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Existence of mild solutions for a class of non-autonomous evolution equations with nonlocal initial conditions

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Abstract

Using the techniques of measures of noncompactness and Schauder fixed point theorem, we present some existence results for mild solutions of a class of nonlocal evolution equations involving causal operators. Moreover, we obtain the compactness of the set of global mild solutions. An example is given to show the efficiency and usefulness of the results.

Keywords: Non-autonomous evolution equation, nonlocal condition, evolution semigroups, causal operator, mild solution.
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1. Introduction

The importance of nonlocal initial value problem based on the fact that it is more general and has better effects than the classical initial condition alone. A nonlocal initial condition for a class of autonomous semilinear evolution equations was first time introduce by Byszewski [13] and has been developed by various authors, see [3–6, 8, 9, 11, 12, 25, 39, 56, 58] and references therein. Byszewski and Lakshmikantham [14] obtained the existence and uniqueness of the mild solution for an autonomous semilinear evolution equation in the case that a Lipschitz type condition is satisfied. The study of existence and uniqueness of mild solution for different classes of semilinear evolution equations under various conditions was developed in many papers, such as [2, 16, 17, 24, 28, 32, 37, 38, 47, 48, 58, 59]. Also, for some results on the existence and uniqueness of solutions for autonomous fractional evolution equations with nonlocal conditions, see [7, 10, 18, 19, 22, 26, 27, 43–46] and references therein.

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_+$ be the set of non-negative real numbers. Let E be a real Banach space endowed with the norm $\|\cdot\|$. We denote by $C([0,a],E)$ the Banach space of continuous functions from $[0,a]$ into E endowed with the norm $\|u(\cdot)\| = \sup_{0\leqslant t\leqslant a} \|u(t)\|$. The space of all (equivalence classes of) strongly measurable functions $u(\cdot) : [0,a] \to E$ such that

$$\|u(\cdot)\|_p := \left(\int_0^a \|u(t)\|^p\right)^{1/p} < \infty,$$

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for $1 \leqslant p < \infty$ and $\|u(\cdot)\|_\infty := \operatorname{ess\,sup}_{t\in[0,a]} \|u(t)\| < \infty$, will be denoted by $L^p([0,a],E)$. This is a Banach space with respect to the norm $\|u(\cdot)\|_p$. Let us denote by $\mathcal{F}_1([0,a],X)$ the space of all the functions from $[0,a]$ into X, and by $\mathcal{F}_2([0,a],Y)$ the space of all the functions from $[0,a]$ into Y. Then an operator $\mathfrak{C} : \mathcal{F}_1([0,a],X) \to \mathcal{F}_2([0,a],Y)$ is called a causal operator, if for each $\tau \in (0,a)$ and for all $u(\cdot), v(\cdot) \in \mathcal{F}_1([0,a],X)$ such that $u(t) = v(t)$ for $t \in [0,\tau]$, we have that $(\mathfrak{C}u)(t) = (\mathfrak{C}v)(t)$ for $t \in [0,\tau]$. Two significant examples of causal operators are: the Niemytzki operator

$$(\mathfrak{C}u)(t) = f(t,u(t)),$$

and the Volterra integral operator

$$(\mathfrak{C}u)(t) = \int_0^t k(t,s)f(s,u(s))ds.$$

Also, the Fredholm operator

$$(Qu)(t) = \int_a^b K(t,s,u(s))ds,$$

is a causal operator, if and only if $K(t,s,u(s)) = 0$, for $t < s < a$. This paper is concerned with the following non-autonomous evolution equation with causal operator and nonlocal condition:

$$\begin{cases} u'(t) = A(t)u(t) + F(t,u(t),(\mathfrak{C}u)(t), & \text{for } t \in [0,a], \\ u(0) = g(u(\cdot)), \end{cases}$$

where $\{A(t) : D(A(t)) \subset E \to E; t \in [0,a]\}$ is a family of closed densely defined unbounded linear operators generating an evolution system $\{\mathcal{T}(t,s) : 0 \leqslant s \leqslant t \leqslant a\}$, $\mathfrak{C} : C([0,a],E) \to L^p([0,a],Z)$ is a causal operator and $F(\cdot,\cdot,\cdot) : [0,a] \times E \times Z \longrightarrow E$ is a given function. The general non-autonomous integro-differential evolution equation

$$\begin{cases} u'(t) = A(t)u(t) + F\left(t,u(t),\displaystyle\int_0^t K(t,s,u(s))ds\right), \\ u(0) = g(u(\cdot)), \end{cases}$$

and the evolution equation with "maxima":

$$u'(t) = A(t)u(t) + F\left(t,u(t),\max_{0\leqslant s\leqslant t} u(s)\right), \quad u(0) = g(u(\cdot)),$$

are example of non-autonomous causal evolution equations with nonlocal initial conditions. From the above examples it is clear that the fractional evolution with causal operators cover a large variety of nonlocal evolution equations. Casual operators represent a class of operators which allows us to study unitarily many types of differential equations and integral equations. For theoretical aspects and applications of causal operators see the monographs [20, 35, 36, 42]. The study of a class of evolution equations with causal operators was approached in [1, 15]. For some recent contribution to the study of autonomous fractional evolution equations with causal operators we refer to the following papers [29, 30, 54].

In this paper we bring some contributions to the study of a class of non-autonomous evolution equations involving causal operators. To best of our knowledge, this is the first paper devoted to the study of such class of problems and it can be viewed as an attempt to unify and generalize the non-autonomous versions of some known results in the autonomous case from the works [20, 32, 37, 38, 55, 56, 58, 59] and other ones. In Section 2, we will recall some necessary results on evolution system of linear operators and some properties of the Hausdorff measure of noncompactness. In Section 3, we will obtain the existence of the mild solution for non-autonomous causal evolution equations under a noncompact evolution system. In Section 4, we will give an application.

2. Preliminaries

We denote the space of all bounded linear operators acting on a Banach space E by $\mathcal{L}(E)$. A two parameter family of bounded linear operators $\{\mathcal{T}(t,s), 0 \leqslant s \leqslant t \leqslant a\}$ on E is called an evolution system, if the following three conditions are satisfied:

(a) $\mathcal{T}(s,s) = I$, the identity operator on E;

(b) $\mathcal{T}(t,r)\mathcal{T}(r,s) = \mathcal{T}(t,s)$ for $0 \leqslant s \leqslant r \leqslant t \leqslant a$;

(c) $(t,s) \mapsto \mathcal{T}(t,s) : \Delta \to \mathcal{L}(E)$ is strongly continuous for $0 \leqslant s \leqslant t \leqslant a$;

(d) $t \mapsto \mathcal{T}(t,s) : (s,a] \to \mathcal{L}(E)$ is differentiable, $\frac{\partial}{\partial t}\mathcal{T}(t,s) \in \mathcal{L}(E)$, and

$$\frac{\partial}{\partial t}\mathcal{T}(t,s) = A(t)\mathcal{T}(t,s), \quad 0 \leqslant s < t \leqslant a,$$

where $\Delta := \{(t,s) \in [0,a] \times [0,a] : \ 0 \leqslant s \leqslant t \leqslant a\}$.

Since the evolution system $\mathcal{T} : [0,a] \times [0,a] \to \mathcal{L}(E)$ is strongly continuous on the compact set $[0,a] \times [0,a]$, there exists $M > 0$ such that $\|\mathcal{T}(t,s)\| \leqslant M$ for any $(t,s) \in [0,a] \times [0,a]$. For the existence and construction of an evolution system with the above properties see [40], [57, Chapter 3]. More details as regards this evolution system can be found in [50].

We denote by $\beta(B)$ the Hausdorff measure of non-compactness of a nonempty bounded set $B \subset E$, and it is defined by (see [31]):

$$\beta(B) = \inf\{\varepsilon > 0; B \text{ admits a finite cover by balls of radius} \leqslant \varepsilon\}.$$

We recall some properties of β (see [31]). If A, B are bounded subsets of E, then

(β1) $\beta(B) = 0$, if and only if $\overline{B}$ is compact;

(β2) $\beta(B) = \beta(\overline{B}) = \beta(\overline{conv}(B))$;

(β3) $\beta(\lambda B) = |\lambda|\beta(B)$, for every $\lambda \in \mathbb{R}$;

(β4) $\beta(B) \leqslant \beta(C)$, if $B \subset C$;

(β5) $\beta(\{x\} \cup B) = \beta(B)$, for every $x \in E$;

(β6) $\beta(B + C) = \beta(B) + \beta(C)$.

(β7) Generalized Cantor's intersection property (see [34]):
If $\{B_n\}_{n \geqslant 1}$ is a decreasing sequence of bounded closed nonempty subsets of E and $\lim_{n\to\infty} \beta(B_n) = 0$, then $\bigcap_{n=1}^{\infty} B_n$ is a nonempty and compact subset of E.

Remark 2.1. If $\dim(B) = \sup\{\|x - y\|; x, y \in B\}$ is the diameter of the bounded set A, then we have that $\beta(B) \leqslant \dim(B)$ and $\beta(B) \leqslant 2d$, if $\sup_{x \in B} \|x\| \leqslant d$.

In the following, we denote by β_c the Hausdorff measure of non-compactness in the space $C([0,a], E)$. Then it is well-known that for every bounded set $B \subset C([0,a], E)$ we have (see [31])

$$\beta(B(t)) \leqslant \beta_c(B),$$

for every $t \in [0,a]$, where $B(t) := \{u(t) : u \in B\}$. Moreover, for every bounded and equicontinuous set $B \subset C([0,a], E)$ we have (see [31])

$$\beta_c(B) = \sup_{0 \leqslant t \leqslant a} \beta(B(t)). \tag{2.1}$$

Lemma 2.2 ([33, Lemma 2.2]). *Let $\{u_n(\cdot); n \geqslant 1\}$ be a subset in $L^1([0,a],E)$ for which there exists $m(\cdot) \in L^1([0,a],\mathbb{R}_+)$ such that $\|u_n(t)\| \leqslant m(t)$ for each $n \geqslant 1$ and for a.e. $t \in [0,a]$. Then the function $t \mapsto \beta(t) := \beta(\{u_n(t); n \geqslant 1\})$ is integrable on $[0,a]$ and for each $t \in [0,a]$, we have*

$$\beta\left(\left\{\int_0^t u_n(s)ds; n \geqslant 1\right\}\right) \leqslant \int_0^t \beta(s)ds.$$

3. Existence result

Consider the following evolution equation with causal operator and nonlocal condition:

$$\begin{cases} u'(t) = A(t)u(t) + F(t,u(t),(\mathfrak{C}u)(t)), & \text{for } t \in [0,a], \\ u(0) = g(u(\cdot)), \end{cases} \tag{3.1}$$

where $\{A(t) : D(A(t)) \subset E \to E,\ t \in [0,a]\}$ is a family of closed densely defined unbounded linear operators generating an evolution system $\{\mathcal{T}(t,s) : 0 \leqslant s \leqslant t \leqslant a\}$, $\mathfrak{C} : C([0,a],E) \to L^p([0,a],Z)$ is a causal operator, and $F(\cdot,\cdot,\cdot) : [0,a] \times E \times Z \longrightarrow E$ is a given function. A function $u(\cdot) : [0,a] \to E$, is a mild solution of (3.1), if it satisfies

$$u(t) = \mathcal{T}(t,0)g(u(\cdot)) + \int_o^t \mathcal{T}(t,s)F(s,u(s),(\mathfrak{C}u)(s))ds, \tag{3.2}$$

for all $t \in [0,a]$.

Let us introduce the following conditions:

(H1) $\{A(t) : D(A(t)) \subset E \to E; t \in [0,a]\}$ is a family of closed densely defined unbounded linear operators, $(D(A(t)) = D(A)$ not depending on t and dense subset of $E)$, such that there exists an evolution system $\{\mathcal{T}(t,s); 0 \leqslant s \leqslant t \leqslant a\}$ with the property that

$$\left\|\frac{\partial}{\partial t}\mathcal{T}(t,s)\right\|_{\mathcal{L}(E)} = \|A(t)\mathcal{T}(t,s)\|_{\mathcal{L}(E)} \leqslant C\,(t-s)^{-1},$$

with $C > 0$ and $0 \leqslant s < t \leqslant a$.

(H2) (a) $\mathfrak{C} : C([0,a],E) \to L^p([0,a],Z)$ is a continuous causal operator such that there exists an increasing function $\Lambda(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\|(\mathfrak{C}u)(t)\| \leqslant \Lambda(\|u(t)\|) \text{ a.e. on } [0,a], \tag{3.3}$$

for all $u(\cdot) \in C([0,a],E)$.

(b) There exists a positive constant $k_1 \in \mathbb{R}_+$ such that

$$\beta(\mathfrak{C}V)(t) \leqslant k_1\beta(V(t)),$$

for each bounded set $V \subset C([0,a],E)$.

(H3) $g(\cdot) : C([0,a],E) \longrightarrow D(A)$ and there exists a constant $l_g > 0$ such that $Ml_g < 1$ and

$$\|g(u(\cdot)) - g(v(\cdot))\| \leqslant l_g\,\|u(\cdot) - v(\cdot)\|, \quad u(\cdot), v(\cdot) \in C([0,a],E),$$

where $M = \sup_{(t,s)\in\Delta} \|\mathcal{T}(t,s)\|_{\mathfrak{L}(E)}$.

(H4) $F(\cdot,\cdot,\cdot) : [0,a] \times E \times Z \longrightarrow E$ satisfies:

(a) $t \longmapsto F(t,u,z)$ is strongly measurable for each $u \in E, z \in Z$.

(b) $(u,z) \longmapsto F(t,u,z)$ is continuous for a.e. $t \in [0,a]$.

(c) There exists $\xi(\cdot,\cdot) : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\xi(\cdot,\sigma) \in L^p([0,a],\mathbb{R}_+)$, for every $\sigma \in \mathbb{R}_+$, $\xi(t,\cdot)$ is continuous and increasing for a.e. $t \in [0,a]$,

$$\limsup_{\sigma\to\infty} \frac{M\Lambda(\sigma)}{\sigma} \int_0^a \xi(s,\sigma)ds < 1 - Ml_g, \tag{3.4}$$

for $t \in [0,a]$, and

$$\|F(t,u,z)\| \leqslant \xi(t,\|u\|)\,\|z\|\,, \text{ for a.e. } t \in [0,a], \tag{3.5}$$

for all $u \in E$, $z \in Z$.

(d) There exists a positive constant $k_0 \in \mathbb{R}_+$ such that for bounded subset $B_1 \subset E$ and $B_2 \subset Z$, we have

$$\beta_E(F(t,B_1,B_2)) \leqslant k_0\,[\beta_E(B_1) + \beta_Z(B_2)]\,,$$

for a.e. $t \in [0,a]$.

Theorem 3.1. *If hypothesis* (H1)-(H4) *are satisfied and*

$$ak_0(1+k_1) < \frac{1 - Ml_g}{M}, \tag{3.6}$$

then the solution set of (3.1) *is nonempty and compact in* $C([0,a],E)$.

Proof. First, we show that there exists $r_0 > 0$ such that $\|u(\cdot)\| \leqslant r_0$, for all possible mild solutions of (3.1). Let $u(\cdot) : [0,a] \to E$ be a mild solution of (3.1). Then by (3.2) we have

$$\begin{aligned}
\|u(t)\| &\leqslant \|\mathcal{T}(t,0)g(u(\cdot))\| + \int_0^t \|\mathcal{T}(t,s)F(s,u(s),((\mathfrak{C}u)(s)))\|\,ds \\
&\leqslant M\|g(u(\cdot))\| + M\int_0^t \|F(s,u(s),(\mathfrak{C}u)(s))\|ds \\
&\leqslant M(l_g\|u(\cdot)\| + \|g(0)\|) + M\int_0^t \xi(s,\|u(s)\|\,\|(\mathfrak{C}u)(s)\|)ds \\
&\leqslant M(l_g\|u(\cdot)\| + \|g(0)\|) + M\Lambda(\|u(\cdot)\|)\int_0^t \xi(s,\|u(\cdot)\|)ds,
\end{aligned}$$

so that

$$(1 - Ml_g)\,\|u(\cdot)\| \leqslant M\|g(0)\| + M\Lambda(\|u(\cdot)\|)\int_0^a \xi(s,\|u(\cdot)\|)ds.$$

From the last inequality and (3.4) it follows that there exists $r_0 > 0$ such that $\|u(\cdot)\| \leqslant r_0$. Indeed, if this is not true, then we can find a sequence $\{u_n(\cdot)\}_{n\geqslant 1}$ of mild solutions of (3.1) such that $\|u_n(\cdot)\| \to \infty$ as $n \to \infty$. Then from the last inequality we obtain

$$\begin{aligned}
1 - Ml_g &\leqslant \limsup_{n\to\infty}\left(\frac{M\|g(0)\|}{\|u_n(\cdot)\|} + \frac{M\Lambda(\|u_n(\cdot)\|)}{\|u_n(\cdot)\|}\int_0^a \xi(s,\|u_n(\cdot)\|)ds\right) \\
&\leqslant \limsup_{n\to\infty}\frac{M\Lambda(\|u_n(\cdot)\|)}{\|u_n(\cdot)\|}\int_0^a \xi(s,\|u_n(\cdot)\|)ds \\
&\leqslant \limsup_{\sigma\to\infty}\frac{M\Lambda(\sigma)}{\sigma}\int_0^a \xi(s,\sigma)ds,
\end{aligned}$$

in contradiction with (3.4). Consequently, there exists $r_0 > 0$ such that $\|u(\cdot)\| \leqslant r_0$ for all mild solution of (3.1). Also, we remark that for every $r > r_0$ we have

$$\|g(0)\| + \Lambda(r)\int_0^a \xi(s,r)ds < \frac{1 - Ml_g}{M}r. \tag{3.7}$$

Let

$$\psi(t) := \xi(t,r)\Lambda(r), \quad t \in [0,a], \tag{3.8}$$

where $\Lambda(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous increasing function and $\xi(\cdot,\cdot) : [0,a] \times \mathbb{R}_+$ satisfies (H3) (c). Then $\psi(\cdot) \in L^p([0,a],\mathbb{R}_+)$ and from (3.3), (3.5) and (3.8), it follows that

$$\|F(t,u(t),(\mathfrak{C}u)(t))\| \leqslant \psi(t), \text{ for a.e. } t \in [0,a].$$

Next, let $r > r_0$ and

$$B_0 := \{u(\cdot) \in C([0,a],E); \|u(\cdot)\| \leqslant r\}.$$

A mild solution of (3.1) will be a fixed point of an operator $\mathcal{A}$ defined by

$$(\mathcal{A}u)(t) = \mathcal{T}(t,0)g(u(\cdot)) + \int_0^t \mathcal{T}(t,s)F(s,u(s),(\mathfrak{C}u)(s))ds,$$

for $t \in [0,a]$. Since $\xi(t,\cdot)$ is increasing on $\mathbb{R}_+$ for a.e. $t \in [0,a]$, by using (3.7) we have

$$\begin{aligned}
\|\mathcal{A}u(t)\| &\leqslant \|\mathcal{T}(t,0)g(u(\cdot))\| + \int_0^t \|\mathcal{T}(t,s)F(s,u(s),(\mathfrak{C}u)(s))\|ds \\
&\leqslant M(l_g\|u\| + \|g(0)\|) + M\int_0^t \xi(s,\|u(s)\|)\Lambda(\|u(s)\|)ds \\
&\leqslant M(l_g\|u\| + \|g(0)\|) + M\Lambda(\|u(\cdot)\|)\int_0^t \xi(s,\|u(\cdot)\|)ds \\
&\leqslant M(l_g r + \|g(0)\|) + M\Lambda(r)\int_0^a \xi(s,r)ds \leqslant r,
\end{aligned}$$

for every $u(\cdot) \in B_0$ and thus $\mathcal{A}B_0 \subset B_0$. Now we prove that $\mathcal{A}$ is a continuous operator. For this, let $\{u_n(\cdot)\}_{n\geqslant 1}$ be a convergent sequence in B_0 such that $\lim_{n\to\infty} u_n(\cdot) = u(\cdot)$. Then we have,

$$\begin{aligned}
\|(\mathcal{A}u_n)(t) - (\mathcal{A}u)(t)\| &\leqslant \|\mathcal{T}(t,0)\left[g(u_n(\cdot)) - g(u(\cdot))\right]\| \\
&\quad + \int_0^t \|\mathcal{T}(t,s)\left(F(s,u_n(s),(\mathfrak{C}u_n)(s)) - F(s,u(s),(\mathfrak{C}u)(s))\right)\|ds \\
&\leqslant M\|g(u_n(\cdot)) - g(u(\cdot))\| \\
&\quad + M\int_0^t \|\left(F(s,u_n(s),(\mathfrak{C}u_n)(s)) - F(s,u(s),(\mathfrak{C}u)(s))\right)\|ds \\
&\leqslant Ml_g\|u_n(\cdot) - u(\cdot)\| \\
&\quad + M\int_0^t \|F(s,u_n(s),(\mathfrak{C}u_n)(s)) - F(s,u(s),(\mathfrak{C}u)(s))\|ds.
\end{aligned}$$

Since $\mathfrak{C} : C([0,a],E) \to L^p([0,a],E)$ is continuous, $(\mathfrak{C}u_n)(\cdot) \to (\mathfrak{C}u)(\cdot)$ in $L^p([0,a],E)$, so that $(\mathfrak{C}u_n)(\cdot) \to (\mathfrak{C}u)(\cdot)$ a.e. on $[0,a]$. Thus by the fact that $F(\cdot,\cdot,\cdot)$ is a Carathéodory function, it follows that

$$F(\cdot,u_n(\cdot),(\mathfrak{C}u_n(\cdot))) \longrightarrow F(\cdot,u(\cdot),(\mathfrak{C}u(\cdot))\text{a.e. on}[0,a].$$

Also, since $\|F(s,u_n(s),(\mathfrak{C}u_n)(s)) - F(t,u(t),\mathfrak{C}u(t))\| \leqslant 2\psi(t)$ a.e. on $[0,a]$ by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_0^t F(s,u_n(s),(\mathfrak{C}u_n)(s))ds = \int_0^t F(s,u(s),(\mathfrak{C}u)(s))ds,$$

for all $t \in [0, a]$. It follows that $(\mathcal{A}u_n)(\cdot) \longrightarrow (\mathcal{A}u)(\cdot)$ on $[0, a]$ as $n \longrightarrow \infty$, and so $\mathcal{A}$ is a continuous operator on $C([0, a], E)$. Next, by (H1) and Theorem 3.2 in [49], it follows that the set

$$\mathcal{A}B_0 = \{(\mathcal{A}u)(\cdot); u(\cdot) \in B_0\} \subset C([0, a], E),$$

is equicontinuous.

Now define $B_{n+1} = \overline{conv}(\mathcal{A}B_n)$, $n = 0, 1, 2, \cdots$. From $\mathcal{A}B_0 \subset B_0$, it follows that

$$B_1 = \overline{conv}(\mathcal{A}B_0) \subset \overline{conv}(B_0) = B_0,$$

and thus, $B_1 \subset C([0, a], E)$ is bounded, closed, convex and equicontinuous. By the mathematical induction it is easy to see that $B_{n+1} \subset B_n$ and $B_n \subset C([0, a], E)$ are bounded, closed, convex and equicontinuous for $n = 0, 1, 2, \cdots$. Next, since $C([0, a], E)$ is separable for each $n = 0, 1, 2, \cdots$, there exists a countable set $U^n = \{u_k^n(\cdot); k = 1, 2, \cdots\} \subset B_n$ such that $\overline{U^n} = B_n$. Using properties of the measure of noncompactness, we have

$$\begin{aligned}\beta(B_{n+1}(t)) &= \beta(\overline{conv}((\mathcal{A}B_n)(t))) = \beta((\mathcal{A}B_n)(t)) = \beta((\mathcal{A}\overline{U^n})(t))\\ &\leqslant \beta((\overline{\mathcal{A}U^n})(t)) = \beta((\mathcal{A}U^n)(t)).\end{aligned}$$

Since by (H3) we have $\beta\left(\mathcal{T}(t, 0)g(U^n)\right) \leqslant Ml_g\beta_c\left(U^n\right)$, using properties of the measure of noncompactness and Lemma 2.2, we have

$$\begin{aligned}\beta(B_{n+1}(t)) &\leqslant \beta\left(\mathcal{T}(t, 0)g(U^n)\right) + \int_0^t \beta\left(\mathcal{T}(t, s)F(s, U^n(s), (\mathfrak{C}U^n)(s))\right) ds\\ &\leqslant M\beta\left(g(U^n)\right) + \int_0^t M\beta\left(F(s, U^n(s), (\mathfrak{C}U^n)(s))\right) ds\\ &\leqslant Ml_g\beta_c\left(U^n\right) + Mk_0\int_0^t [\beta\left(U^n(s)\right) + \beta\left((QU^n)(s)\right)]ds\\ &\leqslant Ml_g\beta_c\left(B_n\right) + Mk_0(1 + k_1)\int_0^t \beta\left(B_n(s)\right) ds.\end{aligned} \tag{3.9}$$

This implies

$$\beta_c(B_{n+1}) \leqslant M\left[l_g + ak_0(1 + k_1)\right]\beta_c(B_n).$$

Since $B_{n+1} \subset B_n$, $n = 0, 1, 2, \cdots$, it is easy to see that

$$\beta_c(B_n) \leqslant k^n\beta_c(B_0), \quad n = 0, 1, 2, \cdots,$$

where $k := M\left[l_g + ak_0(1 + k_1)\right]$. Since by (3.6) we have $0 < k < 1$, the last inequality implies that $\lim_{n\to 0}\beta_c(B_n) = 0$. Since $\{B_n\}_{n\geqslant 1}$ is decreasing sequence of bounded and closed sets, using property (β7) of the measure of noncompactness it follows that $B := \bigcap_{n=0}^{\infty} B_n$ is a compact and convex set in $C([0, a], E)$ and $\mathcal{A}B \subset B$. Consequently, by the Schauder fixed point theorem it follows that the operator $\mathcal{A}$ has at least one fixed point $u(\cdot) \in B$, which is a mild solution of (3.1).

Now, let $S(g) \subset C([0, a], E)$ be the nonempty set of mild solutions of (3.1) and let $\{u_n(\cdot)\}_{n\geqslant 1}$ be a sequence in $S(g)$, that is,

$$u_n(t) = \mathcal{T}(t, 0)g(u_n(\cdot)) + \int_o^t \mathcal{T}(t, s)F(s, u_n(s), (\mathfrak{C}u_n)(s))ds,$$

for all $t \in [0, a]$ and all $n \geqslant 1$. Since $u_n(\cdot) \in B$ for all $n \geqslant 1$, it follows that $\{u_n(\cdot)\}_{n\geqslant 1}$ is equicontinuous. Also, it is easy to see that

$$\beta_c\left(\{u_n(t); n \geqslant 1\}\right) \leqslant Ml_g\beta_c\left(\{u_n(\cdot); n \geqslant 1\}\right)$$

$$+Mk_0(1+k_1)\int_0^t \beta\left(\{u_n(s);n\geqslant 1\}\right)ds,$$

so that $\beta_c\left(\{u_n(\cdot);n\geqslant 1\}\right)\leqslant k\beta_c\left(\{u_n(\cdot);n\geqslant 1\}\right)$. Thus $\beta_c\left(\{u_n(\cdot);n\geqslant 1\}\right)=0$, and so $\{u_n(\cdot)\}_{n\geqslant 1}$ is relatively compact in $C([0,a],E)$. Since $S(g)$ is a closed set in $C([0,a],E)$ we can conclude that $S(g)$ is a compact set in $C([0,a],E)$. □

Theorem 3.2. *Suppose that* (H1), (H2), (H4)(a-b) *are satisfied,* $g(\cdot):C([0,a],E)\to D(A)$ *is continuous and compact, and there exist* $b>0$, $c>0$ *such that* $Mb<1$ *and*

$$\|g(u(\cdot))\|\leqslant b\,\|u(\cdot)\|+c,$$

for all $u(\cdot)\in C([0,a],E)$. *Also, assume that*

(H4 − c) *there exists* $\xi(\cdot,\cdot):[0,a]\times\mathbb{R}_+\to\mathbb{R}_+$ *such that* $\xi(\cdot,\sigma)\in L^p([0,a],\mathbb{R}_+)$ *for every* $\sigma\in\mathbb{R}_+$, $\xi(t,\cdot)$ *is continuous and increasing for a.e.* $t\in[0,a]$,

$$\limsup_{\sigma\to\infty}\frac{M\Lambda(\sigma)}{\sigma}\int_0^a \xi(s,\sigma)ds<1-Mb,$$

for $t\in[0,a]$, *and* (3.5) *holds for all* $u\in E$, $z\in Z$;

(H4 − d) *there exists* $k(\cdot)\in L([0,a],\mathbb{R}_+)$ *such that*

$$\beta_E(F(t,B_1,B_2))\leqslant k(t)\left[\beta_E(B_1)+\beta_Z(B_2)\right],$$

for a.e. $t\in[0,a]$.

Then the solution set of (3.1) *is nonempty and compact in* $C([0,a],E)$.

Proof. With the same notations and with the same reasoning as in the proof of Theorem 3.1 and using (H4-c), it is easy to show that there exists an $r_0>0$ such that $\|u(\cdot)\|\leqslant r_0$, for all possible mild solutions of (3.1). If we take $r>r_0$ and put

$$B_0:=\{u(\cdot)\in C([0,a],E);\|u(\cdot)\|\leqslant r\},$$

then $\mathcal{A}B_0\subset B_0$ and $\mathcal{A}$ is a continuous operator on B_0. Also, by (H1), Corollary 1 in [17] and Theorem 3.2 in [49], it follows that the set

$$\mathcal{A}B_0=\{(\mathcal{A}u)(\cdot);u(\cdot)\in B_0\}\subset C([0,a],E),$$

is equicontinuous. Consequently, $B_{n+1}:=\overline{conv}(\mathcal{A}B_n)$, $n=0,1,2,\cdots$, are bounded, closed, convex and equicontinuous, and $B_{n+1}\subset B_n$ for $n=0,1,2,\cdots$. Next, since $C([0,a],E)$ is separable for each $n=0,1,2,\cdots$, there exists a countable set $U^n=\{u_k^n(\cdot);k=1,2,\cdots\}\subset B_n$ such that $\overline{U^n}=B_n$. Since $g(\cdot)$ is compact, $\beta\left(g(U^n)\right)=0$ and with the same reasoning as in (3.9) and using (H4-d) we obtain

$$\beta(B_{n+1}(t))\leqslant M(1+k_1)\int_0^t k(s)\beta\left(B_n(s)\right)ds.$$

As $\{\beta\left(B_n\right)\}_{n\geqslant 1}$ is a decreasing and monotone sequence, there exists $w(t):=\lim_{n\to\infty}\beta\left(B_n(t)\right)$, for all $t\in[0,a]$. Taking the limit in both sides of the above inequality, it follows that

$$w(t)\leqslant M(1+k_1)\int_0^t k(s)w(s)ds,\ t\in[0,a].$$

Also, since B_n, $n=1,2,\cdots$, are bounded and equicontinuous, it follows that $w(\cdot)$ is continuous on $[0,a]$, so that by Gronwall-Bellman's inequality from the last inequality we obtain $w(t)=0$ for $t\in[0,a]$.

Hence, by using (2.1) we obtain that $\lim_{n\to\infty}\beta_c(B_n)=0$. Therefore, using property (β7) of the measure of noncompactness, it follows that $B:=\bigcap_{n=0}^{\infty}B_n$ is a compact convex set in $C([0,a],E)$ and $\mathcal{A}B\subset B$. Consequently, by the Schauder fixed point theorem it follows that the operator $\mathcal{A}$ has at least one fixed point $u(\cdot)\in B$, which is a mild solution of (3.1). Next, let $S(g)\subset C([0,a],E)$ be the nonempty set of mild solutions of (3.1) and let $\{u_n(\cdot)\}_{n\geqslant 1}$ be a sequence in $S(g)$, that is,

$$u_n(t)=\mathcal{T}(t,0)g(u_n(\cdot))+\int_0^t\mathcal{T}(t,s)F(s,u_n(s),(\mathfrak{C}u_n)(s))ds,$$

for all $t\in[0,a]$ and all $n\geqslant 1$. Since $u_n(\cdot)\in B$ for all $n\geqslant 1$, it follows that $\{u_n(\cdot)\}_{n\geqslant 1}$ is equicontinuous, and so $\nu(t):=\beta(\{u_n(t);n\geqslant 1\})$, $t\in[0,a]$, is a continuous function from $[0,a]$ into $\mathbb{R}$. Also, it is easy to see that

$$\beta(\{u_n(t);n\geqslant 1\})\leqslant M(1+k_1)\int_0^t k(s)\beta(\{u_n(s);n\geqslant 1\})\,ds,$$

so that

$$\nu(t)\leqslant M(1+k_1)\int_0^t k(s)\nu(s)ds,\ t\in[0,a].$$

Thus by Gronwall-Bellman's inequality from the last inequality we obtain $\nu(t)=0$ for all $t\in[0,a]$. Consequently, by Arzelà-Ascoli theorem, we can conclude that $\{u_n(\cdot)\}_{n\geqslant 1}$ is relatively compact in $C([0,a],E)$. Since $S(g)$ is a closed set in $C([0,a],E)$ it follows that $S(g)$ is a compact set in $C([0,a],E)$. □

4. An example

As an application of Theorem 3.1, consider the following heat equation with time-varying coefficients. Let $a(\cdot)\in C([0,1],\mathbb{R})$ be a Lipschitz continuous function such that $a(t)>0$ for all $t\in[0,1]$.

$$\begin{cases}\frac{\partial}{\partial t}w(t,x)=\frac{\partial^2}{\partial x^2}w(t,x)-a(t)w(t,x)+f(t,w(t,x),Z(t,x)),\\ w(t,0)=w(t,\pi)=0,\quad t\in[0,1],\\ w(0,x)=\sum_{i=1}^{\nu}c_i(x)\int_0^{\pi}w(t_i,\eta)d\eta,\quad x\in[0,\pi],\end{cases}\tag{4.1}$$

where

$$Z(t,x):=\int_0^t k(t,s)w(s,x)ds,\quad 0\leqslant t\leqslant 1,\quad 0\leqslant x\leqslant\pi,$$

$c_i(\cdot):\mathbb{R}\to\mathbb{R}$, $1\leqslant i\leqslant\nu$, are given functions and $0<t_1<t_2<...<t_\nu<1$. We assume that the following conditions hold.

(Hf) $t\mapsto f(t,u,y)$ is a Lebesgue measurable for all $u,y\in\mathbb{R}$; $(u,y)\mapsto f(t,u,y)$ is continuous for a.e. $t\in[0,1]$ and there exists $\xi(\cdot,\cdot):\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ such that $\xi(\cdot,\sigma)\in L^2([0,1],\mathbb{R}_+)$ for every $\sigma\in\mathbb{R}_+$, $\xi(t,\cdot)$ is continuous increasing for a.e. $t\in[0,1]$, and

$$|f(t,u,y)|\leqslant\xi(t,|u|)|y|,\ \text{for a.e. } t\in[0,1],$$

and for all $u,y\in\mathbb{R}$.

(Hk) $k(\cdot,\cdot):[0,1]\times[0,1]\to\mathbb{R}$ is continuous.

(Hc) $c_i(\cdot):\mathbb{R}\to\mathbb{R}$, $1\leqslant i\leqslant\nu$ are continuous functions such that $\sup_{x\in[0,\pi]}\left(\sum_{i=1}^{\nu}c_i^2(x)\right)^{1/2}:=c_0<\infty$.

Consider $E:=L^2([0,\pi],\mathbb{R})$ endowed with the usual inner product $\langle\cdot,\cdot\rangle$. We will show that the problem (3.1) can be written in the abstract form (4.1) on the space E as follows.

Define the operator $A_0:D(A_0)\subset E\to E$ by

$$A_0y:=\frac{\partial^2 y}{\partial x^2},\quad y\in D(A_0),$$

$$D(A_0) := \{y(\cdot) \in E: \ y(\cdot), y'(\cdot) \text{ are absolutely continuous } y''(\cdot) \in E \text{ and } y(0) = y(1) = 0\}.$$

Then the operator A_0 is the infinitesimal generator of a compact C_0-semigroup (see [50], [51]). Moreover, the operator A_0 has eigenvalues $-n^2$; $n = 1, 2, \cdots$ with the corresponding eigenvectors $y_n(x) = (\sqrt{\frac{2}{\pi}}) \sin nx$; $n = 1, 2, \cdots$, and it can be written as (see [52, Problem 4.2])

$$A_0 y(\cdot) = -\sum_{n=1}^{\infty} n^2 \langle y(\cdot), y_n(\cdot)\rangle y_n(\cdot), \quad y(\cdot) \in D(A_0).$$

Next, let $\{A(t); t \in [0,1]\}$ be a family of operators given by:

$$D(A(t)) := D(A_0),\ t \in [0,1],$$

$$A(t)y(\cdot) = A_0 y(\cdot) - a(t)y(\cdot),\ \ y(\cdot) \in D(A(t)).$$

Then (see [21], [53]) $\{A(t); t \in [0,1]\}$ generates an evolution system $\{\mathcal{T}(t,s), 0 \leqslant s \leqslant t \leqslant a\}$ which satisfies (H1), namely

$$\mathcal{T}(s,t)y(\cdot) := \sum_{n=1}^{\infty} \exp\left[-n^2(t-s) - \int_s^t a(\tau)d\tau\right] \langle y(\cdot), y_n(\cdot)\rangle y_n(\cdot),$$

for $0 \leqslant s \leqslant t \leqslant 1$ and $y(\cdot) \in E$. Next let us define $u(\cdot) : [0,1] \to E$ by $u(t)(\cdot) = w(t, \cdot)$, $\mathfrak{C} : C([0,1], E) \to L^2([0,1], E)$ by

$$(\mathfrak{C}u)(t) = \int_0^t k(t,s)u(s)ds, \quad 0 \leqslant s \leqslant t \leqslant 1,$$

and $g(\cdot) : C([0,1], E) \to E$ by

$$g(u(\cdot))(x) = \sum_{i=1}^{\nu} c_i(x) \int_0^{\pi} u(t_i)(\eta)d\eta,$$

where $0 < t_1 < t_2 < ... < t_\nu < 1$.

From (Hk) and [23, Proposition 9.5.2] it follows that $\mathfrak{C} : L^2([0,1], E) \to L^2([0,1], E)$ is a continuous causal operator and it satisfies (H1-b) (see [41]). Also, using the Minkowski's inequality we have

$$\begin{aligned}
|g(u(\cdot))(x)| &= \left|\sum_{i=1}^{\nu} c_i(x) \int_0^{\pi} [u(t_i)(\eta) - v(t_i)(\eta)]d\eta\right| \\
&\leqslant \left(\sum_{i=1}^{\nu} c_i^2(x)\right)^{1/2} \left(\sum_{i=1}^{\nu} \left(\int_0^{\pi} [u(t_i)(\eta) - v(t_i)(\eta)]d\eta\right)^2\right)^{1/2} \\
&\leqslant c_0 \sum_{i=1}^{\nu} \left(\int_0^{\pi} [u(t_i)(\eta) - v(t_i)(\eta)]^2 d\eta\right)^{1/2} \\
&\leqslant c_0 \sum_{i=1}^{\nu} \|u(t_i)(\cdot) - v(t_i)(\cdot)\|_E \\
&\leqslant c_0 \nu \|u(\cdot) - v(\cdot)\|_{C([0,1],E)},
\end{aligned}$$

so that

$$\begin{aligned}
\|g(u(\cdot)) - g(u(\cdot))\|_E^2 &= \int_0^{\pi} |g(u(\cdot))(x) - g(u(\cdot))(x)|^2 dx \\
&\leqslant c_0^2 \nu^2 \|u(\cdot) - v(\cdot)\|_{C([0,1],E)}^2,
\end{aligned}$$

it follows that

$$\|g(u(\cdot)) - g(u(\cdot))\| \leqslant l_g \|u(\cdot) - v(\cdot)\|_{C([0,1],E)},$$

where $l_g := .c_0 \nu$. Let $M = \sup_{(t,s)\in\Delta} \|\mathcal{T}(t,s)\|_{\mathfrak{L}(E)}$. If we choose $c_0 > 0$ such that $Mc_0\nu < 1$ and

$$\limsup_{\sigma\to\infty} \frac{M}{\sigma}\left(\|g(0)\| + \int_0^1 \xi(s,\sigma)ds\right) < 1 - MC_0\nu,$$

then $Ml_g < 1$ and (3.4) holds.

Next if we put

$$F(t, u(t), (\mathfrak{C}u)(t))(\cdot) = f(t, u(t)(\cdot), (\mathfrak{C}u)(t)(\cdot)), \quad t \in [0,1],$$

then (4.1) can be written in the abstract form

$$\begin{cases} u'(t) = A(t)u(t) + F(t, u(t), (\mathfrak{C}u)(t)), & t \in [0,1], \\ u(0) = g(u(\cdot)). \end{cases} \tag{4.2}$$

If we assume that (H4-d) and (3.6) hold, then by Theorem 3.1 it follows that (4.2) has at last a mild solution on $[0,1]$.

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References

[1] R. P. Agarwal, S. Arshad, V. Lupulescu, D. O'Regan, *Evolution equations with causal operators*, Differ. Equ. Appl., **7** (2015), 15–26. 1
[2] R. P. Agarwal, J. Banaś, B. C. Dhage, S. D. Sarkate, *Attractivity results for a nonlinear functional integral equation*, Georgian Math. J., **18** (2011), 1–19. 1
[3] S. Aizicovici, H.-W. Lee, *onlinear nonlocal Cauchy problems in Banach spaces*, Appl. Math. Lett., **18** (2005), 401–407. 1
[4] S. Aizicovici, M. McKibben, *Existence results for a class of abstract nonlocal Cauchy problems*, Nonlinear Anal., **39** (2000), 649–668.
[5] S. Aizicovici, V. Staicu, *Multivalued evolution equations with nonlocal initial conditions in Banach spaces*, NoDEA Nonlinear Differential Equations Appl., **14** (2007), 361–376.
[6] K. Balachandran, J. J. Trujillo, *The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces*, Nonlinear Anal., **72** (2010), 4587–4593. 1
[7] D. Baleanu, H. Jafari, H. Khan, S. J. Johnston, *Results for mild solution of fractional coupled hybrid boundary value problems*, Open Math., **13** (2015), 601–608. 1
[8] M. Benchohra, E. P. Gatsori, S. K. Ntouyas, *Existence results for semi-linear integrodifferential inclusions with nonlocal conditions*, Rocky Mountain J. Math., **34** (2004), 833–848. 1
[9] M. Benchohra, J. Henderson, S. K. Ntouyas, *Existence results for impulsive multivalued semilinear neutral functional differential inclusions in Banach spaces*, J. Math. Anal. Appl., **263** (2001), 763–780. 1
[10] M. Benchohra, G. M. N'Guérékata, D. Seba, *Measure of noncompactness and nondensely defined semilinear functional differential equations with fractional order*, Cubo, **12** (2010), 35–48. 1
[11] I. Benedetti, N. V. Loi, L. Malaguti, *Nonlocal problems for differential inclusions in Hilbert spaces*, Set-Valued Var. Anal., **22** (2014), 639–656. 1
[12] I. Benedetti, L. Malaguti, V. Taddei, I. I. Vrabie, *Semilinear delay evolution equations with measures subjected to nonlocal initial conditions*, Ann. Mat. Pura Appl., **195** (2016), 1639–1658. 1
[13] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., **162** (1991), 494–505. 1
[14] L. Byszewski, V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal., **40** (1990), 11–19. 1
[15] T. Cardinali, R. Precup, P. Rubbioni, *A unified existence theory for evolution equations and systems under nonlocal conditions*, J. Math. Anal. Appl., **432** (2015), 1039–1057. 1

[16] T. Cardinali, P. Rubbioni, *On the existence of mild solutions of semilinear evolution differential inclusions*, J. Math. Anal. Appl., **308** (2005), 620–635. 1
[17] T. Cardinali, P. Rubbioni, *Corrigendum and addendum to "On the existence of mild solutions of semilinear evolution differential inclusions"*, [J. Math. Anal. Appl., **308** (2005), 620–635], J. Math. Anal. Appl., **438** (2016), 514–517. 1, 3
[18] P.-Y. Chen, Y.-X. Li, Q. Li, *Existence of mild solutions for fractional evolution equations with nonlocal initial conditions*, Ann. Polon. Math., **110** (2014), 13–24. 1
[19] P.-Y. Chen, Y.-X. Li, *Nonlocal problem for fractional evolution equations of mixed type with the measure of noncompactness*, Abstr. Appl. Anal., **2013** (2013), 12 pages. 1
[20] C. Corduneanu, *Functional equations with causal operators*, Stability and Control: Theory, Methods and Applications, Taylor & Francis, London, (2002). 1
[21] R. F. Curtain, H. Zwart, *An introduction to infinite-dimensional linear systems theory*, Texts in Applied Mathematics, Springer-Verlag, New York, (1995). 4
[22] A. Debbouche, D. Baleanu, R. P. Agarwal, *Nonlocal nonlinear integrodifferential equations of fractional orders*, Bound. Value Probl., **2012** (2012), 10 pages. 1
[23] R. E. Edwards, *Functional analysis*, Theory and applications, Holt, Rinehart and Winston, New York-Toronto-London, (1965). 4
[24] Z.-B. Fan, G. Li, *Existence results for semilinear differential equations with nonlocal and impulsive conditions*, J. Funct. Anal., **258** (2010), 1709–1727. 1
[25] X.-L. Fu, K. Ezzinbi, *Existence of solutions for neutral functional differential evolution equations with nonlocal conditions*, Nonlinear Anal., **54** (2003), 215–227. 1
[26] M. A. E. Herzallah, D. Baleanu, *Existence of a periodic mild solution for a nonlinear fractional differential equation*, Comput. Math. Appl., **64** (2012), 3059–3064. 1
[27] L.-Y. Hu, Y. Ren, R. Sakthivel, *Existence and uniqueness of mild solutions for semilinear integro-differential equations of fractional order with nonlocal initial conditions and delays*, Semigroup Forum, **79** (2009), 507–514. 1
[28] S.-C. Ji, G. Li, *Existence results for impulsive differential inclusions with nonlocal conditions*, Comput. Math. Appl., **62** (2011), 1908–1915. 1
[29] J.-F. Jiang, D.-Q. Cao, H.-T. Chen, *The fixed point approach to the stability of fractional differential equations with causal operators*, Qual. Theory Dyn. Syst., **15** (2016), 3–18. 1
[30] J.-F. Jiang, C. F. Li, D.-Q. Cao, H.-T. Chen, *Existence and uniqueness of solution for fractional differential equation with causal operators in Banach spaces*, Mediterr. J. Math., **12** (2015), 751–769. 1
[31] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing multivalued maps and semilinear differential inclusions in Banach spaces*, De Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter & Co., Berlin, (2001). 2, 2
[32] T. D. Ke, V. Obukhovskii, N.-C. Wong, J.-C. Yao, *On semilinear integro-differential equations with nonlocal conditions in Banach spaces*, Abstr. Appl. Anal., **2012** (2012), 26 pages. 1
[33] M. Kisielewicz, *Multivalued differential equations in separable Banach spaces*, J. Optim. Theory Appl., **37** (1982), 231–249. 2.2
[34] K. Kuratowski, *Sur les espaces complets*, Fund. Math., **15** (1930), 301–309. 2
[35] V. G. Kurbatov, *Functional-differential operators and equations*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, (1999). 1
[36] V. Lakshmikantham, S. Leela, Z. Drici, F. A. McRae, *Theory of causal differential equations*, Atlantis Studies in Mathematics for Engineering and Science, Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2009). 1
[37] J. Liang, J. H. Liu, T.-J. Xiao, *Nonlocal Cauchy problems for nonautonomous evolution equations*, Commun. Pure Appl. Anal., **5** (2006), 529–535. 1
[38] Y. P. Lin, J. H. Liu, *Semilinear integrodifferential equations with nonlocal Cauchy problem*, Nonlinear Anal., **26** (1996), 1023–1033. 1
[39] Q. Liu, R. Yuan, *Existence of mild solutions for semilinear evolution equations with non-local initial conditions*, Nonlinear Anal., **71** (2009), 4177–4184. 1
[40] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Verlag, Basel, (1995). 2
[41] V. Lupulescu, *Causal functional differential equations in Banach spaces*, Nonlinear Anal., **69** (2008), 4787–4795. 4
[42] M. Mahdavi, Y.-Z. Li, *Linear and quasilinear equations with abstract Volterra operators*, Volterra equations and applications, Arlington, TX, (1996), 325–330, Stability Control Theory Methods Appl., Gordon and Breach, Amsterdam, (2000). 1
[43] G. M. Mophou, *Existence and uniqueness of mild solutions to impulsive fractional differential equations*, Nonlinear Anal., **72** (2010), 1604–1615. 1
[44] G. M. Mophou, G. M. N'Guérékata, *Existence of the mild solution for some fractional differential equations with nonlocal conditions*, Semigroup Forum, **79** (2009), 315–322.
[45] G. M. Mophou, G. M. N'Guérékata, *Mild solutions for semilinear fractional differential equations*, Electron. J. Differential Equations, **2009** (2009), 9 pages.
[46] G. M. N'Guérékata, *A Cauchy problem for some fractional abstract differential equation with non local conditions*, Nonlinear Anal., **70** (2009), 1873–1876. 1

[47] S. K. Ntouyas, P. Ch. Tsamatos, *Global existence for semilinear evolution equations with nonlocal conditions*, J. Math. Anal. Appl., **210** (1997), 679–687. 1

[48] A. Paicu, I. I. Vrabie, *A class of nonlinear evolution equations subjected to nonlocal initial conditions*, Nonlinear Anal., **72** (2010), 4091–4100. 1

[49] N. S. Papageorgiou, *Mild solutions of semilinear evolution inclusions and optimal control*, Indian J. Pure Appl. Math., **26** (1995), 189–216. 3, 3

[50] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, Springer-Verlag, New York, (1983). 2, 4

[51] H. Tanabe, *Equations of evolution*, Translated from the Japanese by N. Mugibayashi and H. Haneda, Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, Mass.-London, (1979). 4

[52] I. I. Vrabie, C_0*-semigroups and applications*, North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, (2003). 4

[53] R.-N. Wang, K. Ezzinbi, P.-X. Zhu, *Non-autonomous impulsive Cauchy problems of parabolic type involving nonlocal initial conditions*, J. Integral Equations Appl., **26** (2014), 275–299. 4

[54] J.-R. Wang, Y. Zhou, M. Fečkan, *Abstract Cauchy problem for fractional differential equations*, Nonlinear Dynam., **71** (2013), 685–700. 1

[55] S. Xie, *Existence of solutions for nonlinear mixed type integrodifferential functional evolution equations with nonlocal conditions*, Abstr. Appl. Anal., **2012** (2012), 11 pages. 1

[56] X.-M. Xue, *Nonlocal nonlinear differential equations with a measure of noncompactness in Banach spaces*, Nonlinear Anal., **70** (2009), 2593–2601. 1

[57] A. Yagi, *Abstract parabolic evolution equations and their applications*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, (2010). 2

[58] Y.-L. Yang, J.-R. Wang, *On some existence results of mild solutions for nonlocal integrodifferential Cauchy problems in Banach spaces*, Opuscula Math., **31** (2011), 443–455. 1

[59] T. Zhu, C. Song, G. Li, *Existence of mild solutions for abstract semilinear evolution equations in Banach spaces*, Nonlinear Anal., **75** (2012), 177–181. 1