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Vector valued Orlicz-Lorentz sequence spaces and their operator ideals

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Abstract

In the present paper we introduce and study vector valued Orlicz-Lorentz sequence spaces $l_{p,q,\mathcal{M},u,\Delta,A}(X)$ on Banach space X with the help of a Musilak-Orlicz function $\mathcal{M}$ and for different positive indices p and q. We also study their cross and topological duals. Finally, we introduce the operator ideals with the help of the corresponding scalar sequence spaces and s-numbers.

Keywords: Lorentz sequence spaces, s-numbers of operators, Musielak-Orlicz function, difference sequence spaces, operator ideals.

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1. Introduction and preliminaries

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers. Then we say that A defines a matrix mapping from X into Y, if for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, the A-transform of x, is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k, \quad (n \in \mathbb{N}). \tag{1.1}$$

By (X, Y), we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$, if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) : Ax \in X\}.$$

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors (see [20]).

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The study of vector-valued sequence spaces (VVSS) was provoked by the work of Grothendieck in [6]. Since then this theory has developed considerably in different directions, (see [3, 14] and references given therein).

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, nondecreasing and convex function such that $M(0) = 0, M(x) > 0$, for $x > 0$. Such function M always has the integral representation

$$M(x) = \int_0^x p(t)dt,$$

where $p(t)$, known as the kernel of M, is right continuous, non-decreasing function for $t > 0$. It is clear that an Orlicz function M is always increasing as $M(x) \to \infty$ as $x \to \infty$. Also $tp(t) \to \infty$ as $t \to \infty$ and $tp(t) = 0$ for $t = 0$, [11]. However $p(t) > 0$ for $t = 0$ is equivalent to the fact that the Orlicz sequence space l_M is isomorphic to l_1, [8]. Therefore, we presume here that the kernel $p(t)$ has value 0 for $t = 0$ and obviously $p(t) \to \infty$ as $t \to \infty$.

For Orlicz function M and kernel p, we define $q(s) = \sup\{t : p(t) \leqslant s\}$, $s \geqslant 0$. Clearly q possesses the same properties as p and the function N defined as $N(x) = \int_0^x q(t)dt$, is an Orlicz function. The functions M and N are called mutually complementary functions. These functions M and N satisfy Young's inequality: $xy \leqslant M(x) + N(y)$, for $xy \geqslant 0$ and also $M(\alpha x) \leqslant \alpha M(x)$ for $0 < \alpha < 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for small x or at 0, if for each $k > 1$, there exist $R_k > 0$ and $x_k > 0$ such that

$$M(kx) \leqslant R_k M(x), \quad \text{for all } x \in (0, x_k].$$

Suppose X and Y are vector spaces over the same field $\mathbb{K}$ of real or complex numbers, generates a dual system $\langle X, Y\rangle$ with respect to the bilinear functional $\langle x, y\rangle$. We shall denote the vector space of all sequences formed by the elements of X with respect to the operations of pointwise addition and scalar multiplication by $\Omega(X)$ and the space of all finitely non-zero sequences from $\Omega(X)$ by $\phi(X)$. A vector-valued sequence space $\Lambda(X)$ is a subspace of $\Omega(X)$ containing $\phi(X)$. The symbol δ_i^x exists for the sequence $\{0, 0, ..., 0, x, 0, 0, ...\}$, where x is placed at the ith coordinate. The notation $\overline{x}^{(n)}$ denotes the n-th section of $\overline{x}$ given by $\{x_1, x_2, ..., x_n, 0, 0, ...\}$.

A subset M of $\Lambda(X)$ is said to be normal, if for $\{x_i\} \in M$ and $\{\alpha_i\} \in \mathbb{K}$, with $|\alpha_i| \leqslant 1, i \geqslant 1$, the sequence $\{\alpha_i x_i\} \in M$. The generalized Köthe dual of $\Lambda(X)$ is the space

$$\Lambda^{\times}(Y) = \Big\{\overline{y} = \{y_i\} \in Y : \sum_{i \geqslant 1} |\langle x_i, y_i\rangle| < \infty \;\text{ for all } \{x_i\} \in \Lambda(X)\Big\}.$$

The generalized Köthe dual of $\Lambda^{\times}(Y)$ is denoted by $\Lambda^{\times\times}(X)$. The space $\Lambda(X)$ is said to be perfect, if $\Lambda(X) = \Lambda^{\times\times}(X)$.

A vector-valued sequence space $\Lambda(X)$ equipped with a Hausdorff locally convex topology T is called

(i) a GK-space, if the maps $P_{n,\Lambda(X)} : \Lambda(X) \to X$, $P_{n,\Lambda(X)}(\overline{x}) = x_n$, for each $n \geqslant 1$, are continuous;

(ii) a GAK-space, if $\Lambda(X)$ is a GK-space and for each $\{x_i\} \in \Lambda(X)$, $\overline{x}^{(n)} \to \overline{x}$ as $n \to \infty$, in T;

(iii) a GAD-space, if $\overline{x} \in \overline{\phi(X)}$, for every $\overline{x} \in \Lambda(X)$, i.e., $\overline{\phi(X)} = \Lambda(X)$.

Remark 1.1. Every perfect sequence space $\Lambda(X)$ is normal [14].

Let us state here that if the dual system is $\langle X, X^*\rangle$ where X is a Banach space and X^* is its topological dual, then we may interchangeably use the notations $\langle x, f\rangle$ or $f(x)$ for $x \in X$ and $f \in X^*$ in the sequel.

We write w for $\Omega(X)$, ϕ for $\phi(X)$ and λ for $\Lambda(X)$, if we take $X = \mathbb{K}$, the field of scalars. If e_n's are the n-th unit vectors in w, i.e., $e^n = \{\delta_{nj}\}_{j=1}^{\infty}$, where δ_{nj} is the Kronecker delta, ϕ is clearly the subspace of w spanned by e_n's, $n \geqslant 1$.

A sequence space λ is said to be symmetric, if $\overline{\alpha}_\sigma = \{\alpha_{\sigma(i)}\} \in \lambda$ whenever $\overline{\alpha} \in \lambda$ and $\sigma \in \Pi$, where Π is the collection of all permutations of $\mathbb{N}$. The Köthe dual $\lambda^{\times}$ of a symmetric sequence space λ is symmetric [8].

The δ-dual for scalar-valued sequence space λ is defined as

$$\lambda^{\delta} = \Big\{\overline{\alpha} \in w : \sum_{i \geqslant 1} |\alpha_i \beta_{\rho(i)}| < \infty \text{ for all } \overline{\beta} \in \lambda \text{ and } \rho \in \Pi\Big\}.$$

$\lambda^{\times}$ coincides with λ^{δ}, if λ is symmetric.

We define

$$\lambda(X) = \Big\{\{x_n\} : x_n \in X, n \geqslant 1 \text{ and } \{\|x_n\|\} \in \lambda\Big\},$$

for a scalar-valued sequence space λ and a Banach space X. In case, λ equipped with the norm $\|.\|_{\lambda}$, is a Banach space. Therefore, $\lambda(X)$ is also a Banach space with respect to the norm

$$\|\overline{x}\|_{\lambda(X)} = \|\{\|x_n\|\}\|_{\lambda}, \quad \text{(see [1, 3])}.$$

As particular cases, we have $l_{\infty}(X)$ for $\lambda = l_{\infty}$ and $c_0(X)$ corresponding to $\lambda = c_0$.

We define the set $\tilde{l}_M(X)$ as

$$\tilde{l}_M(X) = \{\overline{x} \in \Omega(X) : \sum_{i \geqslant 1} M(\|x_i\|) < \infty\},$$

for a Banach space X corresponding to an Orlicz function M.

The vector-valued Orlicz sequence space is defined as

$$l_M(X) = \Big\{\overline{x} \in \Omega(X) : \sum_{i \geqslant 1} f_i(x_i) \text{ converges for all } \{f_i\} \in \tilde{l}_N(X^*)\Big\},$$

for mutually complementary functions M and N.

A corresponding way of defining $l_M(X)$ is

$$l_M(X) = \Big\{\overline{x} \in \Omega(X) : \sum_{i \geqslant 1} M\Big(\frac{\|x_i\|}{\rho}\Big) < \infty \text{ for some } \rho > 0\Big\}.$$

Two norms

$$\|\overline{x}\|_{(M)} = \sup\Big\{\Big|\sum_{i \geqslant 1} f_i(x_i)\Big| : \sum_{i \geqslant 1} N(\|f_i\|) \leqslant 1\Big\},$$

and

$$\|\overline{x}\|_M = \inf\Big\{\rho > 0 : \sum_{i \geqslant 1} M\Big(\frac{\|x_i\|}{\rho}\Big) \leqslant 1\Big\},$$

are equivalent on l_M and hence we have

$$\|\overline{x}\|_M \leqslant \|\overline{x}\|_{(M)} \leqslant 2\|\overline{x}\|_M, \quad \text{for } \overline{x} \in l_M(X), \quad \text{(see [21])}.$$

We shall write $l_M(X)$ as l_M for $X = \mathbb{K}$. If M satisfies Δ_2-condition at 0 and M, N are mutually complementary Orlicz functions, then $(l_M)^{\times} = l_N$ [8].

A Musielak-Orlicz function $\mathcal{M} = \{M_n\}$ is a sequence of Orlicz functions (see [5, 13]). A Musielak-Orlicz function $\mathcal{M}$ is said to satisfy L1 condition, if $p_n(x) \geqslant p_{n+1}(x)$ for all $x \in [0, \infty)$, where p_n be the kernel of M_n, for all $n \in \mathbb{N}$. A convex modular $\rho_{\mathcal{M}}$ on w for a Musielak-Orlicz function $\mathcal{M}$ is defined as

$$\rho_{\mathcal{M}}(\{\alpha_n\}) = \sup_{\sigma \in \Pi} \sum_{n=1}^{\infty} M_n(\alpha_{\sigma_{(n)}}).$$

Analogous to a convex modular $\rho_{\mathcal{M}}$, we define modular space as

$$\lambda_{\mathcal{M}} = \{\overline{\alpha} = \{\alpha_n\} \in w : \rho_{\mathcal{M}}(\beta\overline{\alpha}) < \infty, \quad \text{for some } \beta > 0\}.$$

This space becomes a normed space under the Luxemburg norm

$$\|\overline{\alpha}\| = \inf\{\beta > 0 : \rho_{\mathcal{M}}(\frac{\overline{\alpha}}{\beta}) \leqslant 1\}.$$

A modular sequence space $\lambda_{\mathcal{M}}$ is always a symmetric sequence space.

The decreasing rearrangement of the absolute values of a sequence $\overline{\alpha} = \{\alpha_n\}$ in l_∞ is given by $\{t_n(\overline{\alpha})\}$, where

$$t_n(\overline{\alpha}) = \inf\{\rho > 0 : \text{card}\{k : |\alpha_k| > \rho\} < n\}.$$

Here card A denotes the cardinality of the set A. The sequence $\{t_n(\overline{\alpha})\}$ satisfies the following properties [16]:

(i) $\|\overline{\alpha}\|_\infty = t_1(\overline{\alpha}) \geqslant t_2(\overline{\alpha}) \geqslant ... \geqslant 0$ for $\overline{\alpha} \in l_\infty$.

(ii) $t_{m+n-1}(\overline{\alpha} + \overline{\beta}) \leqslant t_m(\overline{\alpha}) + t_m(\overline{\beta})$ for $\overline{\alpha}, \overline{\beta} \in l_\infty$.

(iii) $t_{m+n-1}(\overline{\alpha}\overline{\beta}) \leqslant t_m(\overline{\alpha})t_m(\overline{\beta})$ for $\overline{\alpha}, \overline{\beta} \in l_\infty$.

Here $\overline{\alpha}\overline{\beta} = \{\alpha_n\beta_n\}$.

For $\overline{x} = \{x_n\} \in l_\infty(X)$, we denote by

$$t_n(\overline{x}) = t_n(\{x_n\}) = t_n(\{\|x_n\|\}), \quad n \in \mathbb{N}.$$

The Lorentz sequence space $l_{p,q}$ $(0 < p, q \leqslant \infty)$ is given by

$$l_{p,q} = \{\overline{\alpha} = \{\alpha_n\} \in l_\infty : \{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{\alpha})\} \in l_q\}.$$

For $\overline{\alpha} \in l_{p,q}$, let us consider the real-valued function $\|.\|_{p,q}$ as follows

$$\|\overline{\alpha}\|_{p,q} = \begin{cases} \Big\{ \sum\limits_{n\geqslant 1} (n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{\alpha}))^q \Big\}^{\frac{1}{q}} & \text{for } 0 < q < \infty, \\ \sup\limits_{n\geqslant 1} n^{\frac{1}{p}} t_n(\overline{\alpha}) & \text{for } q = \infty. \end{cases}$$

For a convex modular $\rho_{\mathcal{M}}$ defined on w, it has been proved in [5], that

$$\sum_{n\geqslant 1} M_n(t_n(\overline{\alpha})) = \rho_{\mathcal{M}}(\overline{\alpha}), \tag{1.2}$$

for $\overline{\alpha} \in w$, if and only if $\mathcal{M}$ satisfies L1 condition. We see that $(l_{p,q}, \|.\|_{p,q})$ is Banach spaces for $p \geqslant q$ by (1.2). But for $p < q$, it is a quasi-Banach space. Further, they are symmetric sequence spaces [15].

Throughout the paper, we shall denote the Banach spaces over the complex field $\mathbb{C}$ by X and Y and the class of all bounded linear maps from X to Y by $L(X, Y)$.

Let L be the class of all bounded linear operators between any pair of Banach spaces and w^+ be the class of sequences of non-negative real numbers. A mapping $s : L \to w^+$ is called an s-number function, if it satisfies the following conditions:

(i) $\|S\| = s_1(S) \geqslant s_2(S) \geqslant ... \geqslant 0,\ s(S) = \{s_n(S)\},\ S \in L$;

(ii) $s_n(S + T) \leqslant s_n(S) + \|T\|$ for $S, T \in L(X, Y)$ and $n \in \mathbb{N}$;

(iii) $s_n(RST) \leqslant \|R\|s_n(S)\|T\|$ for $T \in L(X_0, X),\ S \in L(X, Y),\ R \in L(Y, Y_0)$ and $n \in \mathbb{N}$;

(iv) if rank $S < n$, then $s_n(S) = 0$, (v), if dim $X \geqslant n$, then $s_n(I_X) = 1$, where I_X denotes the identity map of X.

If the condition (ii) is replaced by

(ii)' $s_{m+n-1}(S+T) \leqslant s_m(S)+s_n(T)$ for $S,T \in L(X,Y)$ and $m,n = 1,2,\cdots$,

then the s-number function is called additive.

An s-number function is called multiplicative. if the condition (iii) is replaced by

(iii)' $s_{m+n-1}(RT) \leqslant s_m(R)s_n(T)$ for $R \in L(Y_0,Y)$ and $T \in L(X,Y_0)$, $m,n = 1,2,\cdots$.

We write $A(X,Y) = A \cap L(X,Y)$ for a subset A of L. An operator ideal is a collection of A, if it satisfies the following:

(i) A contains all finite rank operators;

(ii) $T+S \in A(X,Y)$ for $S,T \in A(X,Y)$;

(iii) if $T \in A(X,Y)$ and $S \in L(Y,Z)$, then $ST \in A(X,Z)$ and also if $T \in L(X,Y)$ and $S \in A(Y,Z)$, then $ST \in A(X,Z)$.

For the Banach spaces X and Y the collection $A(X,Y)$ is called a component of A.

A real-valued function f is said to be an ideal quasi-norm, if f is defined on an operator ideal A and satisfies the following properties:

(i) $0 \leqslant f(T) < \infty$, for each $T \in A$ and $f(T) = 0$, if and only if $T = 0$;

(ii) there exists a constant $\sigma \geqslant 1$ such that $f(S+T) \leqslant \sigma[f(S)+f(T)]$ for $S,T \in A(X,Y)$, where $A(X,Y)$ is any component of A;

(iii) (a) $f(RS) \leqslant \|R\|f(S)$, for $S \in A(X,Z)$, $R \in L(Z,Y)$, and
(b) $f(RS) \leqslant \|S\|f(R)$, for $S \in L(X,Z)$, $R \in A(Z,Y)$.

An operator ideal is said to be quasi-normed operator ideal, if it is equipped with an ideal quasi-norm and a quasi-Banach operator ideal is a quasi-normed operator ideal of which each component is complete with respect to the ideal quasi-norm.

The notion of difference sequence spaces was introduced by Kızmaz [10] who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Let m be a non-negative integer, then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in Z\},$$

where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+v}.$$

Taking $m = 1$, we get the spaces studied by Et and Çolak [4]. For more details about this work one can refer to [1, 2, 9, 12, 16–18].

2. The vector-valued sequence spaces $l_{p,q,\mathcal{M},u,\Delta,A}(X)$ and $h_{p,q,\mathcal{M},u,\Delta,A}(X)$

Let X be a Banach space. Let $\mathcal{M} = (M_k)$ be an Musielak-Orlicz function, that is, $\mathcal{M}$ is a sequence of Orlicz functions, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. In this paper we define the following classes of sequences:

$$l_{p,q,\mathcal{M},u,\Delta,A}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\rho}\Big)\Big] < \infty,\ \text{for some } \rho > 0\Big\},$$

$$h_{p,q,\mathcal{M},u,\Delta,A}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\delta}\Big)\Big] < \infty,\ \text{for all } \delta > 0\Big\}.$$

For $\bar{x} \in l_{p,q,\mathcal{M},u,\Delta,A}(X)$, we define

$$\|\bar{x}\|_{p,q,\mathcal{M},u,\Delta,A}(X) = \inf\Big\{\rho > 0 : \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\rho}\Big)\Big] \leqslant 1\Big\}.$$

If we take $\mathcal{M}(x) = x$ in $l_{p,q,\mathcal{M},u,\Delta,A}(X)$ and $h_{p,q,\mathcal{M},u,\Delta,A}(X)$, then we have the following spaces:

$$l_{p,q,u,\Delta,A}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} u_k\Big[\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\rho}\Big] < \infty,\ \text{for some } \rho > 0\Big\},$$

$$h_{p,q,u,\Delta,A}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} u_k\Big[\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\delta}\Big] < \infty,\ \text{for all } \delta > 0\Big\}.$$

Let $u = (u_k) = 1$, for all $k \in \mathbb{N}$. Then the spaces $l_{p,q,\mathcal{M},u,\Delta,A}(X)$ and $h_{p,q,\mathcal{M},u,\Delta,A}(X)$ are reduced to $l_{p,q,\mathcal{M},\Delta,A}(X)$ and $h_{p,q,\mathcal{M},\Delta,A}(X)$, respectively, as follow:

$$l_{p,q,\mathcal{M},\Delta,A}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} \Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\rho}\Big)\Big] < \infty,\ \text{for some } \rho > 0\Big\},$$

$$h_{p,q,\mathcal{M},\Delta,A}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} \Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\delta}\Big)\Big] < \infty,\ \text{for all } \delta > 0\Big\}.$$

If we take $A = (C, 1)$ in $l_{p,q,\mathcal{M},u,\Delta,A}(X)$ and $h_{p,q,\mathcal{M},u,\Delta,A}(X)$, then we have the following spaces:

$$l_{p,q,\mathcal{M},u,\Delta}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\rho}\Big)\Big] < \infty,\ \text{for some } \rho > 0\Big\},$$

$$h_{p,q,\mathcal{M},u,\Delta}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\delta}\Big)\Big] < \infty,\quad \text{for all } \delta > 0\Big\}.$$

If we take $A = (C, 1)$ and $\mathcal{M}(x) = x$ in $l_{p,q,\mathcal{M},u,\Delta,A}(X)$ and $h_{p,q,\mathcal{M},u,\Delta,A}(X)$, then we have the following spaces:

$$l_{p,q,u,\Delta}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} u_k\Big[\frac{\|\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\rho}\Big] < \infty,\quad \text{for some } \rho > 0\Big\},$$

$$h_{p,q,u,\Delta,}(X) = \Big\{\bar{x} = \{x_k\} \in l_\infty(X) : \sum_{k\geqslant 1} u_k\Big[\frac{\|\ k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\delta}\Big] < \infty,\quad \text{for all } \delta > 0\Big\}.$$

If we take $(M_k) = M$, $A = I$, $(u_k) = 1$ for all $k \in \mathbb{N}$ and $m = 0$, then we get the analogous of the spaces defined by Gupta and Bhar [7]. The aim of this paper is to study the vector-valued Orlicz-Lorentz sequence spaces. We also study their structural properties and investigate cross and topological duals of these spaces. Finally we prove that the operator ideals defined with the help of scalar-valued sequence spaces $l_{p,q,\mathcal{M},u,\Delta,A}$ and additive s-numbers are quasi-Banach operator ideals for $p < q$ and Banach operator ideals for $p \geqslant q$.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. Then the space $l_{p,q,\mathcal{M},u,\Delta,A}(X)$ equipped with $\|.\|_{p,q,\mathcal{M},u,\Delta,A}$ is a quasi-Banach space for $p < q$ and Banach space for $p \geqslant q$. Further for $\overline{x} \in l_{p,q,\mathcal{M},u,\Delta,A}(X)$, we have*

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}}t_k(\Delta^m x_k)\|}{\|\overline{x}\|_{p,q,\mathcal{M},u,\Delta,A}}\Big)\Big] \leqslant 1. \tag{2.1}$$

Proof. We can easily show that $l_{p,q,\mathcal{M},u,\Delta,A}(X)$ is a vector space with usual coordinate wise addition and scalar multiplication. To show that $\|.\|_{p,q,\mathcal{M},u,\Delta,A}$ is a quasi-norm, let $\|\overline{x}\|_{p,q,\mathcal{M},u,\Delta,A} \geqslant 0$, for each $\overline{x} \in l_{p,q,\mathcal{M},u,\Delta,A}(X)$ and $\|\overline{x}\|_{p,q,\mathcal{M},u,\Delta,A} = 0$, for $\overline{x} = 0$. Suppose that $\|\overline{x}\|_{p,q,\mathcal{M},u,\Delta,A} = 0$, for some $\overline{x} = \{x_k\} \in l_{p,q,\mathcal{M},u,\Delta,A}(X)$ and for given $\varepsilon > 0$, we can find $\rho > 0$ such that $\rho < \varepsilon$ and

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}}t_k(\Delta^m x_k)\|}{\rho}\Big)\Big] \leqslant 1.$$

When $\overline{x} \neq 0$, we get $\|x_{k_0}\| \neq 0$ for some $k_0 \in \mathbb{N}$ and so $t_{k_1}(\Delta^m x_k) = \|\Delta^m x_{k_0}\|$, for some $k_1 \in \mathbb{N}$ implies

$$u_k\Big[M_k\Big(\frac{\|A\ k_1^{\frac{1}{p}-\frac{1}{q}}t_{k_1}(\Delta^m x_k)\|}{\varepsilon}\Big)\Big] \leqslant u_k\Big[M_k\Big(\frac{\|A\ k_1^{\frac{1}{p}-\frac{1}{q}}t_{k_1}(\Delta^m x_k)\|}{\rho}\Big)\Big] \leqslant 1,$$

for any $\varepsilon > 0$. We get a contradiction to the fact, so $\overline{x} = 0$.

To prove triangular-type inequality, let us consider $\overline{x} = \{x_k\}$ and $\overline{y} = \{y_k\} \in l_{p,q,\mathcal{M},u,\Delta,A}(X)$. Thus for any $\varepsilon > 0$, there exist ρ_1, $\rho_2 > 0$ such that

$$\rho_1 < \|\overline{x}\|_{p,q,\mathcal{M},u,\Delta,A} + \frac{\varepsilon}{2} \ \text{ with } \ \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}}t_k(\Delta^m x_k)\|}{\rho_1}\Big)\Big] \leqslant 1,$$

and

$$\rho_2 < \|\overline{y}\|_{p,q,\mathcal{M},u,\Delta,A} + \frac{\varepsilon}{2} \ \text{ with } \ \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}}t_k(\Delta^m y_k)\|}{\rho_2}\Big)\Big] \leqslant 1.$$

If $\frac{1}{p} - \frac{1}{q} > 0$, then via properties (i) and (ii) of $\{t_k(\Delta^m x_k)\}$, we get

$$\begin{aligned}
\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}}t_k(\Delta^m(x_k+y_k))\|}{2^{\frac{1}{p}-\frac{1}{q}+1}(\rho_1+\rho_2)}\Big)\Big] &= \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ (2k)^{\frac{1}{p}-\frac{1}{q}}t_{2k}(\Delta^m(x_k+y_k))\|}{2^{\frac{1}{p}-\frac{1}{q}+1}(\rho_1+\rho_2)}\Big)\Big] \\
&\quad + \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ (2k-1)^{\frac{1}{p}-\frac{1}{q}}t_{2k-1}(\Delta^m(x_k+y_k))\|}{2^{\frac{1}{p}-\frac{1}{q}+1}(\rho_1+\rho_2)}\Big)\Big] \\
&\leqslant 2\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}}(t_k(\Delta^m x_k)+t_k(\Delta^m y_k))\|}{2(\rho_1+\rho_2)}\Big)\Big] \\
&\leqslant \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}}t_k(\Delta^m x_k)\|}{\rho_1}\Big) \\
&\quad + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{\|A\ k^{\frac{1}{p}-\frac{1}{q}}t_k(\Delta^m y_k)\|}{\rho_2}\Big)\Big] \\
&\leqslant 1.
\end{aligned}$$

Hence,

$$\|\overline{x}+\overline{y}\|_{p,q,\mathcal{M},u,\Delta,A} \leqslant 2^{\frac{1}{p}-\frac{1}{q}+1}(\rho_1+\rho_2)$$

$$\leqslant 2^{\frac{1}{p}-\frac{1}{q}+1}(\|\overline{x}\|_{p,q,\mathcal{M},u,\Delta,A} + \|\overline{y}\|_{p,q,\mathcal{M},u,\Delta,A} + \varepsilon).$$

Now we prove the completeness of the space $(l_{p,q,\mathcal{M},u,\Delta,A}(X), \|.\|_{p,q,\mathcal{M},u,\Delta,A})$. Let $\{\overline{x}_k\}$ be a Cauchy sequence in $l_{p,q,\mathcal{M},u,\Delta,A}(X)$, as $\overline{x}_k = \{x_k^n\}_{n\geqslant 1}$, $k \in \mathbb{N}$. Hence for $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\|\overline{x}_{k+j} - \overline{x}_k\|_{p,q,\mathcal{M},u,\Delta,A} = \inf\Big\{\rho > 0 : \sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\ n^{\frac{1}{p}-\frac{1}{q}} t_n(\Delta^m(\overline{x}_{k+j} - \overline{x}_k))\|}{\rho}\Big)\Big] \leqslant 1\Big\} < \varepsilon,$$

for each $k \geqslant k_0$ and each $j \in \mathbb{N}$. Thus

$$\sum_{n\geqslant 1}\Big[\frac{\|A\ n^{\frac{1}{p}-\frac{1}{q}} t_n(\Delta^m(\overline{x}_{k+j} - \overline{x}_k))\|}{\varepsilon}\Big] \leqslant 1, \quad \text{for all } k \geqslant k_0,\ j \in \mathbb{N},$$

which implies that

$$\Big\{\{A\ n^{\frac{1}{p}-\frac{1}{q}} t_n(\Delta^m(\overline{x}_{k+j} - \overline{x}_k))/\varepsilon\} : n \in \mathbb{N}\Big\},$$

is a bounded set for $j \in \mathbb{N}$ and for all $k \geqslant k_0$. Therefore $\{x_k^n\}$ is a Cauchy sequence in X, for each $n \in \mathbb{N}$ and so converges to z_n. Let $\overline{z} = \{z_n\}$. Then $t_n(\Delta^m(\overline{x}_{k+j} - \overline{x}_k)) \to t_n(\overline{z} - \overline{x}_k)$ as $j \to \infty$ and hence by continuity of $\mathcal{M}$,

$$\sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\ n^{\frac{1}{p}-\frac{1}{q}} t_n(\Delta^m(\overline{z} - \overline{x}_k))\|}{\varepsilon}\Big)\Big] \leqslant 1, \quad \text{for all } k \geqslant k_0.$$

This implies that $\overline{z} \in l_{p,q,\mathcal{M},u,\Delta,A}(X)$ and $\|\overline{z} - \overline{x}_k\|_{p,q,\mathcal{M},u,\Delta,A} \to 0$ as $k \to \infty$. Also, inequality (2.1) is directly obtained from the definition of the quasi-norm $\|.\|_{p,q,\mathcal{M},u,\Delta,A}$. This completes the proof. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. Then $h_{p,q,\mathcal{M},u,\Delta,A}(X)$ is a closed subspace of $l_{p,q,\mathcal{M},u,\Delta,A}(X)$. Moreover, if $\mathcal{M} = (M_k)$ satisfies Δ_2-condition at 0, then $l_{p,q,\mathcal{M},u,\Delta,A}(X) = h_{p,q,\mathcal{M},u,\Delta,A}(X)$.*

Proof. First of all it is without a doubt that $h_{p,q,\mathcal{M},u,\Delta,A}(X)$ is a subspace of $l_{p,q,\mathcal{M},u,\Delta,A}(X)$. Now we prove that $h_{p,q,\mathcal{M},u,\Delta,A}(X)$ is closed in $l_{p,q,\mathcal{M},u,\Delta,A}(X)$. Suppose $\overline{x} = \{x_k\} \in \overline{h}_{p,q,\mathcal{M},u,\Delta,A}(X)$, the closure of $h_{p,q,\mathcal{M},u,\Delta,A}(X)$ in $l_{p,q,\mathcal{M},u,\Delta,A}(X)$. So there exists a sequence $\{\overline{y}_k\} = \{\{y_k^n\}\} \in h_{p,q,\mathcal{M},u,\Delta,A}(X)$, $k \geqslant 1$ and we have $\|\overline{y}_k - \overline{x}\|_{p,q,\mathcal{M},u,\Delta,A} \to 0$ as $k \to \infty$. Take any $\delta > 0$. Thus for $\delta_1 = \min\{2^{\frac{1}{p}-\frac{1}{q}}\delta,\ \delta\}$, we get $k_0 \in \mathbb{N}$ such that

$$\|\overline{y}_k - \overline{x}\|_{p,q,\mathcal{M},u,\Delta,A} < \frac{\delta_1}{2}, \quad \text{for all } k \geqslant k_0. \tag{2.2}$$

When $\frac{1}{p} - \frac{1}{q} \geqslant 0$, we have

$$\begin{aligned}\sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\ n^{\frac{1}{p}-\frac{1}{q}} t_n(\Delta^m\overline{x})\|}{\delta}\Big)\Big] &\leqslant 2\sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\ n^{\frac{1}{p}-\frac{1}{q}}\big(t_n(\Delta^m(\overline{x} - \overline{y}_{k_0})) + t_n(\Delta^m\overline{y}_{k_0})\big)\|}{\delta_1}\Big)\Big] \\ &\leqslant \sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\ n^{\frac{1}{p}-\frac{1}{q}}\big(t_n(\Delta^m(\overline{x} - \overline{y}_{k_0}))\big)\|}{\delta_1/2}\Big)\Big] \\ &\quad + \sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\ n^{\frac{1}{p}-\frac{1}{q}} t_n(\Delta^m\overline{y}_{k_0})\|}{\delta_1/2}\Big)\Big] \\ &\leqslant \sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\ n^{\frac{1}{p}-\frac{1}{q}}\big(t_n(\Delta^m(\overline{x} - \overline{y}_{k_0}))\big)\|}{\|\overline{x} - \overline{y}_{k_0}\|_{p,q,\mathcal{M},u,\Delta,A}}\Big)\Big]\end{aligned}$$

$$+\sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\, n^{\frac{1}{p}-\frac{1}{q}} t_n(\Delta^m \overline{y}_{k_0})\|}{\delta_1/2}\Big)\Big]$$
$$<\infty.$$

In the case when $\frac{1}{p}-\frac{1}{q}<0$, we get

$$\begin{aligned}\sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\, n^{\frac{1}{p}-\frac{1}{q}} t_n(\Delta^m \overline{x})\|}{\delta}\Big)\Big] &\leqslant \sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\, n^{\frac{1}{p}-\frac{1}{q}} (t_n(\Delta^m(\overline{x}-\overline{y}_{k_0})))\|}{\delta_1/2}\Big)\Big]\\ &\quad+\sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\, n^{\frac{1}{p}-\frac{1}{q}} t_n(\Delta^m \overline{y}_{k_0})\|}{\delta_1/2}\Big)\Big]\\ &<\infty,\end{aligned}$$

by the relation (2.2). Clearly $\overline{x}\in h_{p,q,\mathcal{M},u,\Delta,A}(X)$ and so the subspace $h_{p,q,\mathcal{M},u,\Delta,A}(X)$ is closed. Now we suppose that $\mathcal{M}$ satisfies Δ_2-condition at 0. Let $\overline{x}\in l_{p,q,\mathcal{M},u,\Delta,A}(X)$, we have

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\rho_0}\Big)\Big]<\infty, \quad \text{for some } \rho_0>0.$$

To show that $\overline{x}\in h_{p,q,\mathcal{M},u,\Delta,A}(X)$, choose any $\eta>0$. If $\eta\geqslant\rho_0$, then

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\eta}\Big)\Big]<\infty.$$

Now presume $\eta<\rho_0$ and suppose $K=\frac{\rho_0}{\eta}$. Since $\mathcal{M}$ satisfies the Δ_2-condition, so we can find $R_K>0$ and $x_K>0$ such that $\mathcal{M}(Kx)\leqslant R_K\mathcal{M}(x)$ for all $x\in(0,x_K]$

$$\Longrightarrow \sum_{k\geqslant k_0} u_k\Big[M_k\Big(\frac{\|KA\, k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\rho_0}\Big)\Big]\leqslant R_K\sum_{k\geqslant k_0} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\rho_0}\Big)\Big]<\infty,$$

for some $k_0\in\mathbb{N}$. Hence

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}} t_k(\Delta^m x_k)\|}{\eta}\Big)\Big]<\infty, \quad \text{for any } \eta>0,$$

and so $\overline{x}\in h_{p,q,\mathcal{M},u,\Delta,A}(X)$, we have $h_{p,q,\mathcal{M},u,\Delta,A}(X)=l_{p,q,\mathcal{M},u,\Delta,A}(X)$. □

Proposition 2.3. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $u=(u_k)$ be a sequence of strictly positive real numbers and $A=(a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. If $Y=h_{p,q,\mathcal{M},u,\Delta,A}(X)\cap c_0(X)$, $0<p,q\leqslant\infty$. Then Y equipped with the subspace topology of $h_{p,q,\mathcal{M},u,\Delta,A}(X)$ is a GAD-space.*

Proof. Obviously $\phi(X)\subset Y$. Suppose $\overline{x}\in Y$. Now for any $\varepsilon>0$, we can find $k_0\in\mathbb{N}$, such that

$$\sum_{k\geqslant k_0} u_k\Big[M_k\Big(\frac{\|A\, k^{1/p-1/q} t_k(\Delta^m x_k)\|}{\varepsilon}\Big)\Big]\leqslant 1.$$

Let $I_k=\{i\in\mathbb{N}:\|x_i\|>\frac{1}{k}\}$, $k\in\mathbb{N}$ and $\overline{v}_k=\sum_{i\in I_k}\delta_i^{x_i}$. Since $\overline{x}\in c_0(X)$, I_k is finite and so $\overline{v}_k\in\phi(X)$. Set $n_k=\text{card}\, I_k$. Then take $m_0\in\mathbb{N}$ such that

$$\sum_{k\geqslant m_0} u_k\Big[M_k\Big(\frac{\|A\, k^{1/p-1/q} t_k(\Delta^m x_k)\|}{\varepsilon}\Big)\Big]\leqslant\frac{1}{2}.$$

Take k so large that

$$\frac{1}{k}\sum_{i=1}^{m_0} u_k\Big[M_k\Big(\frac{\|A\ i^{1/p-1/q}\|}{\varepsilon}\Big)\Big] \leqslant \frac{1}{2}.$$

Thus,

$$\begin{aligned}\sum_{i\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ i^{1/p-1/q}t_{n_k+i}(\Delta^m x_k)\|}{\varepsilon}\Big)\Big] &\leqslant \frac{1}{k}\sum_{i=1}^{m_0} u_k\Big[M_k\Big(\frac{\|A\ i^{1/p-1/q}\|}{\varepsilon}\Big)\Big]\\ &\quad + \sum_{i\geqslant m_0+1} u_k\Big[M_k\Big(\frac{\|A\ k^{1/p-1/q}t_i(\Delta^m x_k)\|}{\varepsilon}\Big)\Big]\\ &\leqslant \frac{1}{2}+\frac{1}{2}=1.\end{aligned}$$

This implies that $\|\bar{x}-\bar{v}_k\|_{p,q,\mathcal{M},u,\Delta,A} \leqslant \varepsilon$, for sufficiently large k. Hence Y is a GAD-space. □

Proposition 2.4. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function satisfying Δ_2-condition at 0, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. If $l_{p,q,\mathcal{M},u,\Delta,A}(X) \subset c_0(X)$, $0 < p \leqslant q \leqslant \infty$. Then $l_{p,q,\mathcal{M},u,\Delta,A}$ is a GAD-space.*

Remark 2.5. It is very motivating to know whether the space $h_{p,q,\mathcal{M},u,\Delta,A}(X)$ is a GAK-space, this means that the k^{th} section $\bar{x}^{(k)} = \{x_1, x_2, ..., x_k, 0, 0, 0, 0, ...\}$ of an element $\bar{x} = \{x_i\}$ of $h_{p,q,\mathcal{M},u,\Delta,A}$ converges to $\bar{x}$ with respect to its quasi-norm. Whenever, if $p, q > 0$ with $\frac{1}{p} - \frac{1}{q} \geqslant 0$ and $\bar{x} \in h_{p,q,\mathcal{M},u,\Delta,A}(X)$ such that $\|x_1\| > \|x_2\| > \|x_3\| > \cdots$, then $t_k(\bar{x}) = \|x_k\|$ and in this case, one can easily show that $\|\bar{x}-\bar{x}^{(k)}\|_{p,q,\mathcal{M},u,\Delta,A} \to 0$ as $k \to \infty$.

3. Duals of the space $l_{p,q,\mathcal{M},u,\Delta,A}(X)$, $1 \leq p \leq q \leq \infty$

Suppose that the spaces $l_{p,q,\mathcal{M},u,\Delta,A}(X)$ are symmetric sequence spaces, since the decreasing rearrangement of $\bar{x}$ would be the same as that of $\bar{x}_\pi$ for any permutation π of $\mathbb{N}$ and $\mathcal{M} = (M_k)$ is an increasing function. Thus the δ-dual of the scalar-valued sequence space $l_{p,q,\mathcal{M},u,\Delta,A}$ would coincide with its cross-dual.

Theorem 3.1. *Let $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ be two mutually complementary Musielak-Orlicz functions such that $\mathcal{M}$ satisfies Δ_2-condition at 0, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. Then $(l_{p_1,q_1,\mathcal{M},u,\Delta,A})^\times \supseteq l_{p_2,q_2,\mathcal{N},u,\Delta,A}$, where $1/p_1 + 1/p_2 = 1$ and $1/q_1 + 1/q_2 = 1$. Moreover, $(l_{p_1,q_1,\mathcal{M},u,\Delta,A})^\times = l_{p_2,q_2,\mathcal{N},u,\Delta,A}$ when $1/p_1 - 1/q_1 \geqslant 0$.*

Proof. To show that $l_{p_2,q_2,\mathcal{N},u,\Delta,A} \subset (l_{p_1,q_1,\mathcal{M},u,\Delta,A})^\times$, suppose $\overline{\beta} \in l_{p_2,q_2,\mathcal{N},u,\Delta,A}$. Then, we have

$$\sum_{k\geqslant 1} u_k\Big[N_k\Big(\frac{\|A\ k^{\frac{1}{p_2}-\frac{1}{q_2}}t_k(\Delta^m\overline{\beta})\|}{\delta_0}\Big)\Big] < \infty, \quad \text{for some } \delta_0 > 0.$$

Let $\overline{\alpha} \in l_{p_1,q_1,\mathcal{M},u,\Delta,A}$. Then $\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p_1}-\frac{1}{q_1}}t_k(\Delta^m\overline{\alpha})\|}{\rho}\Big)\Big] < \infty,$ for all $\rho > 0$. Thus,

$$\begin{aligned}\sum_{k\geqslant 1}|\alpha_k\beta_k| &\leqslant \sum_{k\geqslant 1} t_k(\Delta^m\overline{\alpha})t_k(\Delta^m\overline{\beta})\\ &\leqslant \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\ k^{\frac{1}{p_1}-\frac{1}{q_1}}t_k(\Delta^m\overline{\alpha})\|}{1/\delta_0}\Big)\Big] + \sum_{k\geqslant 1} u_k\Big[N_k\Big(\frac{\|A\ k^{\frac{1}{p_2}-\frac{1}{q_2}}t_k(\Delta^m\overline{\beta})\|}{\delta_0}\Big)\Big] < \infty.\end{aligned}$$

Hence $\overline{\beta} \in (l_{p_1,q_1,\mathcal{M},u,\Delta,A})^{\times}$. Now to prove $(l_{p_1,q_1,\mathcal{M},u,\Delta,A})^{\times} = l_{p_2,q_2,\mathcal{N},u,\Delta,A}$, suppose $\overline{\beta} \in (l_{p_1,q_1,\mathcal{M},u,\Delta,A})^{\times}$, then $\sum_{i\geqslant 1} |\alpha_i\beta_i| < \infty$, for all $\{\alpha_i\} \in l_{p_1,q_1,\mathcal{M},u,\Delta,A}$. Since $l_{p_1,q_1,\mathcal{M},u,\Delta,A}$ and $(l_{p_1,q_1,\mathcal{M},u,\Delta,A})^{\times}$ both are symmetric sequence spaces, $\{t_k(\Delta^m\overline{\alpha})\} \in l_{p_1,q_1,\mathcal{M},u,\Delta,A}$, for $\overline{\alpha} \in l_{p_1,q_1,\mathcal{M},u,\Delta,A}$ and $\{t_k(\Delta^m\overline{\beta})\} \in (l_{p_1,q_1,\mathcal{M},u,\Delta,A})^{\times}$, for $\overline{\beta} \in (l_{p_1,q_1,\mathcal{M},u,\Delta,A})^{\times}$. Hence $\sum_{k\geqslant 1} t_k(\Delta^m\overline{\alpha})t_k(\Delta^m\overline{\beta}) < \infty$, for all $\overline{\alpha} \in l_{p_1,q_1,\mathcal{M},u,\Delta,A}$.

Again if $\overline{\gamma} \in l_{\mathcal{M}}$, then $\{t_k(\Delta^m\overline{\gamma})\} \in l_{\mathcal{M}}$ as $l_{\mathcal{M}}$ is symmetric and normal and so

$$\{Ak^{\frac{1}{p_2}-\frac{1}{q_2}}t_k(\Delta^m\overline{\gamma})\} \in l_{p_1,q_1,\mathcal{M},u,\Delta,A}.$$

Hence

$$\sum_{k\geqslant 1}\Big[\|A\, k^{\frac{1}{p_2}-\frac{1}{q_2}}t_k(\Delta^m\overline{\gamma})t_k(\Delta^m\overline{\beta})\|\Big] < \infty, \quad \text{for all } \overline{\gamma} \in l_{\mathcal{M}}.$$

This implies that $\{Ak^{\frac{1}{p_2}-\frac{1}{q_2}}t_k(\Delta^m\overline{\beta})\} \in l_{\mathcal{M}}^{\times} = l_{\mathcal{N}}$ and so $\overline{\beta} \in l_{p_2,q_2,\mathcal{N},u,\Delta,A}$. Thus, we have

$$(l_{p_1,q_1,\mathcal{M},u,\Delta,A})^{\times} = l_{p_2,q_2,\mathcal{N},u,\Delta,A}.$$

□

Proposition 3.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. For positive reals p_1, p_2, q_1, q_2 with $1/p_1 + 1/p_2 = 1$, $1/q_1 + 1/q_2 = 1$ such that $q_1 < p_1$, the spaces $l_{p_1,q_1,\mathcal{M},u,\Delta,A}$ are perfect sequences spaces.*

Proof. In fact in this case $(l_{p_2,q_2,\mathcal{N},u,\Delta,A})^{\times} = l_{p_1,q_1,\mathcal{M},u,\Delta,A}$ and $l_{p_2,q_2,\mathcal{N},u,\Delta,A} \subseteq (l_{p_1,q_1,\mathcal{M},u,\Delta,A})^{\times}$. So we have $l_{p_1,q_1,\mathcal{M},u,\Delta,A} \subset (l_{p_1,q_1,\mathcal{M},u,\Delta,A})^{\times\times} \subset (l_{p_2,q_2,\mathcal{N},u,\Delta,A})^{\times} = l_{p_1,q_1,\mathcal{M},u,\Delta,A}$. □

Proposition 3.3. *Let $\mathcal{M} = (M_k)$ be an Musielak-Orlicz function satisfying Δ_2-condition at 0, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. Let X be a Banach space and p_1, p_2, q_1, q_2 are such that $1/p_1 + 1/p_2 = 1$, $1/q_1 + 1/q_2 = 1$ and $1/p_1 - 1/q_1 > 0$. Then $(l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X))^{\times} = l_{p_2,q_2,\mathcal{N},u,\Delta,A}(X^*)$.*

Proof. One can easily prove it by using Theorem 3.1, so we omit the proof. □

Theorem 3.4. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. Suppose p_1, p_2, q_1, q_2 are real numbers with $1 < p_1, q_1, p_2, q_2 < \infty$ and $1/p_1 + 1/p_2 = 1$, $1/q_1 + 1/q_2 = 1$. Then the dual of $l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X)$ is topologically isomorphic to $l_{p_2,q_2,\mathcal{N},u,\Delta,A}(X^*)$, if and only if the sequence $\{f_i\} \in l_{p_2,q_2,\mathcal{N},u,\Delta,A}(X^*)$ is identified with the linear functional F given by*

$$F(\{x_i\}) = \sum_{i\geqslant 1}\langle x_i, f_i\rangle, \quad \textit{for each } \{x_i\} \in l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X). \tag{3.1}$$

Proof. Subsequently for $\{f_i\} \in l_{p_2,q_2,\mathcal{N},u,\Delta,A}(X^*)$, we define a linear functional F on $l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X)$ as in (+) where convergence of the series is being guaranteed by Proposition 3.3. For $k \in \mathbb{N}$, let

$$F_k(\{x_i\}) = \sum_{i=1}^{k}\langle x_i, f_i\rangle, \quad \{x_i\} \in l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X).$$

Obviously, $\{F_k\}$ is a sequence of continuous linear functionals on $l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X)$ converging pointwise to F. Thus F is continuous by Banach-Steinhaus Theorem (see [19]). Hence, $F \in (l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X))^*$. Next, for $\overline{x} \in l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X)$, we get

$$\begin{aligned}|f(\overline{x})| &\leqslant \sum_{i\geqslant 1}|\langle x_i, f_i\rangle|\\ &\leqslant \sum_{i\geqslant 1} t_i(\overline{x})\ t_i(\overline{f})\\ &\leqslant \|\overline{f}\|_{p_2,q_2,\mathcal{N},u,\Delta,A}\sum_{i\geqslant 1}u_i\Big[M_i\Big(|A(\ i^{1/p_1-1/q_1}t_i(\Delta^m\overline{x}))\|\frac{\|(i^{1/p_2-1/q_2}t_i(\Delta^m\overline{f}))\|}{\|\overline{f}\|_{p_2,q_2,\mathcal{N},u,\Delta,A}}\Big)\Big]\\ &\leqslant \|\overline{f}\|_{p_2,q_2,\mathcal{N},u,\Delta,A}\ \|u_iA\{i^{1/p_1-1/q_1}t_i(\Delta^m\overline{x})\}\|_{(\mathcal{M})},\end{aligned}$$

since $\sum_{i\geqslant 1} u_i\Big[N_i\Big(\frac{\|A\ i^{1/p_2-1/q_2}t_i(\Delta^m\overline{f})\|}{\|\overline{f}\|_{p_2,q_2,\mathcal{N},u,\Delta,A}}\Big)\Big]\leqslant 1$. Therefore,

$$|F(\overline{x})| \leqslant 2\|\overline{f}\|_{p_2,q_2,\mathcal{N},u,\Delta,A}\ \|\overline{x}\|_{p_1,q_1,\mathcal{M},u,\Delta,A},$$

for any $\overline{x}\in l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X)$. Thus,

$$\|f\| \leqslant 2\|\{f_i\}\|_{p_2,q_2,\mathcal{N},u,\Delta,A}. \tag{3.2}$$

Conversely, suppose $F\in(l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X))^*$. Define $f_i\in X^*, i\in\mathbb{N}$ as $f_i(x)=F(\delta_i^x)$. Now to prove $\{f_i\}\in l_{p_2,q_2,\mathcal{N},u,\Delta,A}(X^*)$ we choose $\{\alpha_i\}\in l_{p_1,q_1,\mathcal{M},u,\Delta,A}$. Take $\{x_i\}\subseteq X$ with $\|x_i\|=1$ and $\|f_i\|<f_i(x_i)+1/2^i$, for all $i\in\mathbb{N}$. Let $\{\beta_i\}\subset\mathbb{C}$ be such that $|f_i(\alpha_i x_i)|=f_i(\alpha_i\beta_i x_i)$, for all $i\in\mathbb{N}$. Obviously, $|\beta_i|=1$, for all $i\in\mathbb{N}$, and so $\{\alpha_i\beta_i x_i\}\in l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X)$. Suppose

$$\begin{aligned}\sum_{i\geqslant 1}|\alpha_i|\|f_i\| &< \sum_{i\geqslant 1} f_i(\alpha_i\beta_i x_i)+\sum_{i\geqslant 1}\frac{\alpha_i}{2^i}\\ &= \sum_{i\geqslant 1}F(\delta^{\alpha_i\beta_i x_i})+K\\ &= \lim_{k\to\infty}\sum_{i=1}^{k}F(\delta^{\alpha_i\beta_i x_i})+K\\ &= F(\{\delta^{\alpha_i\beta_i x_i}\})+K,\end{aligned}$$

where $K=\sum_{i\geqslant 1}\frac{\alpha_i}{2^i}$. Therefore, $\sum_{i\geqslant 1}|\alpha_i|\|f_i\|<\infty$, for all $\{\alpha_i\}\in l_{p_1,q_1,\mathcal{M},u,\Delta,A}$ and hence $\{f_i\}\in l_{p_2,q_2,\mathcal{N},u,\Delta,A}(X^*)$ by Theorem 3.1. To prove that F has the form as given in equation (3.1), suppose for $\{x_i\}\in l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X)$

$$\begin{aligned}\sum_{i\geqslant 1}|\langle x_i, f_i\rangle| &= \sum_{i\geqslant 1}|F(\delta^{x_i})|\\ &= \lim_{k\to\infty}\sum_{i=1}^{k}F(\delta^{\beta_i x_i}) = F(\{\beta_i x_i\}),\end{aligned}$$

where β_i are taken as above. Thus $\sum_{i\geqslant 1}|\langle x_i, f_i\rangle|<\infty$. Hence,

$$\sum_{i\geqslant 1}|\langle x_i, f_i\rangle| \quad \text{is unconditionally convergent.} \tag{3.3}$$

Now if $0<p_1<q_1\leqslant\infty$, $l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X)$ is a GAD-space. We write $t_i(\overline{x})=\|x_{\phi}(i)\|$, for some $\phi\in\pi$ and $\overline{u}_k=\sum_{i=1}^{k}\delta_{\phi_{(i)}}^{x_{\phi_{(i)}}}$, for $k\in\mathbb{N}$. Therefore $\overline{u}_k\in\phi(X)$ and $\|\overline{x}-\overline{u}_k\|_{p_1,q_1,\mathcal{M},u,\Delta,A}\to 0$ as $k\to\infty$ by Proposition 3.3. Then,

$$F(\{x_i\}) = F(\lim_{k\to\infty}\overline{u}_k) = \sum_{i\geqslant 1}\langle x_i, f_i\rangle,$$

by (3.3). Hence the mapping $R : l_{p_2,q_2,\mathcal{N},u,\Delta,A}(X^*) \to (l_{p_1,q_1,\mathcal{M},u,\Delta,A}(X))^*$ defined by $R(\bar{f}) = F$, with $\bar{f} = \{f_i\}$, $f_{i(x)} = F(\delta_i^x)$, $i \in \mathbb{N}$ is a topological isomorphism from equations (3.1), (3.2) and the open mapping theorem (see [19]). This completes the proof. □

4. The operator ideals $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$, $0 < p, q \leq \infty$

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and X, Y are Banach spaces.

Definition 4.1. Let $T : X \to Y$ be a bounded linear operator. Then T is said to be of type $l_{p,q,\mathcal{M},u,\Delta,A}$, if $\{s_k(T)\} \in l_{p,q,\mathcal{M},u,\Delta,A}$. We shall denote the set of all above mappings by $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$ where

$$L^{(s)}_{p,q,\mathcal{M},u,\Delta,A} = \{T \in L : \{s_k(T)\} \in l_{p,q,\mathcal{M},u,\Delta,A}\}.$$

We define the norm for any $T \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$ as

$$\|T\|_{p,q,\mathcal{M},u,\Delta,A} = \inf\Big\{\rho > 0 : \sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T)\|}{\rho}\Big)\Big] \leqslant 1\Big\}.$$

Theorem 4.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix. Then for $p < q$, $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$ equipped with $\|.\|_{p,q,\mathcal{M},u,\Delta,A}$ is a quasi-Banach operator ideal and for $p \geqslant q$ it is a Banach ideal.*

Proof. To show that $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$ is an operator ideal, firstly note that all finite rank operators are contained in $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$, since $s_k(T) = 0$ for $k \geqslant k_0$, if rank $T < k_0$. For $T_1, T_2 \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}(X, Y)$, we have

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_1)\|}{\rho_1}\Big)\Big] < \infty,$$

and

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_2)\|}{\rho_2}\Big)\Big] < \infty,$$

for some $\rho_1, \rho_2 > 0$. Firstly, we consider the condition when $\frac{1}{p} - \frac{1}{q} \geqslant 0$

$$\begin{aligned}\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_1+T_2)\|}{2^{\frac{1}{p}-\frac{1}{q}+1}(\rho_1+\rho_2)}\Big)\Big] &\leqslant \sum_{k\geqslant 1} \frac{\rho_1}{\rho_1+\rho_2} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_1)\|}{\rho_1}\Big)\Big] \\ &\quad + \sum_{k\geqslant 1} \frac{\rho_2}{\rho_1+\rho_2} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_2)\|}{\rho_2}\Big)\Big] \\ &< \infty.\end{aligned}$$

Again, if $\frac{1}{p} - \frac{1}{q} < 0$, then

$$\begin{aligned}\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_1+T_2)\|}{(\rho_1+\rho_2)}\Big)\Big] &\leqslant \sum_{k\geqslant 1} \frac{\rho_1}{\rho_1+\rho_2} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_1)\|}{\rho_1}\Big)\Big] \\ &\quad + \sum_{k\geqslant 1} \frac{\rho_2}{\rho_1+\rho_2} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_2)\|}{\rho_2}\Big)\Big] \\ &< \infty.\end{aligned}$$

This implies that $T_1 + T_2 \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}(X, Y)$. Now, we want to show that for $T \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}(E, F)$,

$R \in L(F, Y)$ and $S \in L(X, E)$, $RTS \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}(X, Y)$. Thus for $T \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}(E, F)$, we have

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T)\|}{\rho_0}\Big)\Big] < \infty,$$

for some $\rho_0 > 0$ and hence

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(RTS)\|}{\|R\|\|S\|\rho_0}\Big)\Big] < \infty,$$

by the property (iii) of s-number function. Thus $RTS \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}(X, Y)$. Therefore, $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$ is an operator ideal.

The proof of the function $\|.\|_{p,q,\mathcal{M},u,\Delta,A}$ is a quasi-norm (or, norm) defined on $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$ is similar to one defined on $L_{p,q,\mathcal{M},u,\Delta,A}(X)$ and so excluded. To prove the completeness, suppose $\{T_k\}$ is a Cauchy sequence in component of $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}(X, Y)$ of $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$. Thus for $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\|T_{k+j} - T_k\|_{p,q,\mathcal{M},u,\Delta,A} < \varepsilon, \quad \text{for all } k \geqslant k_0 \text{ and } j \in \mathbb{N}.$$

This implies that there exists $\rho > 0$ such that $\rho < \varepsilon$ and

$$\sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\, n^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_n(T_{k+j} - T_k)\|}{\varepsilon}\Big)\Big] \leqslant 1, \text{ for all } k \geqslant k_0,\ j \in \mathbb{N}. \tag{4.1}$$

Thus,

$$\Big\{\frac{A\, n^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_n(T_{k+j} - T_k)}{\varepsilon}; n \geqslant 1\Big\},$$

is a bounded sequence for each $k \geqslant k_0$ and $j \in \mathbb{N}$. Therefore for some constant $K > 0$, we get

$$\|T_{k+j} - T_k\| < \varepsilon K, \quad \text{for all } k \geqslant k_0,\ j \in \mathbb{N}.$$

Thus, $\{T_k\}$ is a Cauchy sequence in $L(X, Y)$. So there exists a $T \in L(X, Y)$ such that $\|T_k - T\| \to 0$ as $k \to \infty$. As $s_n(T_k - T) \leqslant \|T_k - T\|$, for all $k \geqslant 1$, we have $s_n(T_k - T) \to 0$ as $k \to \infty$. Also,

$$|s_n(T_{k+j} - T_k) - s_n(T - T_k)| \leqslant \|T_{k+j} - T_k\|,$$

which implies

$$s_n(T_{k+j} - T_k) \to s_n(T - T_k) \text{ as } j \to \infty.$$

Now, we have from (4.1),

$$\sum_{n\geqslant 1} u_n\Big[M_n\Big(\frac{\|A\, n^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_n(T - T_k)\|}{\varepsilon}\Big)\Big] \leqslant 1, \quad \text{for all } k \geqslant k_0.$$

This implies that $T - T_k \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}(X, Y)$ and $\|T - T_k\|_{p,q,\mathcal{M},u,\Delta,A} < \varepsilon$, for all $k \geqslant k_0$. Therefore, $T \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$ and $T_k \to T \in L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$, which shows that $L^{(s)}_{p,q,\mathcal{M},u,\Delta,A}$ is a quasi-Banach operator ideal. □

Theorem 4.3. *Let $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ be two complementary Musielak-Orlicz functions, $u = (u_k)$ be a sequence of strictly positive real numbers, $A = (a_{nk})$ be a nonnegative two-dimensional bounded-regular matrix and s is a multiplicative s-number function. If $0 < p_1, p_2, p, q_1, q_2, q < \infty$ are such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, then*

$$L^{(s)}_{p_1,q_1,\mathcal{M},u,\Delta,A} \circ L^{(s)}_{p_2,q_2,\mathcal{N},u,\Delta,A} \subset L^{(s)}_{p,q,1,u,\Delta,A},$$

where

$$L^{(s)}_{p,q,1,u,\Delta,A} = \Big\{T \in L : \sum_{k\geqslant 1} u_k\Big[\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k T\|\Big] \in l_1\Big\}.$$

Proof. Suppose $T \in L^{(s)}_{p_1,q_1,\mathcal{M},u,\Delta,A} \circ L^{(s)}_{p_2,q_2,\mathcal{N},u,\Delta,A}(X,Y)$. Then $T = T_1T_2$, where $T_1 \in L^{(s)}_{p_1,q_1,\mathcal{M},u,\Delta,A}(Z,Y)$ and $T_2 \in L^{(s)}_{p_2,q_2,\mathcal{N},u,\Delta,A}(X,Z)$ and here Z is a Banach space. Thus

$$\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p_1}-\frac{1}{q_1}}\Delta^m s_k(T_1)\|}{\rho_1}\Big)\Big] < \infty, \quad \text{for some } \rho_1 > 0,$$

$$\sum_{k\geqslant 1} u_k\Big[N_k\Big(\frac{\|A\, k^{\frac{1}{p_2}-\frac{1}{q_2}}\Delta^m s_k(T_2)\|}{\rho_2}\Big)\Big] < \infty, \quad \text{for some } \rho_2 > 0.$$

If $\frac{1}{p}-\frac{1}{q} \geqslant 0$, we have

$$\begin{aligned}&\sum_{k\geqslant 1} u_k\Big[\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_1T_2)\|}{2^{\frac{1}{p}-\frac{1}{q}}\rho_1\rho_2}\Big]\\ &\quad\leqslant 2\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p_1}-\frac{1}{q_1}}\Delta^m s_k(T_1)\|}{\rho_1}\Big)\Big] + \sum_{k\geqslant 1} u_k\Big[N_k\Big(\frac{\|A\, k^{\frac{1}{p_2}-\frac{1}{q_2}}\Delta^m s_k(T_2)\|}{\rho_2}\Big)\Big] < \infty,\end{aligned}$$

and for $\frac{1}{p}-\frac{1}{q} < 0$, we have

$$\begin{aligned}\sum_{k\geqslant 1} u_k\Big[\frac{\|A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_1T_2)\|}{\rho_1\rho_2}\Big] &\leqslant 2\sum_{k\geqslant 1} u_k\Big[M_k\Big(\frac{\|A\, k^{\frac{1}{p_1}-\frac{1}{q_1}}\Delta^m s_k(T_1)\|}{\rho_1}\Big)\Big]\\ &\quad + \sum_{k\geqslant 1} u_k\Big[N_k\Big(\frac{\|A\, k^{\frac{1}{p_2}-\frac{1}{q_2}}\Delta^m s_k(T_2)\|}{\rho_2}\Big)\Big]\\ &< \infty.\end{aligned}$$

This implies that $\{A\, k^{\frac{1}{p}-\frac{1}{q}}\Delta^m s_k(T_1T_2)\} \in l_1$ or $T_1T_2 \in L^{(s)}_{p,q,1,u,\Delta,A}$. This completes the proof. □

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References

[1] L. R. Acharya, *Linear operators and approximation quantities*, Dissertation, I.I.T. Kanpur, India, (2008). 1, 1
[2] A. Alotaibi, K. Raj, S. A. Mohiuddine, *Some generalized difference sequence spaces defined by a sequence of moduli in n-normed spaces*, J. Funct. Spaces, **2015** (2015), 8 pages. 1
[3] N. De Grande-De Kimpe, *Generalized sequence spaces*, Bull. Soc. Math. Belg., **23** (1971), 123–166. 1, 1
[4] M. Et, R. Çolak, *On some generalized difference sequence spaces*, Soochow J. Math., **21** (1995), 377–386. 1
[5] P. Foralewski, H. Hudzik, L. Szymaszkiewicz, *On some geometric and topological properties of generalized Orlicz-Lorentz sequence spaces*, Math. Nachr., **281** (2008), 181–198. 1
[6] A. Grothendieck, *Sur une notion de produit tensoriel topologique d'espaces vectoriels topologiques, et une classe remarquable d'espaces vectoriels liée á cette notion*, (French) C. R. Acad. Sci. Paris, **233** (1951), 1556–1558. 1

[7] M. Gupta, A. Bhar, *Generalized Orlicz-Lorentz sequence spaces and corresponding operator ideals*, Math. Slovaca, **64** (2014), 1475–1496. 2

[8] P. K. Kamthan, M. Gupta, *Sequence spaces and series*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, (1981). 1, 1

[9] M. Kato, *On Lorentz spaces* $l_{p,q}\{E\}$, Hiroshima Math. J., **6** (1976), 73–93. 1

[10] H. Kızmaz, *On certain sequence spaces*, Canad. Math. Bull., **24** (1981), 169–176. 1

[11] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces*, I, Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin-New York, (1977). 1

[12] S. A. Mohiuddine, K. Raj, A. Alotaibi, *Generalized spaces of double sequences for Orlicz functions and bounded-regular matrices over* n*-normed spaces*, J. Inequal. Appl., **2014** (2014), 16 pages. 1

[13] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, (1983). 1

[14] J. Patterson, *Generalized sequence spaces and matrix transformations*, PhD diss., Dissertation, I.I.T. Kanpur, India, (1980). 1, 1.1

[15] A. Pietsch, *s-numbers of operators in Banach spaces*, Studia Math., **51** (1974), 201–223. 1

[16] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, (1987). 1, 1

[17] K. Raj, S. Pandoh, *Some vector-valued statistical convergent sequence spaces*, Malaya J. Mat., **3** (2015), 161–167.

[18] K. Raj, S. K. Sharma, *Some vector-valued sequence spaces defined by a Musielak-Orlicz function*, Rev. Roumaine Math. Pures Appl., **57** (2012), 383–399. 1

[19] W. Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Dsseldorf-Johannesburg, (1976). 3, 3

[20] E. Savaş, M. Mursaleen, *Matrix transformations in some sequence spaces*, İstanbul Üniv. Fen Fak. Mat. Derg., **52** (1993), 1–5. 1

[21] Y. Yılmaz, M. K. Özdemir, İ. Solak, M. Candan, *Operators on some vector valued Orlicz sequence spaces*, F. Ü. Fen ve Mühendislik Bilimleri Dergisi, **17** (2005), 59–71. 1