Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 263–277
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

A modified iterative algorithm for finding a common element in Hilbert space

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Communicated by Y. H. Yao

Abstract

In this paper, a modified iterative algorithm for finding a common element of the solutions of a equilibrium problem, the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem is constructed in Hilbert spaces, and the strong convergence of the generated iterative sequence to the common element is proved under some mild conditions. The main result proposed in this paper extends and improves some recent results in the literature.

Keywords: Common element, iterative algorithm, Hilbert space, strong convergence.
2010 MSC: 47H10, 55M20.

1. Introduction

Let H be a Hilbert space, and C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathfrak{R}$, where $\mathfrak{R}$ is the set of real numbers.

The equilibrium problem $F : C \times C \to \mathfrak{R}$ is to find an element $x \in C$ such that $F(x, y) \geqslant 0$ for all $y \in C$, and the set of such solutions is denoted by $EP(F)$.

Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leqslant \alpha\|x - y\|$ for all $x, y \in C$.

A mapping $S : C \to C$ is called nonexpansive if for all $x, y \in C$, $\|Sx - Sy\| \leqslant \|x - y\|$, and the set of fixed points of S is denoted by $\mathrm{Fix}(S)$. It is well-known that if C is bounded closed convex and $S : C \to C$ is nonexpansive, then $\mathrm{Fix}(S) \neq \phi$.

In 2007, Takahashi and Takahashi [14] introduced an iterative scheme using the viscosity approximation method in a Hilbert space as follows:

$$\begin{cases} F(y_n, u) + \dfrac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geqslant 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sy_n, \end{cases}$$

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doi:10.22436/jnsa.010.01.25

Received 2016-11-16

and the strong convergence to a common element $q \in \mathrm{Fix}(S) \cap EP(F)$ was obtained under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, where $q = P_{\mathrm{Fix}(S)\cap EP(F)} f(q)$.

Let A be a strongly positive bounded linear operator on H, i.e., there exists a constant $\gamma > 0$ such that

$$\langle Ax, x\rangle \geqslant \gamma \|x\|^2 \quad \text{for all} \quad x \in H.$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where b is a given point in H.

In 2006, Marino and Xu [8] proposed the following iterative algorithm:

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geqslant 0. \tag{1.1}$$

Under some appropriate conditions on parameter $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.1) was proved to converge strongly to the unique solution of the following variational inequality

$$\langle (A - \gamma f)q, x - q\rangle \geqslant 0, \qquad x \in \mathrm{Fix}(S),$$

which is the optimality condition for the minimization problem

$$\min_{x\in \mathrm{Fix}(S)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2007, Plubtieng and Punpaeng [10] introduced and considered the following two iterative schemes for finding a common element of the set of solutions of equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space:

$$\begin{cases} F(y_n, u) + \dfrac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geqslant 0, & \forall u \in H, \\ x_n = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)Sy_n, & \forall n \geqslant 1, \end{cases} \tag{1.2}$$

and

$$\begin{cases} F(y_n, u) + \dfrac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geqslant 0, & \forall u \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)Sy_n, & \forall n \geqslant 1. \end{cases} \tag{1.3}$$

The sequences $\{x_n\}$ generated by (1.2) and (1.3) were proved to converge strongly to the unique solution of the following variational inequality under some appropriate conditions:

$$\langle (A - \gamma f)q, x - q\rangle \geqslant 0, \quad \forall x \in \mathrm{Fix}(S) \cap EP(F),$$

which is the optimality condition for the minimization problem

$$\min_{x\in \mathrm{Fix}(S)\cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf.

For finding a common element of the set of the fixed points of nonexpansive mappings and the set of the solutions to variational inequalities for α-cocoercive map, Takahashi and Toyoda proposed the following iterative process in [15]:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \tag{1.4}$$

for every $n = 0, 1, 2, \cdots$, where A is α-cocoercive, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in (0,1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. If the set $\mathrm{Fix}(S) \cap VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.4) was proved to converge weakly to some $q \in \mathrm{Fix}(S) \cap VI(C, A)$.

In 2005, Iiduka and Takahashi [7] proposed another iterative scheme as follows:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \tag{1.5}$$

for every $n = 0, 1, 2, \cdots$, where A is α-cocoercive, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in (0,1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. And, the sequence $\{x_n\}$ generated by (1.5) was proved to converge strongly to $q \in \mathrm{Fix}(S) \cap VI(C, A)$.

In 2007, Yao and Yao [17] extended (1.5) to the following iterative scheme:

$$\begin{cases} y_n = P_c(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n A y_n), \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in [0,1] and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. And, the sequence $\{x_n\}$ defined by (1.6) was proved to converge strongly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings under some parameters controlling conditions.

In 2012, Piri [9] proposed an iteration method for finding an infinite family of nonexpansive mappings, the set of solutions of systems of equilibrium problems and the set of solutions of systems of variational inequalities for two strongly monotone mappings in a real Hilbert space. In 2014, Bnouhachem [2] proposed a modified projection method for computing a common solution of a system of variational inequalities, a split equilibrium problem, and a hierarchical fixed-point problem in Hilbert space, and proved the strong convergence of the iteration sequences. Since the iterative algorithms played an important role for solving integral and differential equations, optimization problems, image reconstruction problems, game theory and other fields such as [5, 16, 19, 20, 22], the convergence and construction of the iteration algorithm for computing fixed points has attracted more and more attentions see, e.g., [3, 6, 12, 18, 21].

Motivated by the above related results in this field, a new general iterative process is constructed:

$$\begin{cases} F(y_n, u) + \dfrac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geqslant 0, \quad \forall u \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A))SP_C(I - s_n B)y_n, \quad \forall n \geqslant 1, \end{cases} \tag{1.7}$$

where A is a linear bounded operator and B is relaxed cocoercive. The strong convergence on the sequence $\{x_n\}$ generated by (1.7) to a common element of the set of fixed points of a nonexpansive mapping, the set of solution of the variational inequalities for a relaxed cocoercive mapping and the set of solutions of the equilibrium problem will be proved, and the common element also solves another variational inequality:

$$\langle \gamma f(q) - Aq, q - p\rangle \leqslant 0, \quad \forall p \in \mathscr{F},$$

where $\mathscr{F} = \mathrm{Fix}(S) \cap VI(S, C) \cap EP(F)$ and is also the optimality condition for the minimization problem $\min\limits_{x \in \mathscr{F}} \frac{1}{2}\langle Ax, x\rangle - h(x)$, where h is a potential function for γf.

2. Preliminaries

Let H be a Hilbert space, whose inner product and norm are denoted by $\langle \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and let $A : C \to H$ be a nonlinear map. Let P_C be the projection of H onto the convex subset C. The classical variational inequality which is denoted by $VI(A, C)$ is used to find $u \in C$ such that

$$\langle Au, v - u\rangle \geqslant 0,$$

for all $v \in C$. For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geqslant 0, \quad \forall v \in C,$$

if and only if $u = P_c z$. It is known that projection operator P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_c(x) \Leftrightarrow \langle x - P_c x, P_c x - y \rangle \geqslant 0, \quad \forall y \in C.$$

It is also known that P_C satisfies

$$\langle x - y, P_c x - P_c y \rangle \geqslant \|P_c x - P_c y\|^2 \text{ for all } \quad x, y \in H.$$

Recall that

(1) B is called ν-strongly monotone, if for each $x, y \in C$, we have

$$\langle Bx - By, x - y \rangle \geqslant \nu \|x - y\|^2,$$

for a constant $\nu > 0$. This implies that

$$\|Bx - By\| \geqslant \nu \|x - y\|,$$

that is, B is ν-expansive and when $\nu = 1$, it is expansive.

(2) B is called μ-cocoercive [15], if for each $x, y \in C$, we have $\langle Bx - By, x - y \rangle \geqslant \mu \|Bx - By\|^2$, for a constant $\mu > 0$. Clearly, every μ-cocoercive map B is $1/\mu$-Lipschitz continuous.

(3) B is called $-\mu$-cocoercive, if there exists a constant $\mu > 0$ such that

$$\langle Bx - By, x - y \rangle \geqslant (-\mu)\|Bx - By\|^2, \quad \forall x, y \in C.$$

(4) B is said to be relaxed (μ, ν)-cocoercive, if there exist two constants $\mu, \nu > 0$ such that

$$\langle Bx - By, x - y \rangle \geqslant (-\mu)\|Bx - By\|^2 + \nu\|x - y\|^2, \quad \forall x, y \in C.$$

For $\mu = 0$, B is ν-strongly monotone. This class of maps is more general than the class of strongly monotone maps. We can have the following implication: ν-strongly monotone $\Rightarrow$ relaxed (μ, ν)-cocoercivity.

(5) A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geqslant 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g \rangle \geqslant 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w \rangle \geqslant 0, \forall u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_c v, & v \in C, \\ \phi, & v \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(B, C)$; the relative content can be found in [11].

In this paper, for solving the equilibrium problems for a bifunction $F : C \times C \to R$, the following assumptions on F will be used:

(C1) $F(x, x) = 0$ for all $x \in C$;

(C2) F is monotone, i.e., $F(x,y)+F(y,x) \leqslant 0$ for all $x,y \in C$;
(C3) For each $x,y,z \in C$, $\lim_{t\downarrow 0} F(tz+(1-t)x,y) \leqslant F(x,y)$;
(C4) For each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

If an equilibrium bifunction $F : C \times C \to R$ satisfies conditions (C1)-(C4), then we have the following two important results.

Lemma 2.1 ([1]). *Let C be a nonempty closed convex subset of H and let F be an equilibrium bifunction of $C \times C$ into R satisfying conditions (C1)-(C4). Let $r > 0$ and $x \in C$. Then, there exists $y \in C$ such that*

$$F(y,z)+\frac{1}{r}\langle z-y, y-x\rangle \geqslant 0, \quad \textit{for all} \;\; z \in C.$$

Lemma 2.2 ([4]). *Assume that F satisfies the same assumptions as Lemma 2.1. For $r > 0$ and $x \in C$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{y \in C : F(y,z)+\frac{1}{r}\langle z-y, y-x\rangle \geqslant 0, \forall z \in C\},$$

for all $y \in H$. Then, the following items hold:

(1) T_r *is single-valued;*
(2) T_r *is firmly nonexpansive, i.e., for any $x,y \in H$, $\|T_r x - T_r y\|^2 \leqslant \langle T_r x - T_r y, x-y\rangle$;*
(3) $\mathrm{Fix}(T_r) = EP(F)$*;*
(4) $EP(F)$ *is closed and convex.*

We also need the following lemmas for proving our main results.

Lemma 2.3 ([13]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geqslant 0$, and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leqslant 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.4 ([8]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F : C \times C \to \mathfrak{R}$ be a bifunction which satisfies (C1)-(C4), S be a nonexpansive mapping of C into H and B be a λ-Lipschitzian, relaxed (μ,ν)-cocoercive map of C into H such that $\mathscr{F} = \mathrm{Fix}(S) \bigcap EP(F) \bigcap VI(B,C) \neq \phi$. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$, and assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let f be a contraction of H into itself with a coefficient $\alpha \in (0,1)$, $\{x_n\}$ and $\{y_n\}$ be sequences generated by (1.7) with $x_1 \in H$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\},\{s_n\} \subset [0,\infty)$ satisfy*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$, *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$.

(iv) $\{s_n\} \subset [a, b]$ *for some* a, b *with* $0 \leqslant a \leqslant b \leqslant \frac{2(\nu-\mu\lambda^2)}{\lambda^2}$.

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in \mathscr{F}$, *where* $q = P_{\mathscr{F}}(\gamma f + (I - A))(q)$ *is a unique solution of the following variational inequality*

$$\langle \gamma f(q) - Aq, p - q\rangle \leqslant 0, \quad \forall p \in \mathscr{F},$$

which is the optimality condition for the minimization problem

$$\min_{x\in\mathscr{F}} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h *is a potential function for* γf.

Proof. For the control conditions (i) and (ii), we may assume, without loss of generality, that $\alpha_n \leqslant (1 - \beta_n)\|A\|^{-1}$. Since A is linear bounded self-adjoint operator on H, then

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\},$$

Observe that

$$\langle((1 - \beta_n)I - \alpha_n A)u, u\rangle = 1 - \beta_n - \alpha_n\langle Au, u\rangle \geqslant 1 - \beta_n - \alpha_n\|A\| \geqslant 0,$$

that is to say $(1 - \beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1 - \beta_n)I - \alpha_n A\| &= \sup\{\langle((1 - \beta_n)I - \alpha_n A)u, u\rangle : u \in H, \|u\| = 1\}\\ &= \sup\{1 - \beta_n - \alpha_n\langle Au, u\rangle : u \in H, \|u\| = 1\}\\ &\leqslant 1 - \beta_n - \alpha_n\overline{\gamma}.\end{aligned}$$

Step 1. We show that $I - s_n B$ is nonexpansive. Indeed, from the relaxed (μ, ν)-cocoercive and λ-Lipschitzian definition on B and condition (iv), we have

$$\begin{aligned}\|(I - s_n B)x - (I - s_n B)y\|^2 &= \|(x - y) - s_n(Bx - By)\|^2\\ &= \|x - y\|^2 - 2s_n\langle x - y, Bx - By\rangle + s_n^2|Bx - B\|^2\\ &\leqslant \|x - y\|^2 - 2s_n[-\mu\|Bx - By\|^2 + \nu\|x - y\|^2] + s_n^2\|Bx - By\|^2\\ &\leqslant \|x - y\|^2 + 2s_n\lambda^2\mu\|x - y\|^2 - 2s_n\nu\|x - y\|^2 + \lambda^2 s_n^2\|x - y\|^2\\ &= (1 + 2s_n\lambda^2\mu - 2s_n\nu + \lambda^2 s_n^2)\|x - y\|^2\\ &\leqslant \|x - y\|^2,\end{aligned}$$

which implies that the mapping $I - s_n B$ is nonexpansive.

Step 2. We show $\{x_n\}$ is bounded. Picking $p \in \mathscr{F}$, by the definition of T_r, and noting that $y_n = T_{r_n}x_n$, we have that

$$\|y_n - p\| = \|T_{r_n\|x_n - T_{r_n}p} \leqslant \|x_n - p\|.$$

Set $\rho_n = P_C(I - s_n B)y_n$, since $p \in VI(B, C)$, we have $p = P_C(I - s_n B)p$. Therefore, we obtain

$$\begin{aligned}\|\rho_n - p\| &= \|P_C(I - s_n B)y_n - P_C(I - s_n B)p\|\\ &\leqslant \|(I - s_n B)y_n - (I - s_n B)p\| \qquad (3.1)\\ &\leqslant \|y_n - p\| \leqslant \|x_n - p\|.\end{aligned}$$

Using (3.1), we have

$$\begin{aligned} \|x_{n+1}-p\| &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n A)(SP_C(I-s_nB)y_n-p)\| \\ &\leqslant (1-\beta_n-\alpha_n\bar{\gamma})\|(I-s_nB)y_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Ap\| \\ &\leqslant (1-\beta_n-\alpha_n\bar{\gamma})\|y_n-p\|+\beta_n\|x_n-p\|+\alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Ap\| \\ &\leqslant (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\| \\ &\leqslant (1-(\bar{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|. \end{aligned} \tag{3.2}$$

It follows from (3.2) and induction, we can get

$$\|x_n-p\| \leqslant \max\{\|x_0-p\|, \frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\}, \quad n \geqslant 0.$$

Therefore, $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{f(x_n)\}$, and $\{\rho_n\}$ are all bounded.

Step 3. We show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Observing that $y_n=T_{r_n}x_n$ and $y_{n+1}=T_{r_{n+1}}x_{n+1}$, we have

$$F(y_n,u)+\frac{1}{r_n}\langle u-y_n, y_n-x_n\rangle \geqslant 0 \ \text{ for all } \ u\in C. \tag{3.3}$$

and

$$F(y_{n+1},u)+\frac{1}{r_{n+1}}\langle u-y_{n+1}, y_{n+1}-x_{n+1}\rangle \geqslant 0 \ \text{ for all } \ u\in C. \tag{3.4}$$

Putting $u=y_{n+1}$ in (3.3) and $u=y_n$ in (3.4), we get

$$F(y_n,y_{n+1})+\frac{1}{r_n}\langle y_{n+1}-y_n, y_n-x_n\rangle \geqslant 0$$

and

$$F(y_{n+1},y_n)+\frac{1}{r_{n+1}}\langle y_n-y_{n+1}, y_{n+1}-x_{n+1}\rangle \geqslant 0.$$

It follows from (C2) that

$$\langle y_{n+1}-y_n, \frac{y_n-x_n}{r_n}-\frac{y_{n+1}-x_{n+1}}{r_{n+1}}\rangle \geqslant 0.$$

So we can get,

$$\langle y_{n+1}-y_n, y_n-y_{n+1}+y_{n+1}-x_n-\frac{r_n}{r_{n+1}}(y_{n+1}-x_{n+1})\rangle \geqslant 0.$$

Without loss of generality, we assume that there exists a real number m such that $r_n>m>0$ for all n, it follows that

$$\|y_{n+1}-y_n\|^2 \leqslant \|y_{n+1}-y_n\|(\|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|y_{n+1}-x_{n+1}\|)$$

i.e.,

$$\begin{aligned} \|y_{n+1}-y_n\| &\leqslant \|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|y_{n+1}-x_{n+1}\| \\ &\leqslant \|x_{n+1}-x_n\|+\frac{M}{m}|r_{n+1}-r_n|, \end{aligned} \tag{3.5}$$

where M is an appropriate constant such that $M \geqslant \sup_{n\geqslant 1}\|y_n-x_n\|$. Note that

$$\begin{aligned} \|\rho_{n+1}-\rho_n\| &= \|P_C(I-s_{n+1}B)y_{n+1}-P_C(I-s_nB)y_n\| \\ &\leqslant \|(I-s_{n+1}B)y_{n+1}-(I-s_nB)y_n\| \\ &= \|(I-s_{n+1}B)y_{n+1}-(I-s_{n+1}B)y_n+(s_n-s_{n+1})By_n\| \\ &\leqslant \|y_{n+1}-y_n\|+|s_n-s_{n+1}|\|By_n\|. \end{aligned} \tag{3.6}$$

Substituting (3.5) into (3.6) yields that

$$\|\rho_{n+1}-\rho_n\| \leqslant \|x_{n+1}-x_n\| + M_1(|r_{n+1}-r_n| + |s_n - s_{n+1}|), \tag{3.7}$$

where M_1 is an appropriate constant such that $M_1 = \max\{\sup_{n\geqslant 1} \|By_n\|, \frac{M}{m}\}$.

We set $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ for all $n \geqslant 0$. From the definition of z_n, we obtain

$$\begin{aligned} z_{n+1}-z_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)S\rho_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)S\rho_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - AS\rho_{n+1}) + \frac{\alpha_n}{1-\beta_n}(\gamma f(x_n) - AS\rho_n) + S\rho_{n+1} - S\rho_n. \end{aligned}$$

By using (3.7), it follows that

$$\begin{aligned} \|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| &\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AS\rho_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\| + \|AS\rho_n\|) \\ &\quad + \|\rho_{n+1}-\rho_n\| - \|x_{n+1}-x_n\| \\ &\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AS\rho_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\| + \|AS\rho_n\|) \\ &\quad + \|x_{n+1}-x_n\| + M_1(|r_{n+1}-r_n| + |s_{n+1}-s_n|) - \|x_{n+1}-x_n\| \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AS\rho_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\| + \|AS\rho_n\|) \\ &\quad + M_1(|r_{n+1}-r_n| + |s_{n+1}-s_n|). \end{aligned}$$

From the conditions (i) and (iii), the last inequality implies that

$$\lim_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leqslant 0.$$

Hence by Lemma 2.3, we have $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.8}$$

Step 4. We show that $\|x_n - y_n\| \to 0$. Since $x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)S\rho_n$, we have

$$\|x_n - S\rho_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - S\rho_n\| \leqslant \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - AS\rho_n\| + \beta_n\|x_n - S\rho_n\|.$$

So, we get

$$\|x_n - S\rho_n\| \leqslant \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - AS\rho_n\|.$$

By condition (i) and using (3.8), we obtain

$$\lim_{n\to\infty}\|x_n - S\rho_n\| = 0. \tag{3.9}$$

For $p \in \mathscr{F}$, note that T_r is firmly nonexpansive, then we have

$$\begin{aligned} \|y_n - p\|^2 = \|T_{r_n}x_n - T_{r_n}p\|^2 \leqslant \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle &= \langle y_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2), \end{aligned}$$

and hence $\|y_n - p\|^2 \leqslant \|x_n - p\|^2 - \|x_n - p\|^2$.

Therefore, we have

$$\begin{aligned}
\mathrm{lcl}\|x_{n+1}-p\|^2 &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-S\rho_n)+(I-\alpha_n A)(S\rho_n-p)\|^2\\
&\leqslant \|(I-\alpha_n A)(S\rho_n-p)+\beta_n(x_n-S\rho_n)\|^2+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&\leqslant [\|(I-\alpha_n A)(S\rho_n-p)\|+\|\beta_n(x_n-S\rho_n)\|]^2+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&\leqslant [(1-\alpha_n\overline{\gamma})\|\rho_n-p\|+\beta_n\|x_n-S\rho_n\|]^2+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&= (1-\alpha_n\overline{\gamma})^2\|y_n-p\|^2+\beta_n^2\|x_n-S\rho_n\|^2+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-S\rho_n\|\\
&\quad+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&\leqslant (1-\alpha_n\overline{\gamma})^2(\|x_n-p\|^2-\|x_n-y_n\|^2)+\beta_n^2\|x_n-S\rho_n\|^2\\
&\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-S\rho_n\|+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&= (1-2\alpha_n\overline{\gamma}+(\alpha_n\overline{\gamma})^2)\|x_n-p\|^2-(1-\alpha_n\overline{\gamma})^2\|x_n-y_n\|^2+\beta_n^2\|x_n-S\rho_n\|^2\\
&\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-S\rho_n\|+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&\leqslant \|x_n-p\|^2+\alpha_n\overline{\gamma}^2\|x_n-p\|^2-(1-\alpha_n\overline{\gamma})^2\|x_n-y_n\|^2+\beta_n^2\|x_n-S\rho_n\|^2\\
&\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-S\rho_n\|+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|.
\end{aligned}$$

Then, we have

$$\begin{aligned}
(1-\alpha_n\overline{\gamma})^2\|x_n-y_n\|^2 &\leqslant \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\overline{\gamma}^2\|x_n-p\|^2+\beta_n^2\|x_n-S\rho_n\|^2\\
&\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-S\rho_n\|+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&\leqslant (\|x_n-p\|+\|x_{n+1}-p\|)\times\|x_{n+1}-x_n\|+\alpha_n\overline{\gamma}^2\|x_n-p\|^2\\
&\quad+\beta_n^2\|x_n-S\rho_n\|^2+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&\quad+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-S\rho_n\|.
\end{aligned}\tag{3.10}$$

From (3.8)-(3.10) and condition (i), we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0.$$

Step 5. We will show that $P_{\mathscr{F}}(\gamma f+(I-A))$ has a unique fixed point. For $p\in\mathscr{F}$, we have

$$\begin{aligned}
\|\rho_n-p\|^2 &= \|P_C(I-s_nB)y_n-P_C(I-s_nB)p\|^2\\
&\leqslant \|(y_n-p)-s_n(By_n-Bp)\|^2\\
&= \|y_n-p\|^2-2s_n\langle y_n-p,By_n-Bp\rangle+s_n^2\|By_n-Bp\|^2\\
&\leqslant \|x_n-p\|^2-2s_n[-\mu\|By_n-Bp\|^2+\nu\|y_n-p\|^2]+s_n^2\|By_n-Bp\|^2\\
&\leqslant \|x_n-p\|^2+2s_n\mu\|By_n-Bp\|^2-2s_n\nu\|y_n-p\|^2+s_n^2\|By_n-Bp\|^2\\
&\leqslant \|x_n-p\|^2+(2s_n\mu+s_n^2-\frac{2s_n\nu}{\lambda^2})\|By_n-Bp\|^2.
\end{aligned}\tag{3.11}$$

Observe that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-S\rho_n)+(I-\alpha_n A)(S\rho_n-p)\|^2\\
&\leqslant \|(I-\alpha_n A)(S\rho_n-p)+\beta_n(x_n-S\rho_n)\|^2+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&\leqslant [\|(I-\alpha_n A)(S\rho_n-p)\|+\|\beta_n(x_n-S\rho_n)\|]^2+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&\leqslant [(1-\alpha_n\overline{\gamma})\|\rho_n-p\|+\beta_n\|x_n-S\rho_n\|]^2+2\alpha_n|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&= (1-\alpha_n\overline{\gamma})^2\|\rho_n-p\|^2+\beta_n^2\|x_n-S\rho_n\|^2+2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n-p\|\|x_n-S\rho_n\|\\
&\quad+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|\\
&\leqslant \|\rho_n-p\|^2+\beta_n^2\|x_n-S\rho_n\|^2+2\beta_n\|\rho_n-p\|\|x_n-S\rho_n\|\\
&\quad+2\alpha_n\|\gamma f(x_n)-Ap\|\|x_{n+1}-p\|.
\end{aligned}\tag{3.12}$$

Substituting (3.11) into (3.12), we can get

$$\begin{aligned}\|x_{n+1}-p\|^2 \leqslant \|x_n-p\|^2 + (2s_n\mu + s_n^2 - \frac{2s_n\nu}{\lambda^2})\|By_n - Bp\|^2 + \beta_n^2\|x_n - S\rho_n\|^2 \\ + 2\beta_n\|\rho_n - p\|\|x_n - S\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1}-p\|.\end{aligned}$$

From condition (iv), we can get

$$\begin{aligned}(\frac{2a\nu}{\lambda^2} - 2b\mu - b^2)\|By_n - Bp\|^2 &\leqslant \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \beta_n^2\|x_n - S\rho_n\|^2 \\ &\quad + 2\beta_n\|\rho_n - p\|\|x_n - S\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1}-p\| \\ &\leqslant (\|x_n-p\| + \|x_{n+1}-p\|)\|x_n - x_{n+1}\| + \beta_n^2\|x_n - S\rho_n\|^2 \\ &\quad + 2\beta_n\|\rho_n - p\|\|x_n - S\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1}-p\|.\end{aligned}$$

From (3.8), (3.9), and condition (i), we have

$$\lim_{n\to\infty}\|By_n - Bp\| = 0. \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}\|\rho_n - p\|^2 &= \|P_C(I-s_nB)y_n - P_C(I-s_nB)p\|^2 \\ &\leqslant \langle (I-s_nB)y_n - (I-s_nB)p, \rho_n - p\rangle \\ &= \frac{1}{2}[\|(I-s_nB)y_n - (I-s_nB)p\|^2 + \|\rho_n-p\|^2 - \|(I-s_nB)y_n - (I-s_nB)p - (\rho_n - p)\|^2] \\ &\leqslant \frac{1}{2}[\|y_n-p\|^2 + \|\rho_n-p\|^2 - \|(y_n-\rho_n) - s_n(By_n - Bp)\|^2] \\ &= \frac{1}{2}[\|y_n-p\|^2 + \|\rho_n-p\|^2 - \|y_n-\rho_n\|^2 - s_n^2\|By_n-Bp\|^2 \\ &\quad + 2s_n\langle y_n - \rho_n, By_n - Bp\rangle],\end{aligned}$$

which yields that

$$\|\rho_n - p\|^2 \leqslant \|x_n-p\|^2 - \|y_n-\rho_n\|^2 + 2s_n\|y_n-\rho_n\|\|By_n-Bp\|. \tag{3.14}$$

Submitting (3.14) into (3.12) yields that

$$\begin{aligned}\|x_{n+1}-p\|^2 \leqslant \|x_n-p\|^2 - \|y_n-\rho_n\|^2 + 2s_n\|y_n-\rho_n\|\|By_n-Bp\| + \beta_n^2\|x_n - S\rho_n\|^2 \\ + 2\beta_n\|\rho_n - p\|\|x_n - S\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1}-p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|y_n-\rho_n\|^2 &\leqslant \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2s_n\|y_n-\rho_n\|\|By_n-Bp\| + \beta_n^2\|x_n - S\rho_n\|^2 \\ &\quad + 2\beta_n\|\rho_n - p\|\|x_n - S\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1}-p\| \\ &\leqslant (\|x_n-p\| + \|x_{n+1}-p\|)\|x_n - x_{n+1}\| + 2s_n\|y_n-\rho_n\|\|By_n-Bp\| \\ &\quad + \beta_n^2\|x_n - S\rho_n\|^2 + 2\beta_n\|\rho_n - p\|\|x_n - S\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\|x_{n+1}-p\|.\end{aligned}$$

From condition (i), (3.8), (3.9), and (3.13), we have that

$$\lim_{n\to\infty}\|y_n - \rho_n\| = 0. \tag{3.15}$$

Then, we can get

$$\begin{aligned}\|y_n - Sy_n\| &\leqslant \|Sy_n - S\rho_n\| + \|S\rho_n - x_n\| + \|x_n - y_n\| + \|y_n - \rho_n\| \\ &\leqslant 2\|y_n-\rho_n\| + \|S\rho_n - x_n\| + \|x_n - y_n\|.\end{aligned}$$

From (3.9), (3.10), and (3.15), we have

$$\lim_{n\to\infty} \|y_n - Sy_n\| = 0. \tag{3.16}$$

Observe that $P_{\mathscr{F}}(\gamma f + (I - A))$ is a contraction. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}\|P_{\mathscr{F}}(\gamma f + (I-A))(x) - P_{\mathscr{F}}(\gamma f + (I-A))(y)\| &\leqslant \|(\gamma f + (I-A))(x) - (\gamma f + (I-A))(y)\| \\ &\leqslant \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\ &\leqslant \gamma\alpha\|x-y\| + (1-\overline{\gamma})\|x-y\| \\ &< \|x-y\|.\end{aligned}$$

From the famous Banach's contraction mapping principle, we get that $P_{\mathscr{F}}(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$, that is, $q = P_{\mathscr{F}}(\gamma f + (I - A))(q)$.

Step 6. We show that $\limsup\limits_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle \leqslant 0$. To get this result, we choose a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle = \limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_{n_i} - q\rangle.$$

Correspondingly, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to ω. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \omega$. Next, we will show that $\omega \in \mathscr{F}$.

Firstly, we prove $\omega \in EP(F)$. Since $y_n = T_{r_n}x_n$, we have $F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geqslant 0$, for all $u \in C$. From (C2), we have $\langle u - y_n, \frac{y_n - x_n}{r_n}\rangle \geqslant F(u, y_n)$. It follows that,

$$\langle u - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geqslant F(u, y_{n_i}).$$

Since $\frac{y_{n_i} - x_{n_i}}{r_{n_i}} \to \infty, y_{n_i} \rightharpoonup \omega$ and (C4), we have $F(u, \omega) \leqslant 0$ for all $u \in C$. For $t \in (0, 1]$ and $u \in C$, let $u_t = tu + (1-t)\omega$. Since $u \in C$ and $\omega \in C$, we have $u_t \in C$ and hence $F(u_t, \omega) \leqslant 0$. So, from (C1) and (C4), we have

$$0 = F(u_t, u) \leqslant tF(u_t, u) + (1-t)F(u_t, \omega) \leqslant tF(u_t, u).$$

That is, $F(u_t, u) \geqslant 0$. It follows from (C3) that $F(\omega, u) \geqslant 0$ for all $u \in C$ and hence $\omega \in EP(F)$.

Secondly, since Hilbert spaces satisfy Opial's condition, from (3.16), we have

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i} - \omega\| < \liminf_{i\to\infty}\|y_{n_i} - S\omega\| &= \liminf_{i\to\infty}\|y_{n_i} - Sy_{n_i} + Sy_{n_i} - S\omega\| \\ &\leqslant \liminf_{i\to\infty}\|Sy_{n_i} - S\omega\| \\ &\leqslant \liminf_{i\to\infty}\|y_{n_i} - \omega\|.\end{aligned}$$

which derives a contraction. Thus, we have $\omega \in \mathscr{F}$.

Thirdly, we show $\omega \in VI(B, C)$. Put

$$T\omega_1 = \begin{cases} B\omega_1 + N_c\omega_1, & \omega_1 \in C, \\ \phi, & \omega_1 \notin C. \end{cases}$$

From B is relaxed (μ, ν)-cocoercive and condition (iv), we have

$$(Bx - By, x - y) \geqslant (-\mu)\|Bx - By\|^2 + \nu\|x - y\|^2 \geqslant (\nu - \mu\lambda^2)\|x - y\|^2,$$

which yields that B is monotone. Thus, T is maximal monotone. Let $(\omega_1, \omega_2) \in G(T)$. Since $\omega_2 - B\omega_1 \in N_C\omega_1$ and $\rho_n \in C$, we have

$$\langle \omega_1 - \rho_n, \omega_2 - B\omega_1\rangle \geqslant 0.$$

On the other hand, from $\rho_n = P_C(I - s_n B)y_n$, we have

$$\langle \omega_1 - \rho_n, \rho_n - (I - s_n B)y_n \rangle \geqslant 0$$

and hence

$$\langle \omega_1 - \rho_n, \frac{\rho_n - y_n}{s_n} + By_n \rangle \geqslant 0.$$

It follows that

$$\begin{aligned}
\langle \omega_1 - \rho_{n_i}, \omega_2 \rangle &\geqslant \langle \omega_1 - \rho_{n_i}, B\omega_1 \rangle \\
&\geqslant \langle \omega_1 - \rho_{n_i}, B\omega_1 \rangle - \langle \omega_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} + By_{n_i} \rangle \\
&= \langle \omega_1 - \rho_{n_i}, B\omega_1 - \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} - By_{n_i} \rangle \\
&= \langle \omega_1 - \rho_{n_i}, B\omega_1 - B\rho_{n_i} \rangle + \langle \omega_1 - \rho_{n_i}, B\rho_{n_i} - By_{n_i} \rangle - \langle \omega_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} \rangle \\
&\geqslant \langle \omega_1 - \rho_{n_i}, B\rho_{n_i} - By_{n_i} \rangle - \langle \omega_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} \rangle,
\end{aligned}$$

which implies that $\langle \omega_1 - \omega, \omega_2 \rangle \geqslant 0$. We have $\omega \in T^{-1}0$ and hence $\omega \in VI(B, C)$. That is, $\omega \in \mathscr{F}$.

Finally, since $q = P_{\mathscr{F}}(\gamma f + (I - A))(q)$, we have

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q \rangle = \lim_{n\to\infty} \langle \gamma f(q) - Aq, x_{n_i} - q \rangle = \langle \gamma f(q) - Aq, \omega - q \rangle \leqslant 0.$$

Step 7. We will prove that $\{x_n\}$ and $\{y_n\}$ converge strongly to q.

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n(\gamma f(x_n) - Aq) + \beta_n(x_n - q) + ((1-\beta_n)I - \alpha_n A)S\rho_n\| \\
&\leqslant \|\beta_n(x_n - q) + ((1-\beta_n)I - \alpha_n A)(S\rho_n - q)\|^2 + 2\alpha_n \langle \gamma f(x_n) - Aq, x_{n+1} - q \rangle \\
&\leqslant [\|((1-\beta_n)I - \alpha_n A)(S\rho_n - q)\| + \|\beta_n(x_n - q)\|]^2 \\
&\quad + 2\alpha_n\gamma\langle f(x_n) - f(q), x_{n+1} - q \rangle + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
&\leqslant [(1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - q\| + \beta_n\|x_n - q\|]^2 + 2\alpha_n\gamma\alpha\|x_n - q\|\|x_{n+1} - q\| \\
&\quad + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
&\leqslant (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n\gamma\alpha[\|x_n - q\|^2 + \|x_{n+1} - q\|^2] \\
&\quad + 2\alpha_n \langle \gamma f(q) - Aq, x_{n+1} - q \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leqslant \frac{1 - 2\alpha_n\overline{\gamma} + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha}\|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle \gamma f(q) - Aq, x_{n+1} - q \rangle \\
&= [1 - \frac{2(\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}]\|x_n - p\|^2 + \frac{(\alpha_n\overline{\gamma})^2}{1 - \alpha_n\gamma\alpha}\|x_n - q\| + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle \gamma f(q) - Aq, x_{n+1} \rangle \\
&\leqslant [1 - \frac{2(\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}]\|x_n - q\|^2 \\
&\quad + \frac{2(\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha} \times [\frac{(\alpha_n\overline{\gamma}^2)M_2}{2(\overline{\gamma} - \gamma\alpha)} + \frac{1}{\overline{\gamma} - \gamma\alpha}\langle \gamma f(q) - Aq, x_{n+1} - q \rangle] \\
&= (1 - \delta_n)\|x_n - q\|^2 + \delta_n\sigma_n,
\end{aligned}$$

where

$$M_2 = \sup\{\|x_n - q\|^2 : n \geqslant 1\}, \quad \delta_n = \frac{2(\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha},$$

and
$$\sigma_n = \frac{(\alpha_n \overline{\gamma}^2)M_2}{2(\overline{\gamma} - \gamma\alpha)} + \frac{1}{\overline{\gamma} - \gamma\alpha}\langle \gamma f(q) - Aq, x_{n+1} - q\rangle.$$

It is easy to see that $\delta_n \to 0, \sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \sigma_n \leqslant 0$. Hence, by Lemma 2.4, the sequence $\{x_n\}$ converges strongly to q. Consequently, we can obtain that $\{y_n\}$ also converges strongly to q. The proof is complete. □

4. Application

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathfrak{R}$ which satisfies (C1)-(C4), let S be a nonexpansive mapping of C into H such that $\mathrm{Fix}(S) \cap EP(F) \neq \phi$. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geqslant 0, & \forall u \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A))Sy_n, & \forall n \geqslant 1. \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [0,\infty)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\liminf_{n\to\infty} r_n > 0, \quad \lim_{n\to\infty}(r_{n+1} - r_n) = 0.$

Then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \mathrm{Fix}(S) \cap EP(F)$, where $q = P_{\mathrm{Fix}(S)\cap EP(F)}(\gamma f + (I - A))(q)$, which solves the following variational inequality

$$\langle \gamma f(q) - Aq, p - q\rangle \leqslant 0, \quad \forall p \in \mathrm{Fix}(S) \cap EP(F).$$

Proof. Putting $\{s_n\} = 0$ in Theorem 3.1, we can get the desired result easily. □

Remark 4.2. If we take $\{s_n\} = 0$ and $\beta_n = 0$ in Theorem 3.1, we can get the results of Marino and Xu [8] and Plubtieng and Punpaeng [10] immediately.

Remark 4.3. If we take $\{s_n\} = 0, \gamma = 1, \beta_n = 0$ and $A = I$ in Theorem 3.1, we can get the result of Takahashi and Takahashi [14] result immediately.

Theorem 4.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F : C \times C \to \mathfrak{R}$ be a bifunction which satisfies (C1)-(C4), let S be a nonexpansive mapping of C into H and let B be a λ-Lipschitzian, relaxed (μ, ν)-cocoercive map of C into H such that $\mathscr{F} = \mathrm{Fix}(S) \bigcap VI(C, B) \neq \phi$. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$ and assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let f be a contraction of H into itself with a coefficient $\alpha \in (0,1)$ and let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A))SP_C(I - s_n)P_C x_n, \quad \forall n \geqslant 1,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\}, \{s_n\} \subset [0,\infty)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\liminf_{n\to\infty} r_n > 0, \ \lim_{n\to\infty}(r_{n+1} - r_n) = 0$ *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(iv) $\{s_n\} \subset [a, b]$ *for some* a, b *with* $0 \leqslant a \leqslant b \leqslant \frac{2(\nu-\mu\lambda^2}{\lambda^2}$.

Then, $\{x_n\}$ *converges strongly to* $q \in \mathscr{F}$*, where* $q = P_{\mathscr{F}}(\gamma f + (I - A))(q)$*, which solves the following variational inequality*

$$\langle \gamma f(q) - Aq, p - q\rangle \leqslant 0, \quad \forall p \in \mathscr{F}.$$

Proof. Putting $F(x, y) = 0$ for all $x, y \in C$ and $\{r_n\} = 1$ for all n in Theorem 3.1, then, we get $y_n = P_C x_n$, and we can obtain the desired conclusion easily. □

Acknowledgment

This work was supported by the China Postdoctoral Science Foundation funded project (No. 2016M591468), the Social Science Foundation of Jilin Province (No. 2016BS13), the Education Department Foundation of Jilin Province (No. 2016068) and the Key Program of Jilin University of Finance and Economics (No. 2016Z10).

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