \infty$, such that

(i) $d(x_n, K(\lambda_n)) \leqslant \epsilon_n, \quad \text{for all } n \in \mathbb{N}$;
(ii) $f(x_n, y_n, \lambda_n) + \epsilon_n e \geqslant_l 0, \ \forall y_n \in K(\lambda_n)$.

Definition 1.2. The problem (LEP) is LP well-posed at $\bar{\lambda}$, if

(i) there exists a unique solution $\bar{x}$ of LEP;
(ii) for any sequence $\{\lambda_n\}$ converging to $\bar{\lambda}$, every LP approximating sequence $\{x_n\}$ with respect to $\{\lambda_n\}$ converges to $\bar{x}$.

Definition 1.3 ([4]). *Let $Q : X \rightrightarrows Y$ be a set-valued mapping between metric spaces.*

(i) Q is upper semicontinuous (usc) at $\bar{x}$, if for any open set $U \supseteq Q(\bar{x})$, there is a neighborhood N of $\bar{x}$ such that $Q(N) \subseteq U$.
(ii) Q is lower semicontinuous (lsc) at $\bar{x}$, if for any open subset U of Y with $Q(\bar{x}) \cap U \neq \emptyset$, there is a neighborhood N of $\bar{x}$ such that $Q(x) \cap U \neq \emptyset$ for all $x \in N$.
(iii) Q is closed at $\bar{x}$, if for any sequences $x_k \to \bar{x}$ and $y_k \to \bar{y}$ with $y_k \in Q(x_k)$, it holds $\bar{y} \in Q(\bar{x})$.

Lemma 1.4 ([4])**.**

   (i) *If Q is usc at $\bar{x}$ and $Q(\bar{x})$ is compact, then for any sequence $x_n \to \bar{x}$, every sequence $\{y_n\}$ with $y_n \in Q(x_n)$ has a subsequence converging to some point in $Q(\bar{x})$. If, in addition, $Q(\bar{x}) = \{\bar{y}\}$ is a singleton, then such a sequence $\{y_n\}$ must converge to $\bar{y}$.*
   (ii) *Q is lsc at $\bar{x}$, if and only if for any sequence $x_n \to \bar{x}$ and any point $y \in Q(\bar{x})$, there is a sequence $\{y_n\}$ with $y_n \in Q(x_n)$ converging to y.*

Definition 1.5 ([1, 2])**.** Let g be an extended real-valued function on a metric space X and ε be a real number.

   (i) g is upper ε-level closed at $\bar{x} \in X$, if for any sequence $x_n \to \bar{x}$,

$$[g(x_n) \geqslant \varepsilon, \ \forall n] \Rightarrow [g(\bar{x}) \geqslant \varepsilon].$$

   (ii) g is strongly upper ε-level closed at $\bar{x} \in X$ if for any sequences $x_n \to \bar{x}$ and $\{\nu_n\} \subset [0, \infty)$ converging to 0,

$$[g(x_n) + \nu_n \geqslant \varepsilon, \ \forall n] \Rightarrow [g(\bar{x}) \geqslant \varepsilon].$$

Let A, B be two subsets of metric space X. The Hausdorff distance between A and B is defined as follows

$$H(A, B) = \max\{H^*(A, B), H^*(B, A)\},$$

where $H^*(A, B) = \sup_{a \in A} d(a, B)$, and $d(x, A) = \inf_{y \in A} d(x, y)$.

2. LP well-posedness for lexicographic vector equilibrium problems

In this section, we shall give some necessary and/or sufficient conditions for (LEP) to be LP well-posed at the reference point $\bar{\lambda} \in \Lambda$. To simplify the presentation, in the sequel, the results will be formulated for the case $n = 2$. For any two positive numbers α, ϵ, the solution set of approximation solutions for the problem $(\text{LEP}_{\lambda, \varepsilon})$ is denoted by

$$\Gamma(\bar{\lambda}, \alpha, \epsilon) = \bigcup_{\lambda \in B(\bar{\lambda}, \alpha) \cap \Lambda} \{x \in X \mid d(x, K(\lambda)) \leqslant \epsilon, \quad \text{and} \quad f(x, y, \lambda) + \varepsilon e \geqslant_l 0, \ \forall y \in K(\lambda)\}, \tag{2.1}$$

where $B(\bar{\lambda}, \alpha)$ denotes the closed ball centered at $\bar{\lambda}$ with radius α. The set-valued mapping $Z : \Lambda \times X \to 2^X$ next defined will play an important role our analysis

$$Z(\lambda, x) = \begin{cases} \{z \in K(\lambda) \mid f_1(x, z, \lambda) = 0\}, & \text{if } (\lambda, x) \in \text{gr } Z_1; \\ X & \text{otherwise,} \end{cases}$$

where $Z_1 : \Lambda \to 2^X$ denotes the solution mapping of the scalar equilibrium problem determined by the real-valued function f_1 :

$$Z_1(\lambda) = \{x \in K(\lambda) \mid f_1(x, y, \lambda) \geqslant 0, \ \forall y \in K(\lambda)\}.$$

Then (2.1) is equivalent to

$$\Gamma(\bar{\lambda}, \alpha, \epsilon)$$
$$= \bigcup_{\lambda \in B(\bar{\lambda}, \alpha) \cap \Lambda} \{x \in X \mid d(x, K(\lambda)) \leqslant \epsilon, \ f_1(x, y, \lambda) \geqslant 0, \ \forall y \in K(\lambda) \ \text{and} \ f_2(x, z, \lambda) + \epsilon \geqslant 0, \ \forall z \in Z(\lambda, x)\}$$
$$= \bigcup_{\lambda \in B(\bar{\lambda}, \alpha) \cap \Lambda} \tilde{S}(\lambda, \varepsilon),$$

where $\tilde{S}$ is the solution map for $(\text{LEP}_{\lambda,\epsilon})$ defined by (1.1). For the solution map $S : \Lambda \to 2^X$ of (LEP), in general, we observe that

$$\Gamma(\bar{\lambda}, 0, 0) = S(\bar{\lambda}) \quad \text{and} \quad S(\bar{\lambda}) \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon), \quad \forall \alpha, \epsilon > 0,$$

and hence

$$S(\bar{\lambda}) \subseteq \bigcap_{\alpha, \epsilon > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon).$$

Next, we provide the sufficient conditions for the two sets to coincide.

Proposition 2.1. *Suppose that the following conditions are satisfied:*

 (i) K *is closed and lsc on* Λ;
 (ii) Z *is lsc on* $\Lambda \times X$;
 (iii) f_1 *is upper 0-level closed on* $X \times X \times \Lambda$;
 (iv) f_2 *is strongly upper 0-level closed on* $X \times X \times \Lambda$;

then

$$\bigcap_{\alpha, \epsilon > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon) = S(\bar{\lambda}).$$

Proof. Let $\bar{x} \in \bigcap_{\alpha, \epsilon > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon)$, then without loss of generality, there exist sequences $\alpha_n > 0, \epsilon_n > 0$ with $\alpha_n \to 0, \epsilon_n \to 0$, such that $\bar{x} \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. Hence, it follows that there exists a sequence $\lambda_n \in B(\bar{\lambda}, \alpha_n) \cap \Lambda$, such that for all $n \in \mathbb{N}$,

$$d(\bar{x}, K(\lambda_n)) \leqslant \epsilon_n, \tag{2.2}$$

and

$$f_1(\bar{x}, y, \lambda_n) \geqslant 0, \quad \forall y \in K(\lambda_n), \quad \text{and} \quad f_2(\bar{x}, z, \lambda_n) + \epsilon_n \geqslant 0, \quad \forall z \in Z(\lambda_n, \bar{x}). \tag{2.3}$$

Since $K(\bar{\lambda})$ is a closed set in X, it follows from (2.2) that we can choose $x_n \in K(\lambda_n)$, such that

$$d(\bar{x}, x_n) \leqslant \epsilon_n, \quad \forall n \in \mathbb{N}.$$

Thus $x_n \to \bar{x}$ as $n \to \infty$. Clearly $\lambda_n \to \bar{\lambda}$ as $n \to \infty$ and also as K is closed at $\bar{\lambda}$, it follows that $\bar{x} \in K(\bar{\lambda})$. As K is lsc at $\bar{\lambda}$ and $\lambda_n \to \bar{\lambda}$ for any $y \in K(\bar{\lambda})$ there exists $y_n \in K(\lambda_n)$ such that $y_n \to y$. Also Z is lsc at $(\bar{\lambda}, \bar{x})$ and $(\lambda_n, x_n) \to (\bar{\lambda}, \bar{x})$, it is clear that for any $z \in Z(\bar{\lambda}, \bar{x})$ there exists a sequence $z_n \in Z(\lambda_n, x_n)$ such that $z_n \to z$. This implies by assumptions (iii), (iv), and (2.3) that $f_1(\bar{x}, y, \bar{\lambda}) \geqslant 0$, $f_2(\bar{x}, z, \bar{\lambda}) \geqslant 0$ and hence, $\bar{x} \in S(\bar{\lambda})$. $\qquad\square$

Theorem 2.2. *Suppose that the conditions* (i)-(iv) *in Proposition 2.1 are satisfied. Then* (LEP) *is LP well-posed at* $\bar{\lambda} \in \Lambda$, *if and only if* $\Gamma(\bar{\lambda}, \alpha, \epsilon) \neq \emptyset$, *for all* $\alpha, \epsilon > 0$ *and* $\text{diam}\,\Gamma(\bar{\lambda}, \alpha, \epsilon) \to 0$ *as* $(\alpha, \epsilon) \to (0, 0)$.

Proof. Suppose that the problem (LEP) is LP well-posed. Hence, it has a unique solution $\bar{x} \in S(\bar{\lambda})$ and hence $\Gamma(\bar{\lambda}, \alpha, \epsilon) \neq \emptyset$, for all $\alpha, \epsilon > 0$ as $S(\bar{\lambda}) \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Suppose to the contrary that $\text{diam}\,\Gamma(\bar{\lambda}, \alpha, \epsilon) \nrightarrow 0$ as $(\alpha, \epsilon) \to (0, 0)$. Then there are positive numbers r, m and sequences $\{\alpha_n\}, \{\epsilon_n\}$ in $(0, \infty)$ with $(\alpha_n, \epsilon_n) \to (0, 0)$ and $x_n, x'_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$ such that

$$d(x_n, x'_n) > r, \quad \forall n \geqslant m. \tag{2.4}$$

By $x_n, x'_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$, there exist $\lambda_n, \lambda'_n \in B(\bar{\lambda}, \alpha_n) \cap \Lambda$ such that

$$d(x_n, K(\lambda_n)) \leqslant \epsilon_n,$$

$$f_1(x_n, y, \lambda_n) \geqslant 0, \quad \forall y \in K(\lambda_n), \quad \text{and} \quad f_2(x_n, z, \lambda_n) + \epsilon_n \geqslant 0, \quad \forall z \in Z(\lambda_n, x_n),$$

and
$$d(x_n', K(\lambda_n')) \leqslant \epsilon_n,$$
$$f_1(x_n', y, \lambda_n') \geqslant 0, \quad \forall y \in K(\lambda_n'), \qquad f_2(x_n', z, \lambda_n') + \epsilon_n \geqslant 0, \quad \forall z \in Z(\lambda_n', x_n).$$

The sequence $\{x_n\}$ and $\{x_n'\}$ are LP approximating sequences for (LEP) corresponding to sequences $\lambda_n \to \bar{\lambda}$ and $\lambda_n' \to \bar{\lambda}'$, respectively. Since (LEP) is LP well-posed, we have that $\{x_n\}$ and $\{x_n'\}$ converge to the unique solution $\bar{x}$, which arrives a contradiction to (2.4). Hence, diam $\Gamma(\bar{\lambda}, \alpha, \epsilon) \to 0$ as $(\alpha, \epsilon) \to (0,0)$.

Conversely, let $\{\lambda_n\}$ be a sequence in Λ converging to $\bar{\lambda}$ and $\{x_n\}$ be a LP approximating sequence with respect to $\{\lambda_n\}$. Then there exists a sequence $\{\epsilon_n\}$ in $(0, \infty)$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$d(x_n, K(\lambda_n)) \leqslant \epsilon_n,$$
$$f_1(x_n, y, \lambda_n) \geqslant 0, \quad \forall y \in K(\lambda_n) \quad \text{and} \quad f_2(x_n, z, \lambda_n) + \epsilon_n \geqslant 0, \quad \forall z \in Z(\lambda_n, x_n). \qquad (2.5)$$

If we choose $\alpha_n = d(\lambda_n, \bar{\lambda})$, then $\alpha_n \to 0$ and $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. Since diam $\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n) \to 0$ as $n \to \infty$, it follows that $\{x_n\}$ is a Cauchy sequence in X and hence it converges to $\bar{x} \in X$. For each positive integer n, $K(\lambda_n)$ is compact. Thus, there exists $x_n' \in K(\lambda_n)$ such that

$$d(x_n, x_n') \leqslant \epsilon_n, \quad \text{for all } n \in \mathbb{N},$$

which implies that $x_n' \to \bar{x}$. Since K is closed at $\bar{\lambda}$, it follows that $\bar{x} \in K(\bar{\lambda})$. Suppose to the contrary $\bar{x} \notin S(\bar{\lambda})$, that is, there exist $\bar{y} \in K(\bar{\lambda})$ and $\bar{z} \in Z(\bar{\lambda}, \bar{x})$ such that

$$f_1(\bar{x}, \bar{y}, \bar{\lambda}) < 0 \quad \text{or} \quad f_2(\bar{x}, \bar{z}, \bar{\lambda}) + \epsilon < 0. \qquad (2.6)$$

Since K is lsc at $\bar{\lambda}$ and $\lambda_n \to \bar{\lambda}$, it is clear that for any $y \in K(\bar{\lambda})$ there exists a sequence $y_n \in K(\lambda_n)$ such that $y_n \to \bar{y}$. Again, since Z is lsc at $(\bar{\lambda}, \bar{x})$ and $(\lambda_n, x_n) \to (\bar{\lambda}, \bar{x})$ there exists a sequence $z_n \in Z(\lambda_n, x_n)$ such that $z_n \to \bar{z}$. Hence, we obtain by assumptions (iv), (v) and (2.5) that,

$$f_1(\bar{x}, \bar{y}, \bar{\lambda}) \geqslant 0 \quad \text{and} \quad f_2(\bar{x}, \bar{z}, \bar{\lambda}) \geqslant 0.$$

This yields a contradiction to (2.6). Hence, we conclude that $\bar{x} \in S(\bar{\lambda})$.

Finally, we show that $\bar{x}$ is the only solution of (LEP). Let x^* be another point in $S(\bar{\lambda})$ $(x^* \neq \bar{x})$. It is clear that they both belong to $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ for any $\alpha, \epsilon > 0$. Then, it follows that

$$0 \leqslant d(\bar{x}, x^*) \leqslant \text{diam } \Gamma(\bar{\lambda}, \alpha, \epsilon) \downarrow 0 \text{ as } (\alpha, \epsilon) \downarrow (0,0).$$

This is impossible and therefore we are done. The proof is completed. □

The following examples show that none of the assumptions in Theorem 2.2 can be dropped.

Example 2.3 (Lower semicontinuity of K). Let $X = \Lambda = [0,2]$ and K and f be defined by

$$K(\lambda) = \begin{cases} [0,1] & \text{if } \lambda \neq 0; \\ [0,2] & \text{if } \lambda = 0, \end{cases}$$

$$f(x, y, \lambda) = (x - y, \lambda).$$

One can check that K is closed but not lsc at $\bar{\lambda} = 0$ and

$$S(\lambda) = Z_1(\lambda) = \begin{cases} \{1\} & \text{if } \lambda \neq 0; \\ \{2\} & \text{if } \lambda = 0, \end{cases}$$

$$Z(\lambda, x) = \{x\}, \quad \forall (\lambda, x) \in \text{ gr } Z_1.$$

Thus, assumptions (iii)-(v) hold true. However, (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, let $\lambda_n := \frac{1}{n}$ and $x_n := 1 + \frac{1}{2n}$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 1 \notin S(0)$.

Example 2.4 (Closedness of K). Let $X = \Lambda = [-2, 2]$, $K(\lambda) = (0, 1]$ (continuous), and a function $f := (f_1, f_2) : X \times X \times \Lambda \to \mathbb{R}^2$ be defined by, for all $x, y \in X$ and $\lambda \in \Lambda$,

$$f(x, y, \lambda) = (x - \frac{y}{2}, \frac{1}{2} - x).$$

It can be calculated that

$$Z(\lambda, x) = \begin{cases} \{1\} & \text{if } x = \frac{1}{2}; \\ \emptyset & \text{if } x \in (\frac{1}{2}, 1]; \\ X & \text{otherwise.} \end{cases}$$

Then, we can conclude that

$$\Gamma(\lambda, \alpha, \epsilon) = \left[\frac{1}{2}, \frac{1}{2} + \min\{\epsilon, \frac{3}{2}\}\right],$$

and

$$\text{diam } \Gamma(\lambda, \alpha, \epsilon) \to 0 \text{ as } (\alpha, \epsilon) \to (0, 0).$$

One can check that,

$$S(\lambda) = \left\{\frac{1}{2}\right\}.$$

We observe that (LEP) is not LP well-posed. Indeed, put $\lambda_n := \frac{1}{n}$, $x_n := 1 + \frac{\epsilon_n}{n}$ for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 1 \notin S(\lambda)$.

Example 2.5 (Lower semicontinuity of Z). Let $X = \Lambda = [0, 1]$, $K(\lambda) = [0, 1]$ (continuous and closed), $\bar{\lambda} = 0$ and $f(x, y, \lambda) = (\lambda x(x - y), y - x)$. One can check that

$$Z_1(\lambda) = \begin{cases} [0, 1] & \text{if } \lambda = 0; \\ \{0, 1\} & \text{if } \lambda \neq 0, \end{cases}$$

and, for each $(\lambda, x) \in \text{gr } Z_1$,

$$Z(\lambda, x) = \begin{cases} [0, 1] & \text{if } \lambda = 0 \text{ or } x = 0; \\ \{1\} & \text{if } \lambda \neq 0 \text{ and } x \neq 0. \end{cases}$$

Z is not lsc at $(0, 1)$. Indeed, by taking $\lambda_n := \frac{1}{2n}$ and $x_n := 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$, we have $(\lambda_n, x_n) \to (0, 1)$ and $Z(\lambda_n, x_n) = \{1\}$, for all n, while $Z(0, 1) = [0, 1]$. Assumptions (iv) and (v) are obviously satisfied. By calculating the solution mapping S explicitly as follows:

$$S(\lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0; \\ \{0, 1\} & \text{if } \lambda \neq 0, \end{cases}$$

we observe that (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, let $\lambda_n := \frac{1}{2n}$ and $x_n := 1 + \frac{1}{n}$, for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 1 \notin S(0)$.

Example 2.6 (Upper 0-level closedness of f_1). Let $X = \Lambda = [0, 1]$, $K(\lambda) = [0, 1]$ (continuous and closed), $\bar{\lambda} = 0$ and

$$f(x, y, \lambda) = \begin{cases} (x - y, \lambda) & \text{if } \lambda = 0; \\ (y - x, \lambda) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$S(\lambda) = Z_1(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0\} & \text{if } \lambda \neq 0, \end{cases}$$

$$Z(\lambda, x) = \{x\}, \quad \forall (\lambda, x) \in \text{gr } Z_1.$$

Hence, all the assumptions except number (iv) hold true. However, (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, take sequences $\lambda_n := \frac{1}{n+1}$ and $x_n := 0$, for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to $\{\lambda_n\}$ with $\epsilon_n := \frac{1}{n}$, while $x_n \to 0 \notin S(0)$.

Finally, we show that assumption (iv) is not satisfied. Indeed, take $\{x_n\}$ and $\{\lambda_n\}$ as above and $\{y_n := 1\}$, we have $(x_n, y_n, \lambda_n) \to (0, 1, 0)$ and $f_1(x_n, y_n, \lambda_n) = 1 > 0$, for all n, while $f_1(0, 1, 0) = -1 < 0$.

Example 2.7 (Strongly upper 0-level closedness of f_2). Let X, Λ, K be as in Example 2.6 and

$$f(x, y, \lambda) = \begin{cases} (0, x - y) & \text{if } \lambda = 0; \\ (0, x(x - y)) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$Z_1(\lambda) = Z(\lambda, x) = [0, 1], \quad \forall x, \lambda \in [0, 1],$$

$$S(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0, 1\} & \text{if } \lambda \neq 0. \end{cases}$$

Thus, all the assumptions of Theorem 2.2 except (v) are satisfied. However, (LEP) is not LP well-posed at $\bar{\lambda}$. Indeed, take sequences $\lambda_n := \frac{1}{n+1}$ and $x_n := 0$, for all $n \in \mathbb{N}$. Then, $\{x_n\}$ is an LP approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to $\{\lambda_n\}$, while $x_n \to 0 \notin S(0)$.

Finally, we show that assumption (iv) is not satisfied. Indeed, take sequences $x_n := 0, y_n := 1, \lambda_n := \frac{1}{n+1}$ and $\epsilon_n := \frac{1}{n}$, for all $n \in \mathbb{N}$, we have $(x_n, y_n, \lambda_n, \epsilon_n) \to (0, 1, 0, 0)$ and $f_2(x_n, y_n, \lambda_n) + \epsilon_n > 0$, for all n, while $f_2(0, 1, 0)$.

Corollary 2.8. *If the conditions of the previous theorem hold, then (LEP) is LP well-posed, if and only if $S(\bar{\lambda}) \neq \emptyset$ and*

$$\text{diam } \Gamma(\bar{\lambda}, \alpha, \epsilon) \to 0 \quad as \quad (\alpha, \epsilon) \to (0, 0).$$

Then (LEP) is LP well-posed, if and only if $\Gamma(\bar{\lambda}, \alpha, \epsilon) \neq \emptyset$, $\forall \alpha, \epsilon > 0$, and $\text{diam } \Gamma(\bar{\lambda}, \alpha, \epsilon) \to 0$ as $(\alpha, \epsilon) \to (0, 0)$.

Theorem 2.9. *Suppose that the conditions* (i)-(iv) *in Proposition 2.1 are satisfied. Then (LEP) is LP well-posed, if and only if it has a unique solution.*

Proof. By the definition, we know that LP well-posedness for (LEP) implies that it has a unique solution. For the converse, suppose that the problem (LEP) has a unique solution x'. Let $\{\lambda_n\}$ be a sequence in Λ converging to $\bar{\lambda}$ and $\{x_n\}$ an LP approximating sequence with respect to $\{\lambda_n\}$. Then, there exists a sequence $\{\epsilon_n\}$ in $(0, \infty)$ with $\epsilon_n \to 0$, as $n \to \infty$, such that

$$d(x_n, K(\lambda_n)) \leqslant \epsilon_n, \quad \text{for all } n \in \mathbb{N}, \tag{2.7}$$

and

$$f_1(x_n, y, \lambda_n) \geqslant 0, \quad \forall y \in K(\lambda_n), \qquad f_2(x_n, z, \lambda_n) + \epsilon_n \geqslant 0, \quad \forall z \in Z(\lambda_n, x_n).$$

By (2.7) and the closedness of $K(\lambda_n)$ in X, for each positive integer n, we can choose $x'_n \in K(\lambda_n)$ such that

$$d(x_n, x'_n) \leqslant \epsilon_n. \tag{2.8}$$

Since X is a compact set, the sequence $\{x'_n\}$ has a subsequence $\{x'_{n_k}\}$ which converges to a point $\bar{x} \in X$. By using (2.8), we conclude that the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x}$. Again as K is closed at $\bar{\lambda}$, it follows that $\bar{x} \in K(\bar{\lambda})$. Proceeding along the lines of converse part in the proof of Theorem 2.2, we can show that $\bar{x} \in S(\bar{\lambda})$. Consequently, $\bar{x}$ coincides with $x'(\bar{x} = x')$. Again, by the uniqueness of the solution, it is obvious that every possible subsequence converges to the unique solution x' and hence the whole sequence $\{x_n\}$ converges to x', thus yielding the LP well-posedness of (LEP). □

To weaken the assumption of LP well-posednes in Theorem 2.2, we are going to use the notions of measures of noncompactness in a metric space X.

Definition 2.10. Let M be a nonempty subset of a metric space X.

(i) The Kuratowski measure of M is

$$\mu(M) = \inf \left\{ \epsilon > 0 \mid M \subseteq \bigcup_{k=1}^{n} M_k \text{ and } \text{diam } M_k \leqslant \epsilon, k = 1, \dots, n, \ \exists n \in \mathbb{N} \right\}.$$

(ii) The Hausdorff measure of M is

$$\eta(M) = \inf\left\{\varepsilon > 0 | M \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon), x_k \in X, \text{ for some } n \in \mathbb{N}\right\}.$$

(iii) The Istrătescu measure of M is

$$\iota(M) = \inf\left\{\varepsilon > 0 | M \text{ have no infinite } \varepsilon - \text{ discrete subset }\right\}.$$

Daneš [13] obtained the following inequalities:

$$\eta(M) \leqslant \iota(M) \leqslant \mu(M) \leqslant 2\eta(M). \tag{2.9}$$

The measures μ, η and ι share many common properties and we will use γ in the sequel to denote either one of them. γ is a regular measure (see [5, 38]), i.e., it enjoys the following properties.

Lemma 2.11. *Let M be a nonempty subset of a metric space X.*

(i) *$\gamma(M) = +\infty$, if and only if the set M is unbounded;*
(ii) *$\gamma(M) = \gamma(clM)$;*
(iii) *from $\gamma(M) = 0$, it follows that M is totally bounded set;*
(iv) *if X is a complete space and if $\{A_n\}$ is a sequence of closed subsets of X such that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ and $\lim_{n \to +\infty} \gamma(A_n) = 0$, then $K := \bigcap_{n \in \mathbb{N}} A_n$ is a nonempty compact set and $\lim_{n \to +\infty} H(A_n, K) = 0$, where H is the Hausdorff metric;*
(v) *from $M \subseteq N$, it follows that $\gamma(M) \leqslant \gamma(N)$.*

In terms of a measure $\gamma \in \{\mu, \eta, \iota\}$ of noncompactness, we have the following result.

Theorem 2.12. *Let X and Λ be metric spaces.*

(i) *If LEP is LP well-posed at $\bar{\lambda}$, then $\gamma(\Gamma(\bar{\lambda}, \alpha, \varepsilon)) \downarrow 0$ as $(\alpha, \varepsilon) \downarrow (0,0)$.*
(ii) *Conversely, suppose that $S(\bar{\lambda})$ has a unique point and $\gamma(\Gamma(\bar{\lambda}, \alpha, \varepsilon)) \downarrow 0$ as $(\alpha, \varepsilon) \downarrow (0,0)$, and the following conditions hold:*
   (a) *X is complete and Λ is compact or a finite dimensional normed space;*
   (b) *K is continuous, closed and compact-valued on Λ;*
   (c) *Z is lsc on $\Lambda \times X$;*
   (d) *f_1 is upper 0-level closed on $X \times X \times \Lambda$;*
   (e) *f_2 is upper b-level closed on $X \times X \times \Lambda$ for every negative b close to zero.*
   Then LEP is LP well-posed at $\bar{\lambda}$.

Proof. By the relationship (2.9), the proof is similar for the three mentioned measures of noncompactness. We discuss only the case $\gamma = \mu$, the Kuratowski measure.

(i) Suppose that (LEP) is LP-well posed at $\bar{\lambda}$.

By applying Proposition 3.2, we can conclude that $S(\bar{\lambda})$ is compact, and hence $\mu(S(\bar{\lambda})) = 0$. Let $\epsilon > 0$ and assume that

$$S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k \quad \text{with} \quad \mathrm{diam}M_k \leqslant \epsilon, \quad \text{for all } k = 1, \ldots, n.$$

We set

$$N_k = \{y \in X | \, d(y, M_k) \leqslant H(\Gamma(\bar{\lambda}, \alpha, \epsilon), \, S(\bar{\lambda}))\},$$

and we want to show that $\Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \bigcup_{k=1}^{n} N_k$. For any $x \in \Gamma(\bar{\lambda}, \alpha, \epsilon)$, we have

$$d(x, S(\bar{\lambda})) \leqslant H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})).$$

Due to $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k$, one has

$$d\left(x, \bigcup_{k=1}^{n} M_k\right) \leqslant H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})).$$

Then, there exists $\bar{k} \in \{1, 2, \ldots, n\}$ such that

$$d(x, M_{\bar{k}}) \leqslant H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})),$$

i.e., $x \in N_{\bar{k}}$. Thus, $\Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \bigcup_{k=1}^{n} N_k$. Because $\mu(S(\bar{\lambda})) = 0$ and

$$\text{diam} N_k = \text{diam} M_k + 2H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) \leqslant \epsilon + 2H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})),$$

it holds

$$\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon)) \leqslant 2H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})).$$

Note that $H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) = H^*(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda}))$ since $S(\bar{\lambda}) \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon)$ for all $\alpha, \epsilon > 0$. Now, we claim that $H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) \downarrow 0$ as $\alpha, \epsilon \downarrow 0$ and . Indeed, if otherwise, we can assume that there exist $r > 0$ and sequences $\alpha_n, \epsilon_n \downarrow 0$, and $\{x_n\}$ with $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$ such that

$$d(x_n, S(\bar{x})) \geqslant r, \quad \forall n. \tag{2.10}$$

Since $\{x_n\}$ is an approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to some $\{\lambda_n\}$ with $\lambda_n \in B(\bar{\lambda}, \alpha_n) \cap \Lambda$, it has a subsequence $\{x_{n_k}\}$ converging to some $x \in S(\bar{\lambda})$, which gives a contradiction with (2.10). Therefore, we conclude that $\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon))$ as $\xi \downarrow 0$ and $\varepsilon \downarrow 0$.

(ii) Suppose that $\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon)) \to 0$ as $(\alpha, \epsilon) \to (0, 0)$. First, we show that $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is closed for any $\alpha, \epsilon > 0$. Let $\{x_n\} \subseteq \Gamma(\bar{\lambda}, \alpha, \epsilon)$, with $x_n \to \bar{x}$. Then for each $n \in \mathbb{N}$, there exists $\lambda_n \in B(\bar{\lambda}, \alpha) \cap \Lambda$ such that

$$d(x_n, K(\lambda_n)) \leqslant \epsilon,$$

and

$$f_1(x_n, y, \lambda_n) \geqslant 0, \quad \forall y \in K(\lambda_n) \text{ and } f_2(x_n, z, \lambda_n) + \epsilon \geqslant 0, \quad \forall z \in Z(\lambda_n, x_n), \text{ for all } n \in \mathbb{N}.$$

By the assumption of Λ, this implies that $B(\bar{\lambda}, \alpha)$ is compact. We can assume $\{\lambda_n\}$ converges to some $\lambda \in B(\bar{\lambda}, \alpha) \cap \Lambda$. First, we claim that $d(\bar{x}, K(\lambda)) \leqslant \epsilon$. Since $K(\lambda_n)$ is compact, there exists $x_n' \in K(\lambda_n)$ such that $d(x_n, x_n') \leqslant \epsilon$, for all $n \in \mathbb{N}$. By the upper continuity and compactness of K, there exists a subsequence $\{x_{n_j}'\}$ of $\{x_n'\}$ such that $x_{n_j}' \to x' \in K(\lambda)$. Consequently,

$$d(\bar{x}, K(\lambda)) \leqslant d(\bar{x}, x') = \lim_{n \to \infty} d(x_n, x_n') \leqslant \epsilon. \tag{2.11}$$

For each $y \in K(\lambda)$, the lower semicontinuity of K at λ, there exists a sequence $\{y_n\} \subseteq K(\lambda_n)$ such that $y_n \to y$. It follows from the upper 0-level closedness of f_1 that

$$f_1(\bar{x}, y, \lambda) \geqslant 0,$$

that is

$$f_1(\bar{x}, y, \lambda) \geqslant 0, \ \forall y \in K(\lambda). \tag{2.12}$$

Next, we show that

$$f_2(\bar{x}, z, \lambda) + \epsilon \geqslant 0, \quad \forall z \in Z(\lambda, \bar{x}). \tag{2.13}$$

Suppose to the contrary that there exists $\bar{z} \in Z(\lambda, \bar{x})$ such that

$$f_2(\bar{x}, \bar{z}, \lambda) + \epsilon < 0.$$

Since Z is lower semicontinuous at $(\lambda, \bar{x})$, we have for all n, there is $z_n \in Z(\lambda_n, x_n)$ such that $z_n \to \bar{z}$ as $n \to \infty$. It follows from the upper ($-\epsilon$)-level closedness f_2 at $(\bar{x}, \bar{z}, \lambda)$ that

$$f_2(x_n, z_n, \lambda_n) < -\epsilon,$$

when n is sufficiently large which leads to a contradiction. By (2.11), (2.12) and (2.13), we can conclude that $\bar{x} \in \tilde{S}(\lambda, \epsilon)$, and so $\bar{x} \in \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Therefore $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is closed for any $\alpha, \epsilon > 0$. Now we show that

$$S(\bar{\lambda}) = \bigcap_{\alpha, \epsilon > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon).$$

It is clear that, $S(\bar{\lambda}) \subseteq \bigcap_{\alpha, \epsilon > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Next, we first check that, for each $\epsilon > 0$,

$$\bigcap_{\alpha > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \tilde{S}(\bar{\lambda}, \epsilon).$$

For any $x \in \bigcap_{\alpha > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Then for each $\{\alpha_n\} \downarrow 0$, there exists a sequence $\{\lambda_n\}$ with $\lambda_n \in B(\bar{\lambda}, \alpha_n) \cap \Lambda$ such that $x \in \tilde{S}(\lambda_n, \epsilon)$, for all $n \in \mathbb{N}$, which gives that

$$d(x, K(\lambda_n)) \leqslant \epsilon,$$

$$f_1(x, y, \lambda_n) \geqslant 0, \ \forall y \in K(\lambda_n), \ \text{and} \ f_2(x, z, \lambda_n) + \epsilon \geqslant 0, \ \forall z \in Z(\lambda_n, x).$$

Since $K(\lambda_n)$ is compact, we can choose $x_n \in K(\lambda_n)$ such that

$$d(x, x_n) \leqslant \epsilon, \ \forall n \in \mathbb{N}.$$

By the upper continuity and compactness of K, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x' \in K(\lambda)$, which arrives that

$$d(x, K(\bar{\lambda})) \leqslant d(x, x') = \lim_{n \to \infty} d(x, x_n) \leqslant \epsilon. \tag{2.14}$$

By assumptions on K and f_1 again, we have $x \in Z_1(\bar{\lambda})$, that is,

$$f_1(x, y, \bar{\lambda}) \geqslant 0. \tag{2.15}$$

Next, for each $z \in Z(\bar{\lambda}, x)$, there exists $z_n \in Z(\lambda_n, x)$ such that $z_n \to z$, since Z is lsc at $(\bar{\lambda}, x)$. As $x \in \tilde{S}(\lambda_n, \epsilon)$, it holds

$$f_2(x, z_n, \lambda_n) + \epsilon \geqslant 0, \ \ \forall n \in \mathbb{N}.$$

Since f_2 is upper $-\epsilon$-level closed at $(x, z, \bar{\lambda})$, we have

$$f_2(x, z, \bar{\lambda}) + \epsilon \geqslant 0. \tag{2.16}$$

From (2.14), (2.15), (2.16), we get that $x \in \tilde{S}(\bar{\lambda}, \epsilon)$. We obtain that $\bigcap_{\alpha > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \tilde{S}(\bar{\lambda}, \epsilon)$ for every $\epsilon > 0$. Consequently,

$$\bigcap_{\alpha, \epsilon > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon) \subseteq \bigcap_{\epsilon > 0} \tilde{S}(\bar{\lambda}, \epsilon) = S(\bar{\lambda}).$$

Therefore, we obtain that $S(\bar{\lambda}) = \bigcap_{\alpha, \epsilon > 0} \Gamma(\bar{\lambda}, \alpha, \epsilon)$. Further, since $\mu(\Gamma(\bar{\lambda}, \alpha, \epsilon)) \to 0$ as $(\alpha, \epsilon) \to (0, 0)$. By applying Lemma 2.11 (iv), we get that $S(\bar{\lambda})$ is compact and $H(\Gamma(\bar{\lambda}, \alpha, \epsilon), S(\bar{\lambda})) \longrightarrow 0$ as $(\alpha, \epsilon) \to (0, 0)$.

Finally, we prove that LEP is LP well-posedness. Indeed, let $\{x_n\}$ be an LP-approximating sequence of $(\text{LEP}_{\bar{\lambda}})$ corresponding to some $\lambda_n \longrightarrow \bar{\lambda}$. Then there exists a sequence $\{\epsilon_n\}$ in $(0, \infty)$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$d(x_n, K(\lambda_n)) \leqslant \epsilon_n,$$

$$f_1(x_n, y, \lambda_n) \geqslant 0, \ \forall y \in K(\lambda_n) \ \text{and} \ f_2(x_n, z, \lambda_n) + \epsilon_n \geqslant 0, \ \forall z \in Z(\lambda_n, x_n).$$

If we choose $\alpha_n = d(\lambda_n, \bar{\lambda})$, then $\alpha_n \to 0$ and $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. We see that

$$d(x_n, S(\bar{\lambda})) \leqslant H(\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n), S(\bar{\lambda})) \longrightarrow 0 \ \ \text{as} \ \ n \to \infty.$$

Hence, there exists a sequence $\{\bar{x}_n\}$ in $S(\bar{\lambda})$ such that $d(x_n, \bar{x}_n) \to 0$ as $n \to \infty$. By the compactness of $S(\bar{\lambda})$, there is a subsequence $\{\bar{x}_{n_j}\}$ of $\{\bar{x}_n\}$ converging to a point $\bar{x}$ in $S(\bar{\lambda})$. Consequently, the corresponding subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges to $\bar{x}$. Hence, LEP is LP well-posedness. The proof is completed. $\quad\square$

3. LP well-posedness in the generalized sense

In many practical situations, the problem (LEP) may not always possess a unique solution. Hence, in this section, we introduce a generalization of LP well-posedness for (LEP).

Definition 3.1. The problem (LEP) is said to be LP well-posed in the generalized sense at $\bar{\lambda}$, if
(i) the solution set $S(\bar{\lambda})$ is nonempty;
(ii) for any sequence $\{\lambda_n\}$ converging to $\bar{\lambda}$, every LP approximating sequence $\{x_n\}$ with respect to $\{\lambda_n\}$ has a subsequence converging to some point of $S(\bar{\lambda})$.

Proposition 3.2. *If (LEP) is LP well-posed in the generalized sense at $\bar{\lambda}$, then its solution set $S(\bar{\lambda})$ is a nonempty compact set.*

Proof. Let $\{x_n\}$ be any sequence in $S(\bar{\lambda})$. Then, of course, it is an LP approximating sequence with respect to sequences $\lambda_n := \bar{\lambda}$ and $\epsilon_n := \frac{1}{n}$, for every $n \in \mathbb{N}$. The generalized LP well-posedness of (LEP) ensures the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point of in $S(\bar{\lambda})$. Therefore, we conclude that $S(\bar{\lambda})$ is a nonempty compact set. The proof is completed. $\square$

Next, we present a metric characterization for the generalized LP well-posedness of (LEP) in terms of the upper semicontinuity of the approximate solution set.

Theorem 3.3. (LEP) *is LP well-posed in the generalized sense, if and only if $S(\bar{\lambda})$ is a nonempty, compact set and $\Gamma(\bar{\lambda}, \cdot, \cdot)$ is usc at $(\alpha, \epsilon) := (0, 0)$.*

Proof. Suppose that (LEP) is LP well-posed in the generalized sense. Therefore, $S(\bar{\lambda}) \neq \emptyset$ and further on using Proposition 3.2, we have $S(\bar{\lambda})$ is compact. Next, we assume, on the contrary, that $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is not usc at $(0, 0)$. Consequently, there exists an open set U containing $\Gamma(\bar{\lambda}, 0, 0) = S(\bar{\lambda})$ and positive sequences $\{\alpha_n\}$ and $\{\epsilon_n\}$ satisfying $\alpha_n \to 0$ and $\epsilon_n \to 0$ such that

$$\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n) \subsetneq U, \quad \text{for all } n \in \mathbb{N}.$$

Thus, there exists a sequence $\{x_n\}$ in $\Gamma(\bar{\lambda}, \alpha_n, \epsilon_n) \setminus S(\bar{\lambda})$. Therefore, of course, $\{x_n\}$ is an LP approximating sequence for (LEP), such that none of its subsequence converges to a point of $S(\bar{\lambda})$, which is a contradiction.

Conversely, let $\{\lambda_n\}$ be a sequence in Λ converging to $\bar{\lambda}$ and $\{x_n\}$ be an LP approximating sequence with respect to $\{\lambda_n\}$. If we choose a sequence $\alpha_n = d(\lambda_n, \bar{\lambda})$ then $\alpha_n \to 0$ and $x_n \in \Gamma(\bar{\lambda}, \alpha_n, \epsilon_n)$. As $\Gamma(\bar{\lambda}, \alpha, \epsilon)$ is usc at $(\alpha, \epsilon) = (0, 0)$ and $S(\bar{\lambda}) \neq \emptyset$, it follows that for every $\delta > 0$, $\Gamma(\bar{\lambda}, \delta_n, \epsilon_n) \subset S(\bar{\lambda}) + B(0, \delta)$, for n sufficiently large. Thus $x_n \in S(\bar{\lambda}) + B(0, \delta)$, for n sufficiently large and hence there exists a sequence $\bar{x}_n \in S(\bar{\lambda})$, such that

$$d(x_n, \bar{x}_n) \leqslant \delta. \tag{3.1}$$

Since $S(\bar{\lambda})$ is compact, there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ converging to $\bar{x} \in S(\bar{\lambda})$. By using (3.1), we conclude that the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x} \in S(\bar{\lambda})$. $\square$

The following result illustrates the fact that LP well-posedness in the generalized sense of LEP ensures the stability, in terms of the upper semi-continuity of the solution set S.

Theorem 3.4. *If (LEP) is LP well-posed in the generalized sense, then the solution mapping S is usc at $\bar{\lambda}$.*

Proof. Suppose on the contrary, S is not usc at $\bar{\lambda}$. Then there exists an open set U containning $S(\bar{\lambda})$ such that for every sequence $\lambda_n \to \bar{\lambda}$, there exists $x_n \in S(\lambda_n)$ such that $x_n \notin U$, for every n. Since $\lambda_n \to \bar{\lambda}$, $\{x_n\}$ is an LP approximating sequence for (LEP) and none of its subsequnces converges to a point of $S(\bar{\lambda})$, hence we have a contradiction to the fact that (LEP) is LP well-posed in the generalized sense. $\square$

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