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Integral inequalities of extended Simpson type for (α, m)-ε-convex functions

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Abstract

In the paper, the authors establish some integral inequalities of extended Simpson type for (α, m)-ε-convex functions.

Keywords: Integral inequality, extended Simpson type, (α, m)-ε-convex function.

2010 MSC: 26A51, 26D15, 41A55.

1. Introduction

The following definition is well-known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f(tx + (1 - t)y) \leqslant tf(x) + (1 - t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2 ([4]). Let X be a real linear space, $D \subseteq X$ a convex set, and $f : D \to \mathbb{R}$ a mapping on D. For any constant $\varepsilon \geqslant 0$, the mapping $f(x)$ is said to be ε-convex on D, if it satisfies

$$f(tx + (1 - t)y) \leqslant tf(x) + (1 - t)f(y) + \varepsilon,$$

for all $x, y \in D$ and $t \in [0, 1]$.

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In [11] the concept of m-convex functions was innovated as follows.

Definition 1.3 ([11])**.** For $f : [0, b] \to \mathbb{R}$ with $b > 0$ and $m \in (0, 1]$, if

$$f(tx + m(1-t)y) \leqslant tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m-convex function on $[0, b]$.

Definition 1.4 ([7])**.** For $f : [0, b] \to \mathbb{R}$ with $b > 0$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(\lambda x + m(1-\lambda)y) \leqslant \lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b]$.

The following inequalities are known.

Theorem 1.5 ([1, Theorem 2.2])**.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.6 ([8, Theorems 1 and 2])**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$ and $q \geqslant 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}.$$

Theorem 1.7 ([2])**.** *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be m-convex and $m \in (0, 1]$. If $f \in L_1([a, b])$ for $0 \leqslant a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\right\}.$$

For more information on this topic, please refer to the papers [3, 5, 6, 9, 10, 12–15] and closely-related references therein.

In this paper, we will introduce a new concept "(α, m)-ε-convex function" and establish some integral inequalities of the Simpson type for (α, m)-ε-convex functions.

2. Definition and lemmas

Now we give a definition of the so-called (α, m)-ε-convex functions.

Definition 2.1. For $f : [0, b^*] \to \mathbb{R}$, $(\alpha, m) \in (0, 1]^2$, and $\varepsilon \geqslant 0$, the function f is said to be (α, m)-ε-convex on I, if

$$f(tx + m(1-t)y) \leqslant t^\alpha f(x) + m(1-t^\alpha)f(y) + \varepsilon$$

holds for all $x, y \in [0, b^*]$ and $t \in [0, 1]$.

Remark 2.2. We give two remarks as follows.

1. If f is (α, m)-ε-convex on $[0, b^*]$ and $\alpha = 1$, then we say that f is m-ε-convex on $[0, b^*]$.
2. If f is (α, m)-ε-convex on $[0, b^*]$ and $\alpha = m = 1$, then it is ε-convex on $[0, b^*]$.

To establish some new extended Simpson type inequalities for (α, m)-ε-convex functions, we need the following lemmas.

Lemma 2.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a, b])$, $\lambda \geqslant 0$, and $n \in \mathbb{N}_+$, then*

$$\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1} f(x_{2k})+\lambda\sum_{k=1}^{n} f(x_{2k-1})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx$$
$$=\frac{b-a}{4n^2}\left[\sum_{k=1}^{n}\int_0^1\left(\frac{\lambda}{\lambda+2}-t\right)f'(tx_{2k-2}+(1-t)x_{2k-1})\,dt+\sum_{k=1}^{n}\int_0^1\left(\frac{2}{\lambda+2}-t\right)f'(tx_{2k-1}+(1-t)x_{2k})\,dt\right].$$

In particular,

1. *if $\lambda = 4$, then*

$$\frac{1}{6n}\left[f(a)+2\sum_{k=1}^{n-1} f(x_{2k})+4\sum_{k=1}^{n} f(x_{2k-1})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx$$
$$=\frac{b-a}{4n^2}\left[\sum_{k=1}^{n}\int_0^1\left(\frac{2}{3}-t\right)f'(tx_{2k-2}+(1-t)x_{2k-1})\,dt+\sum_{k=1}^{n}\int_0^1\left(\frac{1}{3}-t\right)f'(tx_{2k-1}+(1-t)x_{2k})\,dt\right];$$

2. *if $\lambda = 0$, then*

$$\frac{1}{2n}\left[f(a)+2\sum_{k=1}^{n-1} f(x_{2k})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx$$
$$=\frac{b-a}{4n^2}\left[\sum_{k=1}^{n}\int_0^1(-t)f'(tx_{2k-2}+(1-t)x_{2k-1})\,dt+\sum_{k=1}^{n}\int_0^1(1-t)f'(tx_{2k-1}+(1-t)x_{2k})\,dt\right],$$

where $x_k = a + \frac{k(b-a)}{2n}$ for $k = 0, 1, \dots, 2n$.

Proof. By integration by parts, the result is followed immediately. □

By taking $n = 1$ in Lemma 2.3, we have the following identities.

Lemma 2.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a, b])$ and $\lambda \geqslant 0$, then*

$$\frac{1}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx$$
$$=\frac{b-a}{4}\left[\int_0^1\left(\frac{\lambda}{\lambda+2}-t\right)f'\left(ta+(1-t)\frac{a+b}{2}\right)dt+\int_0^1\left(\frac{2}{\lambda+2}-t\right)f'\left(t\frac{a+b}{2}+(1-t)b\right)dt\right].$$

Letting $\lambda = 0$ in Lemma 2.4, we can obtain

Lemma 2.5 ([1, p. 91, Lemma 2.1])**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I°. If $f' \in L_1([a, b])$ for $a, b \in I$ with $a < b$, then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)\,dt.$$

3. Some new integral inequalities of the Simpson type

In this section, integral inequalities of the Simpson type related to (α, m)-ε-convex functions are discussed.

Theorem 3.1. *Let $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, $a, b \in \mathbb{R}_0$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ is (α, m)-ε-convex on $\left[0, \frac{b}{m}\right]$ for $\varepsilon \geqslant 0$, $\lambda \geqslant 0$, $(\alpha, m) \in (0, 1]^2$, and $q \geqslant 1$, then*

$$\left|\frac{1}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{4}C^{1-1/q}(\lambda)\left\{\left[A(\alpha,\lambda)|f'(a)|^q+m(C(\lambda)-A(\alpha,\lambda))\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon C(\lambda)\right]^{1/q} + \left[B(\alpha,\lambda)\left|f'\left(\frac{a+b}{2}\right)\right|^q+m(C(\lambda)-B(\alpha,\lambda))\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon C(\lambda)\right]^{1/q}\right\},$$

where

$$A(\alpha,\lambda)=\frac{[2\alpha(\lambda+2)+4](\lambda+2)^\alpha+\lambda^2[2\lambda^\alpha-(\lambda+2)^\alpha]}{(\alpha+1)(\alpha+2)(\lambda+2)^{\alpha+2}}, \tag{3.1}$$

$$B(\alpha,\lambda)=\frac{(\alpha\lambda+\lambda-2)(\lambda+2)^{\alpha+1}+2^{\alpha+3}}{(\alpha+1)(\alpha+2)(\lambda+2)^{\alpha+2}}, \quad \text{and} \quad C(\lambda)=\frac{\lambda^2+4}{2(\lambda+2)^2}. \tag{3.2}$$

Proof. Since $|f'|^q$ is an (α, m)-ε-convex function on $\left[0, \frac{b}{m}\right]$, from Lemma 2.4 and Hölder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leqslant \frac{b-a}{4}\left[\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt+\int_0^1\left|t-\frac{2}{\lambda+2}\right|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt\right] \\
&\leqslant \frac{b-a}{4}\left\{\left(\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|dt\right)^{1-1/q}\left[\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|\left(t^\alpha|f'(a)|^q\right.\right.\right. \\
&\quad \left.\left.+m(1-t^\alpha)\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon\right)dt\right]^{1/q}+\left(\int_0^1\left|t-\frac{2}{\lambda+2}\right|dt\right)^{1-1/q} \\
&\quad \left.\times\left[\int_0^1\left|t-\frac{2}{\lambda+2}\right|\left(t^\alpha\left|f'\left(\frac{a+b}{2}\right)\right|^q+m(1-t^\alpha)\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon\right)dt\right]^{1/q}\right\} \\
&= \frac{b-a}{4}C^{1-1/q}(\lambda)\left\{\left[A(\alpha,\lambda)|f'(a)|^q+m(C(\lambda)-A(\alpha,\lambda))\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon C(\lambda)\right]^{1/q}\right. \\
&\quad \left.+\left[B(\alpha,\lambda)\left|f'\left(\frac{a+b}{2}\right)\right|^q+m(C(\lambda)-B(\alpha,\lambda))\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon C(\lambda)\right]^{1/q}\right\}.
\end{aligned}$$

The proof of Theorem 3.1 is thus completed. □

Corollary 3.2. *Under the assumptions of Theorem 3.1, if $\lambda = 0$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{8}\left\{\left[\frac{2}{\alpha+2}|f'(a)|^q+\frac{\alpha m}{\alpha+2}\left|f'\left(\frac{a+b}{2m}\right)\right|^q+\varepsilon\right]^{1/q} + \left[\frac{2}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\alpha m(\alpha+2)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon\right]^{1/q}\right\}.$$

Theorem 3.3. *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, $a, b \in \mathbb{R}_0$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ is (α, m)-ε-convex on $\left[0, \frac{b}{m}\right]$ for $\varepsilon \geqslant 0$, $(\alpha, m) \in (0, 1]^2$, and $q \geqslant 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{4}\left[\frac{2(2^\alpha\alpha+1)|f'(a)|^q + m(2^\alpha(\alpha^2+\alpha+2)-2)|f'(b/m)|^q}{2^{\alpha+1}(\alpha+1)(\alpha+2)} + \frac{\varepsilon}{2}\right]^{1/q}.$$

Proof. Since $|f'|^q$ is an (α, m)-ε-convex function on $\left[0, \frac{b}{m}\right]$, by Lemma 2.5 and Hölder's integral inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leqslant \frac{b-a}{2}\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|^q dt \\ &\leqslant \frac{b-a}{2}\left[\int_0^1 |1-2t|\,dt\right]^{1-1/q} \\ &\quad \times \left[\int_0^1 |1-2t|\left[t^\alpha|f'(a)|^q + m(1-t^\alpha)|f'(b/m)|^q + \varepsilon\right]dt\right]^{1/q} \\ &= \frac{b-a}{4}\left[\frac{2(2^\alpha\alpha+1)|f'(a)|^q + m(2^\alpha(\alpha^2+\alpha+2)-2)|f'(b/m)|^q}{2^{\alpha+1}(\alpha+1)(\alpha+2)} + \frac{\varepsilon}{2}\right]^{1/q}.\end{aligned}$$

Theorem 3.1 is thus proved. □

Theorem 3.4. *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, $a, b \in \mathbb{R}_0$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ is (α, m)-ε-convex on $\left[0, \frac{b}{m}\right]$ for $\varepsilon \geqslant 0$, $\lambda \geqslant 0$, $(\alpha, m) \in (0, 1]^2$, and $q > 1$, then*

$$\begin{aligned}&\left|\frac{1}{\lambda+2}\left[f(a) + \lambda f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{4}\left[\frac{\lambda^{\frac{2q-1}{q-1}} + 2^{\frac{2q-1}{q-1}}}{(\lambda+2)^{\frac{2q-1}{q-1}}}\right]^{1-1/q} \\ &\quad \times \left\{\left[\frac{1}{\alpha+1}|f'(a)|^q + \frac{\alpha m}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q + \varepsilon\right]^{1/q} + \left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{\alpha m}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q + \varepsilon\right]^{1/q}\right\}.\end{aligned}$$

Proof. From Lemma 2.4 and Hölder's integral inequality, by the (α, m)-ε-convexity of $|f'|^q$, we have

$$\begin{aligned}&\left|\frac{1}{\lambda+2}\left[f(a) + \lambda f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leqslant \frac{b-a}{4}\left\{\left(\int_0^1 \left|t - \frac{\lambda}{\lambda+2}\right|^{q/(q-1)} dt\right)^{1-1/q}\left[\int_0^1 \left(t^\alpha|f'(a)|^q + m(1-t^\alpha)\left|f'\left(\frac{a+b}{2m}\right)\right|^q + \varepsilon\right)dt\right]^{1/q}\right. \\ &\quad \left. + \left(\int_0^1 \left|t - \frac{2}{\lambda+2}\right|^{q/(q-1)} dt\right)^{1-1/q}\left[\int_0^1 \left(t^\alpha\left|f'\left(\frac{a+b}{2}\right)\right|^q + m(1-t^\alpha)\left|f'\left(\frac{b}{m}\right)\right|^q + \varepsilon\right)dt\right]^{1/q}\right\} \\ &= \frac{b-a}{4}\left[\frac{\lambda^{\frac{2q-1}{q-1}} + 2^{\frac{2q-1}{q-1}}}{(\lambda+2)^{\frac{2q-1}{q-1}}}\right]^{1-1/q}\left\{\left[\frac{1}{\alpha+1}|f'(a)|^q + \frac{\alpha m}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q + \varepsilon\right]^{1/q}\right. \\ &\quad \left. + \left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{\alpha m}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q + \varepsilon\right]^{1/q}\right\}.\end{aligned}$$

Theorem 3.4 is thus proved. □

Corollary 3.5. *Under the assumptions of Theorem 3.4, if $\lambda = 0$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leqslant \frac{b-a}{4}\left\{\left[\frac{1}{\alpha+1}|f'(a)|^q + \frac{\alpha m}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q + \varepsilon\right]^{1/q}\right. \\ &\quad \left. + \left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{\alpha m}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q + \varepsilon\right]^{1/q}\right\}.\end{aligned}$$

Theorem 3.6. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is (α,m)-ε-convex on $\left[0,\frac{b}{m}\right]$ for $\varepsilon\geqslant 0$, $(\alpha,m)\in(0,1]^2$, and $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leqslant\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\frac{|f'(a)|^q+\alpha m|f'(b/m)|^q}{\alpha+1}+\varepsilon\right]^{1/q}.$$

Proof. Since $|f'|^q$ is an (α,m)-ε-convex function on $\left[0,\frac{b}{m}\right]$, from Lemma 2.5 and Hölder's integral inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|&\leqslant\frac{b-a}{2}\int_0^1|1-2t|\big|f'\big(ta+(1-t)b\big)\big|^q\,dt\\&\leqslant\frac{b-a}{2}\left[\int_0^1|1-2t|^{q/(q-1)}\,dt\right]^{1-1/q}\\&\quad\times\left(\int_0^1\left[t^\alpha|f'(a)|^q+m(1-t^\alpha)\left|f'\left(\frac{b}{m}\right)\right|^q+\varepsilon\right]dt\right)^{1/q}\\&=\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\frac{|f'(a)|^q+\alpha m|f'(b/m)|^q}{\alpha+1}+\varepsilon\right]^{1/q}.\end{aligned}$$

Theorem 3.6 is thus proved. □

Theorem 3.7. *For $n\in\mathbb{N}_+$ and $n\geqslant 2$, let $f:\mathbb{R}_0\to\mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, $a,b\in\mathbb{R}_0$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|$ is (α,m)-ε-convex on $\left[0,\frac{b}{m}\right]$ for $\varepsilon\geqslant 0$, $\lambda\geqslant 0$, and $(\alpha,m)\in(0,1]^2$, then*

$$\begin{aligned}&\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f(x_{2k})+\lambda\sum_{k=1}^{n}f(x_{2k-1})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leqslant\frac{b-a}{4n^2}\sum_{k=1}^{n}\left[A(\alpha,\lambda)\big|f'(x_{2k-2})\big|+m(C(\lambda)-A(\alpha,\lambda))\left|f'\left(\frac{x_{2k-1}}{m}\right)\right|\right.\\&\quad\left.+B(\alpha,\lambda)\big|f'(x_{2k-1})\big|+m(C(\lambda)-B(\alpha,\lambda))\left|f'\left(\frac{x_{2k}}{m}\right)\right|+2\varepsilon C(\lambda)\right],\end{aligned}$$

where $x_k=a+\frac{k(b-a)}{2n}$ for $k=0,1,\dots,2n$ and $A(\alpha,\lambda)$, $B(\alpha,\lambda)$, and $C(\alpha)$ are defined by (3.1) and (3.2), respectively.

Proof. Since $|f'|$ is an (α,m)-ε-convex function on $\left[0,\frac{b}{m}\right]$, from Lemma 2.3 and Hölder's integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f(x_{2k})+\lambda\sum_{k=1}^{n}f(x_{2k-1})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leqslant\frac{b-a}{4n^2}\left[\sum_{k=1}^{n}\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|\big|f'(tx_{2k-2}+(1-t)x_{2k-1})\big|\,dt\right.\\&\quad\left.+\sum_{k=1}^{n}\int_0^1\left|t-\frac{2}{\lambda+2}\right|\big|f'(tx_{2k-1}+(1-t)x_{2k})\big|\,dt\right]\\&\leqslant\frac{b-a}{4n^2}\left\{\sum_{k=1}^{n}\int_0^1\left|t-\frac{\lambda}{\lambda+2}\right|\left[t^\alpha\big|f'(x_{2k-2})\big|+m(1-t^\alpha)\left|f'\left(\frac{x_{2k-1}}{m}\right)\right|+\varepsilon\right]dt\right.\\&\quad\left.+\sum_{k=1}^{n}\int_0^1\left|t-\frac{2}{\lambda+2}\right|\left[t^\alpha\big|f'(x_{2k-1})\big|+m(1-t^\alpha)\left|f'\left(\frac{x_{2k}}{m}\right)\right|+\varepsilon\right]dt\right\}\end{aligned}\tag{3.3}$$

$$
\begin{aligned}
&= \frac{b-a}{4n^2}\sum_{k=1}^{n}\bigg[A(\alpha,\lambda)\big|f'(x_{2k-2})\big| + m(C(\lambda)-A(\alpha,\lambda))\bigg|f'\bigg(\frac{x_{2k-1}}{m}\bigg)\bigg| \\
&\quad + B(\alpha,\lambda)\big|f'(x_{2k-1})\big| + m(C(\lambda)-B(\alpha,\lambda))\bigg|f'\bigg(\frac{x_{2k}}{m}\bigg)\bigg| + 2\varepsilon C(\lambda)\bigg].
\end{aligned}
$$

The proof of Theorem 3.7 is thus completed. □

Corollary 3.8. *Under the assumptions of Theorem 3.4, then*

$$
\begin{aligned}
&\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f(x_{2k})+\lambda\sum_{k=1}^{n}f(x_{2k-1})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad \leqslant \frac{b-a}{4n}\Big\{\big[(1-m)(A(\alpha,\lambda)+B(\alpha,\lambda))+2mC(\lambda)\big]\sup_{a\leqslant x\leqslant b/m}|f'(x)|+2\varepsilon C(\lambda)\Big\}.
\end{aligned}
$$

Theorem 3.9. *For $n \in \mathbb{N}_+$ and $n \geqslant 2$, let $f : \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, $a, b \in \mathbb{R}_0$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|$ is m-ε-convex on $\left[0, \frac{b}{m^2}\right]$ for $\varepsilon \geqslant 0$ and $m \in (0, 1]$, then*

$$
\begin{aligned}
&\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f(x_{2k})+\lambda\sum_{k=1}^{n}f(x_{2k-1})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad \leqslant \frac{b-a}{8n^2}\bigg[\frac{3n(\lambda^2+4)(\lambda+2)+\lambda^3+12\lambda+16}{6(\lambda+2)^3}|f'(a)| \\
&\qquad + m\frac{3n(\lambda^2+4)(\lambda+2)-(\lambda^3+12\lambda+16)}{6(\lambda+2)^3}\bigg(\bigg|f'\bigg(\frac{a}{m}\bigg)\bigg|+\bigg|f'\bigg(\frac{b}{m}\bigg)\bigg|\bigg) \\
&\qquad + m^2\frac{3n(\lambda^2+4)(\lambda+2)+\lambda^3+12\lambda+16}{6(\lambda+2)^3}\bigg|f'\bigg(\frac{b}{m^2}\bigg)\bigg|+\frac{2\varepsilon(\lambda^2+4)}{(\lambda+2)^2}\bigg].
\end{aligned}
$$

Proof. By the equation (3.3) and the m-ε-convexity of the function $|f'|$, we have

$$
\begin{aligned}
&\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f(x_{2k})+\lambda\sum_{k=1}^{n}f(x_{2k-1})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad \leqslant \frac{b-a}{4n^2}\sum_{k=1}^{n}\bigg[A(\alpha,\lambda)\big|f'(x_{2k-2})\big| + m(C(\lambda)-A(\alpha,\lambda))\bigg|f'\bigg(\frac{x_{2k-1}}{m}\bigg)\bigg| \\
&\qquad + B(\alpha,\lambda)\big|f'(x_{2k-1})\big| + m(C(\lambda)-B(\alpha,\lambda))\bigg|f'\bigg(\frac{x_{2k}}{m}\bigg)\bigg| + 2\varepsilon C(\lambda)\bigg] \\
&\quad \leqslant \frac{b-a}{4n^2}\sum_{k=1}^{n}\bigg\{\frac{\lambda^3+12\lambda+16}{6(\lambda+2)^3}\bigg[\frac{2n-(2k-2)}{2n}|f'(a)|+\frac{m(2k-2)}{2n}\bigg|f'\bigg(\frac{b}{m}\bigg)\bigg|\bigg] \\
&\qquad + \frac{m(\lambda^3+3\lambda^2+4)}{3(\lambda+2)^3}\bigg[\frac{2n-(2k-1)}{2n}\bigg|f'\bigg(\frac{a}{m}\bigg)\bigg|+\frac{m(2k-1)}{2n}\bigg|f'\bigg(\frac{b}{m^2}\bigg)\bigg|\bigg] \\
&\qquad + \frac{\lambda^3+3\lambda^2+4}{3(\lambda+2)^3}\bigg[\frac{2n-(2k-1)}{2n}|f'(a)|+\frac{m(2k-1)}{2n}\bigg|f'\bigg(\frac{b}{m}\bigg)\bigg|\bigg] \\
&\qquad + \frac{m(\lambda^3+12\lambda+16)}{6(\lambda+2)^3}\bigg[\frac{2n-2k}{2n}\bigg|f'\bigg(\frac{a}{m}\bigg)\bigg|+\frac{2km}{2n}\bigg|f'\bigg(\frac{b}{m^2}\bigg)\bigg|\bigg]+\frac{2\varepsilon(\lambda^2+4)}{2(\lambda+2)^2}\bigg\} \\
&\quad = \frac{b-a}{8n^2}\bigg[\frac{3n(\lambda^2+4)(\lambda+2)+\lambda^3+12\lambda+16}{6(\lambda+2)^3}|f'(a)| \\
&\qquad + m\frac{3n(\lambda^2+4)(\lambda+2)-(\lambda^3+12\lambda+16)}{6(\lambda+2)^3}\bigg(\bigg|f'\bigg(\frac{a}{m}\bigg)\bigg|+\bigg|f'\bigg(\frac{b}{m}\bigg)\bigg|\bigg)
\end{aligned}
$$

$$+m^2\frac{3n(\lambda^2+4)(\lambda+2)+\lambda^3+12\lambda+16}{6(\lambda+2)^3}\left|f'\left(\frac{b}{m^2}\right)\right|+\frac{2\varepsilon(\lambda^2+4)}{(\lambda+2)^2}\Bigg].$$

The proof of Theorem 3.9 is thus complete. □

Corollary 3.10. *Under the assumptions of Theorem 3.4,*

1. *if* $\lambda = 4$, *then*

$$\left|\frac{1}{n(\lambda+2)}\left[f(a)+2\sum_{k=1}^{n-1}f(x_{2k})+4\sum_{k=1}^{n}f(x_{2k-1})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leqslant \frac{b-a}{1536n^2}\left[(45n+16)|f'(a)|+m(45n-16)\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)+m^2(45n+16)\left|f'\left(\frac{b}{m^2}\right)\right|+180\varepsilon\right];$$

2. *if* $\lambda = 0$, *then*

$$\left|\frac{1}{2n}\left[f(a)+2\sum_{k=1}^{n-1}f(x_{2k})+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leqslant \frac{b-a}{48n^2}\left[(3n+2)|f'(a)|+m(3n-2)\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)+m^2(3n+2)\left|f'\left(\frac{b}{m^2}\right)\right|+12\varepsilon\right].$$

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