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Almost e^*-continuous functions and their characterizations

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Abstract

The main goal of this paper is to introduce and investigate a new class of functions called almost e^*-continuous functions containing the class of almost e-continuous functions defined by Özkoç and Kına. Several characterizations concerning almost e^*-continuous functions are obtained. Furthermore, we investigate the relationships between almost e^*-continuous functions and separation axioms and almost e^*-closedness of graphs of functions.

Keywords: e^*-open, e^*-continuity, almost e^*-continuity, weakly e^*-continuity, faintly e^*-continuity, e^*-closed graph.
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1. Introduction

The notion of continuity on topological spaces, as significant and fundamental subject in the study of topology, has been researched by many mathematicians. Several investigations related to almost continuity which is a generalization of continuity have been published. The study of almost continuity was initiated by Singal and Singal [17] in 1968. Next, almost α-continuous functions were introduced and investigated by Thakur and Paik [19]. In 1998, Noiri and Popa [11] defined the concept of almost β-continuous functions. Later, almost b-continuous functions introduced by Şengül [15] and almost e-continuous functions defined by Özkoç and Kına [12] and studied these concept.

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In this paper, we introduce some new types of continuity such as almost e^*-continuity, weakly e^*-continuity and faintly e^*-continuity via e^*-open set defined by Ekici [5]. Also we look into several properties of these new concepts. Moreover, we investigate not only some of their properties but also their relationships with other types of already existing topological function.

2. Preliminaries

In this section let us recall some definitions and results which are used in this paper. All through this paper, X and Y represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. The family of all closed (open) sets of X is denoted by $C(X)$ $(O(X))$. A subset A is said to be regular open [18] (resp. regular closed [18]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A point $x \in X$ is said to be δ-cluster point [20] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighborhood U of x. The set of all δ-cluster points of A is called the δ-closure [20] of A and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then A is called δ-closed [20], and the complement of a δ-closed set is called δ-open [20]. The set $\{x|(U \in O(X,x))(int(cl(U)) \subset A)\}$ is called the δ-interior of A and is denoted by $int_\delta(A)$.

A subset A is called α-open [9] (resp. semiopen [7], δ-semiopen [13], preopen [8], δ-preopen [14], b-open [1], e-open [4], e^*-open [5], a-open [3]) if $A \subset int(cl(int(A)))$ (resp. $A \subset cl(int(A))$, $A \subset cl(int_\delta(A))$, $A \subset int(cl(A))$, $A \subset int(cl_\delta(A))$, $A \subset cl(int(A)) \cup int(cl(A))$, $A \subset cl(int_\delta(A)) \cup int(cl_\delta(A))$, $A \subset cl(int(cl_\delta(A)))$, $A \subset int(cl(int_\delta(A)))$). The complement of an α-open (resp. semiopen, δ-semiopen, preopen, δ-preopen, b-open, e-open, e^*-open, a-open) set is called α-closed [9] (resp. semiclosed [7], δ-semiclosed [13], preclosed [8], δ-preclosed [14], b-closed [1], e-closed [4], e^*-closed [5], a-closed [3]). The intersection of all e^*-closed (resp. a-closed, semi-closed, δ-semi-closed, pre-closed, δ-pre-closed) sets of X containing A is called the e^*-closure [5] (resp. a-closure [3], semi-closure [7], δ-semi-closure [13], pre-closure [8], δ-pre-closure [14]) of A and is denoted by e^*-$cl(A)$ (resp. a-$cl(A)$, $scl(A)$, δ-$scl(A)$, $pcl(A)$, δ-$pcl(A)$). The union of all e^*-open (resp. a-open, semiopen, δ-semiopen, preopen, δ-preopen) sets of X contained in A is called the e^*-interior [5] (resp. a-interior [3], semi-interior [7], δ-semi-interior [13], pre-interior [8], δ-pre-interior [14]) of A and is denoted by e^*-$int(A)$ (resp. a-$int(A)$, $sint(A)$, δ-$sint(A)$, $pint(A)$, δ-$pint(A)$).

A point x of X is called a θ-cluster [20] point of A if $cl(U) \cap A \neq \emptyset$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure [20] of A and is denoted by $cl_\theta(A)$. A subset A is said to be θ-closed [20] if $A = cl_\theta(A)$. The complement of a θ-closed set is called a θ-open [20] set. A point x of X is said to be a θ-interior [20] point of a subset A, denoted by $int_\theta(A)$, if there exists an open set U of X containing x such that $cl(U) \subset A$.

The family of all e^*-open (resp. e^*-closed, regular open, regular closed, δ-open, δ-closed, θ-open, θ-closed, semiopen, semiclosed, preopen, preclosed, δ-semiopen, δ-semiclosed, δ-preopen, δ-preclosed, a-open, a-closed) subsets of X is denoted by $e^*O(X)$ (resp. $e^*C(X)$, $RO(X)$, $RC(X)$, $\delta O(X)$, $\delta C(X)$, $\theta O(X)$, $\theta C(X)$, $SO(X)$, $SC(X)$, $PO(X)$, $PC(X)$, $\delta SO(X)$, $\delta SC(X)$, $\delta PO(X)$, $\delta PC(X)$), $aO(X)$, $aC(X)$). The family of all e^*-open (resp. e^*-closed, regular open, regular closed, δ-open, δ-closed, θ-open, θ-closed, semiopen, semiclosed, preopen, preclosed, δ-semiopen, δ-semiclosed, δ-preopen, δ-preclosed, a-open, a-closed) sets of X containing a point x of X is denoted by $e^*O(X,x)$ (resp. $e^*C(X,x)$, $RO(X,x)$, $RC(X,x)$, $\delta O(X,x)$, $\delta C(X,x)$, $\theta O(X,x)$, $\theta C(X,x)$, $SO(X,x)$, $SC(X,x)$, $PO(X,x)$, $PC(X,x)$, $\delta SO(X,x)$, $\delta SC(X,x)$, $\delta PO(X,x)$, $\delta PC(X,x)$, $aO(X,x)$, $aC(X,x)$).

The following basic properties of e^*-closure and e^*-interior are useful in the sequel:

Lemma 2.1 ([5])**.** *Let A be a subset of a space X, then the following hold:*

(a) e^*-$cl(X \setminus A) = X \setminus e^*$-$int(A)$;

(b) $x \in e^*\text{-}cl(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in e^*O(X, x)$;
(c) A *is* $e^*C(X)$ *if and only if* $A = e^*\text{-}cl(A)$;
(d) $e^*\text{-}cl(A) \in e^*C(X)$;
(e) $e^*\text{-}int(A) = A \cap cl(int(cl_\delta(A)))$.

Lemma 2.2 ([3, 10, 13])**.** *Let A be a subset of a space X, then the following properties hold:*
(a) *If A is preopen, then* $scl(A) = int(cl(A))$;
(b) $a\text{-}cl(A) = A \cup cl(int(cl_\delta(A)))$;
(c) $\delta\text{-}scl(A) = A \cup int(cl_\delta(A))$.

Lemma 2.3. *Let A be a subset of a space X. If $A \in e^*O(X)$, then $cl_\delta(A) \in RC(X)$.*

Proof. Let A be any e^*-open set of X.

$$\begin{aligned} A \in e^*O(X) &\Rightarrow A \subset cl(int(cl_\delta(A))) \\ &\Rightarrow cl_\delta(A) \\ &\subset cl_\delta(cl(int(cl_\delta(A)))) \\ &= cl_\delta(cl_\delta(int(cl_\delta(A)))) = cl_\delta(int(cl_\delta(A))) = cl(int(cl_\delta(A))) \ldots, \end{aligned} \tag{2.1}$$

$$A \subset X \Rightarrow int(cl_\delta(A)) \subset cl_\delta(A) \Rightarrow cl(int(cl_\delta(A))) \subset cl(cl_\delta(A)) \subset cl_\delta(cl_\delta(A)) = cl_\delta(A) \ldots, \tag{2.2}$$

$$(2.1), (2.2) \Rightarrow cl_\delta(A) = cl(int(cl_\delta(A))) \Rightarrow cl_\delta(A) \in RC(X).$$

□

3. Almost e^*continuous functions

Definition 3.1. A function $f : X \to Y$ is said to be

(1) almost e^*-continuous (briefly a.e^*.c.) at $x \in X$ if for each open set V containing $f(x)$, there exists an e^*-open set U in X containing x such that $f[U] \subset int(cl(V))$.
(2) weakly e^*-continuous (briefly w.e^*.c.) at $x \in X$ if for each open set V containing $f(x)$, there exists an e^*-open set U in X containing x such that $f[U] \subset cl(V)$.
(3) almost e^*-continuous (resp. weakly e^*-continuous) if it is almost e^*-continuous (resp. weakly e^*-continuous) at every point of X.
(4) e^*-continuous [5] (briefly e^*.c.) if $f^{-1}[V] \in e^*O(X)$ for each open set V of Y.

Theorem 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:*
(a) *f is a.e^*.c. at $x \in X$;*
(b) *for each open neighborhood V of $f(x)$, $x \in cl(int(cl_\delta(f^{-1}[scl(V)])))$;*
(c) *for each open neighborhood V of $f(x)$ and each open neighborhood U of x, there exists a nonempty open set $G \subset U$ such that $G \subset cl_\delta(f^{-1}[scl(V)])$;*
(d) *for each open neighborhood V of $f(x)$, there exists $U \in SO(X, x)$ such that $U \subset cl_\delta(f^{-1}[scl(V)])$.*

Proof. (a)⇒ (b): let V be any open neighborhood of $f(x)$.

$$\begin{aligned} \left.\begin{matrix} V \in O(Y, f(x)) \\ \text{(a)} \end{matrix}\right\} &\Rightarrow (\exists U \in e^*O(X, x))(f[U] \subset int(cl(V)) = scl(V)) \\ &\Rightarrow (\exists U \in e^*O(X, x))(U \subset f^{-1}[scl(V)]) \\ &\Rightarrow (\exists U \in e^*O(X, x))(cl(int(cl_\delta(U))) \subset cl(int(cl_\delta(f^{-1}[scl(V)])))) \\ &\Rightarrow (\exists U \in e^*O(X, x))(U \subset cl(int(cl_\delta(U))) \subset cl(int(cl_\delta(f^{-1}[scl(V)])))) \\ &\Rightarrow x \in cl(int(cl_\delta(f^{-1}[scl(V)]))). \end{aligned}$$

(b)⇒ (c): let V be any open neighborhood of $f(x)$ and U an open set of X containing x.

$$\left.\begin{array}{r} V\in O(Y,f(x)) \\ (b) \end{array}\right\} \Rightarrow \left.\begin{array}{r} x\in cl(int(cl_\delta(f^{-1}[scl(V)]))) \\ U\in O(X,x) \end{array}\right\} \Rightarrow U\cap cl(int(cl_\delta(f^{-1}[scl(V)])))\neq\emptyset$$
$$\Rightarrow cl(U\cap int(cl_\delta(f^{-1}[scl(V)])))\neq\emptyset \Rightarrow \left.\begin{array}{r} U\cap int(cl_\delta(f^{-1}[scl(V)]))\neq\emptyset \\ G:=U\cap int(cl_\delta(f^{-1}[scl(V)])) \end{array}\right\}\Rightarrow$$
$$\Rightarrow (G\in\tau\setminus\{\emptyset\})(G\subset U)(G\subset cl_\delta(f^{-1}[scl(V)])).$$

(c)⇒(d): let V be any open neighborhood of $f(x)$ and U an open set of X containing x.

$$\left.\begin{array}{r} (V\in O(Y,f(x)))(U\in O(X,x)) \\ (c) \end{array}\right\} \Rightarrow \quad \left.\begin{array}{r} (G_U\in\tau\setminus\{\emptyset\})(G_U\subset U)(G_U\subset cl_\delta(f^{-1}[scl(V)])) \\ U_0:=\cup\{G_U|U\in O(X,x)\} \end{array}\right\}$$
$$\Rightarrow (U_0\in SO(X,x))(U_0\subset cl_\delta(f^{-1}[scl(V)])).$$

(d)⇒ (a): Let V be any open neighborhood of $f(x)$.

$$\begin{array}{l} \left.\begin{array}{r} V\in O(Y,f(x)) \\ (d) \end{array}\right\} \Rightarrow (G\in SO(X,x))(G\subset cl_\delta(f^{-1}[scl(V)])) \\ \qquad \Rightarrow (G\in SO(X,x))(f^{-1}[V]\cap G\subset f^{-1}[V]\cap cl(int(G)) \\ \qquad \subset f^{-1}[scl(V)]\cap cl(int(G)) \\ \qquad \subset f^{-1}[scl(V)]\cap cl(int(cl_\delta(f^{-1}[scl(V)]))) \\ \qquad = e^*\text{-}int(f^{-1}[scl(V)])) \\ \qquad \Rightarrow (U:=e^*\text{-}int(f^{-1}[scl(V)])\in e^*O(X,x))(f[U]\subset scl(V)=int(cl(V))). \end{array}$$

This shows that f is a.e^*.c. at x. □

Theorem 3.3. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following are equivalent:*

(a) *f is a.e^*.c.;*
(b) *e^*-$cl(f^{-1}[cl(int(cl_\delta(B)))])\subset f^{-1}[cl_\delta(B)]$ for every subset B of Y;*
(c) *e^*-$cl(f^{-1}[cl(int(F))])\subset f^{-1}[F]$ for every δ-closed set F of Y;*
(d) *e^*-$cl(f^{-1}[cl(V)])\subset f^{-1}[cl(V)]$ for every open set V of Y;*
(e) *$f^{-1}[V]\subset e^*$-$int(f^{-1}[scl(V)])$ for every open set V of Y;*
(f) *$f^{-1}[V]\subset cl(int(cl_\delta(f^{-1}[scl(V)])))$ for every open set V of Y.*

Proof. (a)⇒ (b): let B be any subset of Y.

$$\begin{array}{rl} x\in X\setminus f^{-1}[cl_\delta(B)] & \Rightarrow f(x)\in Y\setminus cl_\delta(B) \\ & \Rightarrow (\exists V\in O(Y,f(x))(int(cl(V))\cap B=\emptyset) \\ & \left.\begin{array}{r}\Rightarrow (\exists V\in O(Y,f(x))(int(cl(V))\cap cl(int(cl_\delta(B)))=\emptyset) \\ f \text{ is a.}e^*\text{.c.}\end{array}\right\} \\ & \Rightarrow (\exists U\in e^*O(X,x))(f[U]\subset int(cl(V)))(int(cl(V))\cap cl(int(cl_\delta(B)))=\emptyset) \\ & \Rightarrow (\exists U\in e^*O(X,x))(f[U]\cap cl(int(cl_\delta(B)))=\emptyset) \\ & \Rightarrow (\exists U\in e^*O(X,x))(U\cap f^{-1}[cl(int(cl_\delta(B)))]=\emptyset) \\ & \Rightarrow x\notin e^*\text{-}cl(f^{-1}[cl(int(cl_\delta(B)))]) \\ & \Rightarrow x\in X\setminus e^*\text{-}cl(f^{-1}[cl(int(cl_\delta(B)))]). \end{array}$$

(b)⇒ (c): let F be any closed set of Y.

$$\left.\begin{array}{r} F \in \delta C(Y) \Rightarrow cl_\delta(F) = F \\ \text{(b)} \end{array}\right\} \Rightarrow e^*\text{-}cl(f^{-1}[cl(int(F))]) \subset f^{-1}[F].$$

(c)⇒(d): let V be any open set of Y.

$$\begin{array}{l} \left.\begin{array}{r} V \in \sigma \Rightarrow cl(V) \in RC(Y) \Rightarrow cl(V) \in \delta C(Y) \\ (c) \end{array}\right\} \\ \quad \Rightarrow e^*\text{-}cl(f^{-1}[cl(V)]) = e^*\text{-}cl(f^{-1}[cl(int(cl(V)))]) \subset f^{-1}[cl(V)]. \end{array}$$

(d)⇒ (e): let V be any open set of Y.

$$\begin{array}{l} \left.\begin{array}{r} V \in \sigma \Rightarrow Y \setminus cl(V) \in \sigma \\ (d) \end{array}\right\} \\ \quad \Rightarrow X \setminus e^*\text{-}int(f^{-1}[scl(V)]) = e^*\text{-}cl(f^{-1}[Y \setminus int(cl(V))]) \subset f^{-1}[cl(Y \setminus cl(V))] \subset X \setminus f^{-1}[cl(V)]. \end{array}$$

(e)⇒ (f): let V be any open set of Y.

$$\left.\begin{array}{r} V \in \sigma \\ (e) \end{array}\right\} \Rightarrow f^{-1}[V] \subset e^*\text{-}int(f^{-1}[scl(V)]) \overset{\text{Lemma 2.1 (e)}}{\subset} cl(int(cl_\delta(f^{-1}[scl(V)]))).$$

(f)⇒(a): let x be any point of X and V any open set Y containing $f(x)$.

$$\left.\begin{array}{r} V \in O(Y, f(x)) \\ (f) \end{array}\right\} \Rightarrow x \in f^{-1}[V] \subset cl(int(cl_\delta(f^{-1}[scl(V)]))).$$

Then from Theorem 3.2 (b) f is a.e^*.c. at any point $x \in X$. Therefore f is a.e^*.c. □

Theorem 3.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:*

(a) *f is a.e^*.c.;*
(b) *for each $x \in X$ and each regular open set V containing $f(x)$, there exists an e^*-open set U in X containing x such that $f[U] \subset V$;*
(c) *e^*-$cl(f^{-1}[V]) \subset f^{-1}[cl_\delta(V)]$ for every e^*-open set V of Y;*
(d) *e^*-$cl(f^{-1}[V]) \subset f^{-1}[cl_\delta(V)]$ for every semi-open set V of Y;*
(e) *$f^{-1}[V] \subset e^*$-$int(f^{-1}[int(cl(V))])$ for every preopen set V of Y;*
(f) *$f^{-1}[V] \in e^*O(X)$ for every regular open set V of Y;*
(g) *$f^{-1}[V] \in e^*C(X)$ for every regular closed set V of Y.*

Proof. (a)⇔ (b): it is straightforward. (a)⇒ (c): let V be any e^*-open set of Y and $x \notin f^{-1}[cl_\delta(V)]$.

$$\begin{array}{l} \left.\begin{array}{r} x \notin f^{-1}[cl_\delta(V)] \Rightarrow f(x) \in Y \setminus cl_\delta(V) = int_\delta(Y \setminus V) \\ (a) \end{array}\right\} \\ \quad \Rightarrow (\exists U \in e^*O(X, x))(f[U] \subset int(cl(int_\delta(Y \setminus V)))) \\ \quad \left.\begin{array}{r} \Rightarrow (\exists U \in e^*O(X, x))(f[U] \cap cl(int(cl_\delta(V))) = \emptyset) \\ V \in e^*O(Y) \end{array}\right\} \Rightarrow (\exists U \in e^*O(X, x))(f[U] \cap V = \emptyset) \\ \quad \Rightarrow (\exists U \in e^*O(X, x))(U \cap f^{-1}[V] = \emptyset) \\ \quad \Rightarrow x \notin e^*\text{-}cl(f^{-1}[V]). \end{array}$$

(c)⇒ (d): this is obvious since $SO(Y) \subset e^*O(Y)$.
(d)⇒ (e): let V be any preopen set of Y.

$$\left.\begin{array}{r} V \in PO(Y) \Rightarrow cl(int(Y \setminus V)) \in SO(Y) \\ (d) \end{array}\right\}$$

$$\left.\begin{array}{r} \Rightarrow e^*\text{-}cl(f^{-1}[cl(int(Y \setminus V))]) \subset f^{-1}[cl_\delta(cl(int(Y \setminus V)))] \subset f^{-1}[cl(int(Y \setminus V))] \\ f^{-1}[cl(int(Y \setminus V))] \subset e^*\text{-}cl(f^{-1}[cl(int(Y \setminus V))]) \end{array}\right\}$$
$$\Rightarrow X \setminus f^{-1}[int(cl(V))] = X \setminus e^*\text{-}int(f^{-1}[int(cl(V))])$$
$$\Rightarrow f^{-1}[V] \subset f^{-1}[int(cl(V))] = e^*\text{-}int(f^{-1}[int(cl(V))]).$$

(e)⇒ (f): let V be any regular open set of Y.

$$\left.\begin{array}{r} V \in RO(Y) \Rightarrow V \in PO(Y) \\ (e) \end{array}\right\} \Rightarrow f^{-1}[V] \subset e^*\text{-}int(f^{-1}[int(cl(V))]) = e^*\text{-}int(f^{-1}[V])$$
$$\Rightarrow f^{-1}[V] \in e^*O(X).$$

(f)⇒ (g): it is straightforward.
(f)⇒ (a): let x be any point of X and V any open set Y containing $f(x)$.

$$\left.\begin{array}{r} V \in O(Y, f(x)) \Rightarrow Y \setminus int(cl(V)) \in RC(Y) \\ (f) \end{array}\right\} \Rightarrow f^{-1}[Y \setminus int(cl(V))] \in e^*C(X)$$
$$\left.\begin{array}{r} \Rightarrow f^{-1}[int(cl(V))] \in e^*O(X, x) \\ U := f^{-1}[int(cl(V))] \end{array}\right\}$$
$$\Rightarrow (U \in e^*O(X, x))(f[U] \subset int(cl(V))).$$

□

Theorem 3.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:*

(a) *f is a.e*.c.;*
(b) *$f[e^*\text{-}cl(A)] \subset cl_\delta(f[A])$ for every subset A of X;*
(c) *$e^*\text{-}cl(f^{-1}[B]) \subset f^{-1}[cl_\delta(B)]$ for every subset B of Y;*
(d) *$f^{-1}[F] \in e^*C(X)$ for every δ-closed set F of Y;*
(e) *$f^{-1}[V] \in e^*O(X)$ for every δ-open set V of Y.*

Proof. (a)⇒ (b): let A be a subset of X.

$$\left.\begin{array}{r} A \subset X \Rightarrow cl_\delta(f[A]) = \bigcap\{F | (f[A] \subset F)(F \in RC(Y))\} \in \delta C(Y) \\ (a) \end{array}\right\}$$
$$\left.\begin{array}{l} \Rightarrow f^{-1}[cl_\delta(f[A])] = \bigcap_{(f[A] \subset F)(F \in RC(Y))} f^{-1}[F] \in e^*C(X) \\ A \subset f^{-1}[cl_\delta(f[A])] \Rightarrow e^*\text{-}cl(A) \subset e^*\text{-}cl(f^{-1}[cl_\delta(f[A])]) \end{array}\right\} \Rightarrow e^*\text{-}cl(A) \subset f^{-1}[cl_\delta(f[A])]$$
$$\Rightarrow f[e^*\text{-}cl(A)] \subset cl_\delta(f[A]).$$

(b)⇒ (c): let B be a subset of Y.

$$\left.\begin{array}{r} B \subset Y \Rightarrow f^{-1}[B] \subset X \\ \text{(b)} \end{array}\right\} \Rightarrow f[e^*\text{-}cl(f^{-1}[B])] \subset cl_\delta(f[f^{-1}[B]]) \subset cl_\delta(B)$$

$$\Rightarrow e^*\text{-}cl(f^{-1}[B]) \subset f^{-1}[cl_\delta(B)].$$

(c)⇒(d): let F be any δ-closed set of Y.

$$\left.\begin{array}{r} F \in \delta C(Y) \\ \text{(c)} \end{array}\right\} \Rightarrow e^*\text{-}cl(f^{-1}[F]) \subset f^{-1}[cl_\delta(F)] = f^{-1}[F] \Rightarrow f^{-1}[F] \in e^*C(X).$$

(d)⇒ (e): it is straightforward.
(e)⇒ (a): let V be any regular open set of Y.

$$\left.\begin{array}{r} V \in RO(Y) \Rightarrow V \in \delta O(Y) \\ (e) \end{array}\right\} \Rightarrow f^{-1}[V] \in e^*O(X).$$

□

Lemma 3.6. *Let A be a subset of a space X, then the following hold:*

(1) $a\text{-}cl(A) = cl_\delta(A)$ *for every* $A \in e^*O(X)$;
(2) $a\text{-}cl(A) = \delta\text{-}pcl(A) = cl_\delta(A)$ *for every* $A \in \delta SO(X)$;
(3) $A \in RO(X) \Rightarrow A = int(cl(A)) = int(cl_\delta(A))$.

Proof. (1) Let A be any e^*-open set of X.

$$\begin{aligned} A \in e^*O(X) &\Rightarrow A \subset cl(int(cl_\delta(A))) \\ &\Rightarrow cl_\delta(A) \subset cl_\delta(int(cl_\delta(A))) = cl(int(cl_\delta(A))) \\ &\Rightarrow cl_\delta(A) \subset A \cup cl(int(cl_\delta(A))) \\ &\Rightarrow cl_\delta(A) \subset a\text{-}cl(A) \ldots, \end{aligned} \tag{3.1}$$

$$A \subset X \Rightarrow a\text{-}cl(A) \subset cl_\delta(A) \ldots, \tag{3.2}$$

$$(3.1), (3.2) \Rightarrow a\text{-}cl(A) = cl_\delta(A).$$

(2) Let A be any δ-semiopen set of X.

$$\begin{aligned} A \in \delta SO(X) &\Rightarrow A \subset cl(int_\delta(A)) \\ &\Rightarrow cl_\delta(A) \subset cl_\delta(int_\delta(A)) = cl(int_\delta(A)) \\ &\Rightarrow cl_\delta(A) \subset A \cup cl(int_\delta(A)) \\ &\Rightarrow cl_\delta(A) \subset \delta\text{-}pcl(A) \ldots, \end{aligned} \tag{3.3}$$

$$A \subset X \Rightarrow \delta\text{-}pcl(A) \subset cl_\delta(A) \ldots, \tag{3.4}$$
$$(3.3), (3.4) \Rightarrow \delta\text{-}pcl(A) = cl_\delta(A).$$

(3)

$$A \subset X \Rightarrow int(cl(A)) \subset int(cl_\delta(A)) \ldots, \tag{3.5}$$

$$x \in int(cl_\delta(A)) \Rightarrow (\exists U \in \mathcal{U}(x))(U \subset cl_\delta(A))$$

$$\begin{aligned}
&\Rightarrow (\exists U \in \mathcal{U}(x))(U \cap (X \setminus cl_\delta(A)) = \emptyset)\\
&\left.\begin{array}{r}\Rightarrow (\exists U \in \mathcal{U}(x))(U \cap (int_\delta(X \setminus A)) = \emptyset)\\ A \in RO(X) \Rightarrow X \setminus A \in RC(X)\end{array}\right\} \Rightarrow (\exists U \in \mathcal{U}(x))(U \cap int(X \setminus A) = \emptyset)\\
&\Rightarrow (\exists U \in \mathcal{U}(x))(U \subset cl(A))\\
&\Rightarrow x \in int(cl(A)).
\end{aligned}$$

Then we have

$$int(cl_\delta(A)) \subset int(cl(A)) \ldots, \tag{3.6}$$

$$(3.5), (3.6) \Rightarrow A = int(cl(A)) = int(cl_\delta(A)).$$

□

Theorem 3.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:*

(a) *f is a.e*.c.;*
(b) *e^*-$cl(f^{-1}[V]) \subset f^{-1}[a$-$cl(V)]$ for each $V \in e^*O(Y)$;*
(c) *e^*-$cl(f^{-1}[V]) \subset f^{-1}[\delta$-$pcl(V)]$ for each $V \in \delta SO(Y)$;*
(d) *$f^{-1}[V] \subset e^*$-$int(f^{-1}[\delta$-$scl(V)])$ for each $V \in \delta PO(Y)$.*

Proof. (a) $\Rightarrow$ (b): let V be any e^*-open set of Y.

$$\left.\begin{array}{r}\left.\begin{array}{r} V \in e^*O(Y)\\ \text{Theorem 3.4 } (c)\end{array}\right\} \Rightarrow e^*\text{-}cl(f^{-1}[V]) \subset f^{-1}[cl_\delta(V)]\\ \text{Lemma 3.6 (1)}\end{array}\right\} \Rightarrow e^*\text{-}cl(f^{-1}[V]) \subset f^{-1}[a\text{-}cl(V)].$$

(b) $\Rightarrow$ (c): let V be any δ-semiopen set of Y.

$$\begin{aligned}&\left.\begin{array}{r} V \in \delta SO(Y) \Rightarrow V \in e^*O(Y)\\ (b)\end{array}\right\} \Rightarrow \left.\begin{array}{r} e^*\text{-}cl(f^{-1}[V]) \subset f^{-1}[a\text{-}cl(V)]\\ \text{Lemma 3.6(2)}\end{array}\right\}\\ &\Rightarrow e^*\text{-}cl(f^{-1}[V]) \subset f^{-1}[\delta\text{-}pcl(V)].\end{aligned}$$

(c) $\Rightarrow$ (d): let F be any δ-preopen set of Y.

$$\begin{aligned}&\left.\begin{array}{r} V \in \delta PO(Y) \Rightarrow cl(int_\delta(Y \setminus V)) \in \delta SO(Y)\\ (c)\end{array}\right\}\\
&\Rightarrow e^*\text{-}cl(f^{-1}[cl(int_\delta(\setminus V))]) \subset f^{-1}[\delta\text{-}pcl(cl(int_\delta(\setminus V)))]\\
&\Rightarrow f^{-1}[V] = f^{-1}[\delta\text{-}pint(V)] \subset f^{-1}[\delta\text{-}pint(int(cl_\delta(V)))]\\
&\subset e^*\text{-}int(f^{-1}[int(cl_\delta(V))])\\
&\subset e^*\text{-}int(f^{-1}[V \cup int(cl_\delta(V))])\\
&= e^*\text{-}int(f^{-1}[\delta\text{-}scl(V)]).\end{aligned}$$

$(d) \Rightarrow (a)$: let V be any regular open set of Y.

$$\begin{aligned}&\left.\begin{array}{r} V \in RO(Y) \Rightarrow V \in \delta PO(Y)\\ (d)\end{array}\right\}\\
&\Rightarrow f^{-1}[V] \subset e^*\text{-}int(f^{-1}[\delta\text{-}scl(V)]) = e^*\text{-}int(f^{-1}[V \cup int(cl_\delta(V))]) \overset{\text{Lemma 3.6 (3)}}{=} e^*\text{-}int(f^{-1}[V])\\
&\Rightarrow f^{-1}[V] \in e^*O(X).\end{aligned}$$

□

4. Some fundamental properties

Definition 4.1. A function $f : X \to Y$ is said to be faintly e^*-continuous (briefly f.e^*.c.) if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists an e^*-open set U in X containing x such that $f[U] \subset V$.

Theorem 4.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent:*

(a) *f is f.e^*.c.;*
(b) *$f^{-1}[V] \in e^*O(X)$ for every θ-open set V of Y;*
(c) *$f^{-1}[V] \in e^*C(X)$ for every θ-closed set V of Y.*

Proof. (a)$\Rightarrow$ (b): let V be any θ-open set of Y and $x \in f^{-1}[V]$.

$$\left.\begin{array}{r}(V \in \theta O(Y))(x \in f^{-1}[V]) \Rightarrow V \in \theta O(Y, f(x)) \\ (a)\end{array}\right\} \Rightarrow (\exists U \in e^*O(X, x))(f[U] \subset V)$$
$$\Rightarrow (\exists U \in e^*O(X, x))(U \subset f^{-1}[V])$$
$$\Rightarrow x \in e^*\text{-}int(f^{-1}[V]).$$

This shows that $f^{-1}[V]$ is e^*-open in X.

(b)$\Rightarrow$ (a): let V be any θ-open set of Y.

$$\left.\begin{array}{r}V \in \theta O(Y, f(x)) \\ (b)\end{array}\right\} \Rightarrow \left.\begin{array}{r}f^{-1}[V] \in e^*O(X) \\ U := f^{-1}[V]\end{array}\right\} \Rightarrow (U \in e^*O(X, x))(f[U] \subset V).$$

(b)$\Rightarrow$ (c) and (c)$\Rightarrow$(b): they are straightforward. □

Theorem 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the implications* (a)$\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) hold for the following properties:

(a) f is e^*.c.;
(b) $f^{-1}[cl_\delta(B)]$ is e^*-closed in X for every subset B of Y;
(c) f is a.e^*.c.;
(d) f is w.e^*.c.;
(e) f is f.e^*.c.

Proof. (a)$\Rightarrow$ (b): let B be any subset of Y.

$$\left.\begin{array}{r}B \subset Y \Rightarrow cl_\delta(B) \in C(Y) \\ (a)\end{array}\right\} \Rightarrow f^{-1}[cl_\delta(B)] \in e^*C(X).$$

(b) $\Rightarrow$ (c): let x be any point of X and V any open set Y containing $f(x)$.

$$\left.\begin{array}{r}V \in O(Y, f(x)) \\ (b)\end{array}\right\} \Rightarrow f^{-1}[cl_\delta(V)] \in e^*C(X) \Rightarrow e^*\text{-}cl(f^{-1}[V]) \subset e^*\text{-}cl(f^{-1}[cl_\delta(V)]) = f^{-1}[cl_\delta(V)].$$

Then from Theorem 3.5 (c) f is a.e^*.c.

(c) $\Rightarrow$ (d): it is straightforward.

(d) $\Rightarrow$ (e): let x be any point of X and V any θ-open set Y containing $f(x)$.

$$\left.\begin{array}{r}V \in \theta O(Y, f(x)) \Rightarrow V \in O(Y, f(x)) \\ (d)\end{array}\right\} \Rightarrow (\exists U \in e^*O(X, x))(f[U] \subset cl(V) = V).$$

□

Remark 4.4. If Y is regular, then the above five properties are equivalent. Suppose that Y is regular. We prove that $(e) \Rightarrow (a)$. Let V be any open set of Y. Since Y is regular, V is θ-open in Y. By the faint e^*-continuity of f, $f^{-1}[V]$ is e^*-open in X. Therefore, f is e^*-continuous.

Definition 4.5. A space X is said to be:

(1) almost regular [16] if for any regular closed set F of X and any point $x \notin F$ there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.

(2) semi-regular [11] if for any open set U of X and each point $x \in U$ there exists a regular open set V of X such that $x \in V \subset U$.

Lemma 4.6 ([16]). *For a topological space (X, τ) the following are equivalent:*

(a) *(X, τ) is almost regular,*
(b) *for each point $x \in X$ and each regular open set V containing x, there exists a regular open set U such that $x \in U \subset cl(U) \subset V$,*
(c) *for each point $x \in X$ and each neighborhood M of x, there exists a regular open neighborhood V of x such that $cl(V) \subset int(cl(M))$,*
(d) *for each point $x \in X$ and each neighborhood M of x, there exists an open neighborhood V of x such that $cl(V) \subset int(cl(M))$,*
(e) *for every regular closed set A and each point x not belonging to A, there exist open sets U and V such that $x \in U$, $A \subset V$ and $cl(U) \cap cl(V) = \emptyset$,*
(f) *every regular closed set F is expressible as an intersection of some regular closed neighborhoods of F,*
(g) *every regular closed set F is identical with the intersection of all closed neighborhoods of F,*
(h) *for every set A and every regular open set B such that $A \cap B \neq \emptyset$, there exists an open set G such that $A \cap G \neq \emptyset$ and $cl(G) \subset B$,*
(i) *for every non-empty set A and every regular closed set B satisfying $A \cap B \neq \emptyset$, there exist disjoint open sets G and H such that $A \cap G \neq \emptyset$ and $B \subset H$.*

Theorem 4.7. *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly e^*-continuous function and Y is almost regular, then f is almost e^*-continuous.*

Proof. Let x be any point of X and V any open set Y containing $f(x)$.

$$\left.\begin{array}{r} V \in O(Y, f(x)) \\ Y \text{ is almost regular} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists G \in RO(X, x))(cl(G) \subset int(cl(V))) \\ f \text{ is w.}e^*\text{.c.} \end{array}\right\}$$
$$\Rightarrow (\exists U \in e^*O(X, x))(f[U] \subset cl(G) \subset int(cl(V))).$$

□

Theorem 4.8. *If $f : (X, \tau) \to (Y, \sigma)$ is an almost e^*-continuous and Y is semi-regular, then f is e^*-continuous.*

Proof. Let x be any point of X and V any open set Y containing $f(x)$.

$$\left.\begin{array}{r} V \in O(Y, f(x)) \\ Y \text{ is semi-regular} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists G \in RO(Y, f(x)))(G \subset U) \\ f \text{ is a.}e^*\text{.c.} \end{array}\right\}$$
$$\Rightarrow (\exists U \in e^*O(X, x))(f[U] \subset int(cl(G)) = G).$$

□

Lemma 4.9. *Let X be a topological space. Then X is regular iff X is almost regular and semi-regular.*

Proof. Necessity. This is straightforward.

Sufficiency. Let X be almost regular and semi-regular.

$$\left.\begin{array}{r} x\in U\in\tau \\ X \text{ is semi-regular} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists V\in RO(X,x))(V\subset U) \\ X \text{ is almost regular} \end{array}\right\}$$
$$\Rightarrow (\exists W\in\mathcal{U}(x))(cl(W)\subset int(cl(V))=V\subset U)$$
$$\Rightarrow (\exists W\in\mathcal{U}(x))(cl(W)\subset U).$$

□

Corollary 4.10. *Let $f:X\to Y$ be a function. If Y is an almost regular and semi-regular space, then the following are equivalent*:

(a) *f is e^*.c.;*
(b) *f is a.e^*.c.;*
(c) *f is w.e^*.c.*

Definition 4.11. A function $f:X\to Y$ is said to be almost e^*-open if $f[U]\subset int(cl(f[U]))$ for every e^*-open set U of X.

Theorem 4.12. *If $f:(X,\tau)\to(Y,\sigma)$ is an almost e^*-open and weakly e^*-continuous function, then f is almost e^*-continuous.*

Proof. Let x be any point of X and V any open set Y containing $f(x)$.

$$\left.\begin{array}{r} V\in O(Y,f(x)) \\ f \text{ is w.}e^*\text{.c.} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\exists U\in e^*O(X,x))(f[U]\subset cl(V)) \\ f \text{ is almost } e^*\text{-open} \end{array}\right\} \Rightarrow f[U]\subset int(cl(f[U]))\subset int(cl(V)).$$

□

Definition 4.13. The e^*-frontier [2] of a subset A of X, denoted by e^*-$Fr(A)$, is defined by e^*-$Fr(A)=e^*$-$cl(A)\cap e^*$-$cl(X\setminus A)=e^*$-$cl(A)\setminus e^*$-$int(A)$.

Theorem 4.14. *The set of all points x of X at which a function $f:(X,\tau)\to(Y,\sigma)$ is not almost e^*-continuous is identical with the union of the e^*-frontiers of the inverse images of regular open sets containing $f(x)$.*

Proof. Let $A=\{x|f$ is not a.e^*.c. at $x\}$.

$$\begin{aligned} x\in A &\Rightarrow f \text{ is not a.}e^*\text{.c. at } x \\ &\Rightarrow (\exists V\in RO(Y,f(x))(\forall U\in e^*O(X,x))(f[U]\nsubseteq V) \\ &\Rightarrow (\exists V\in RO(Y,f(x))(\forall U\in e^*O(X,x))(U\cap\big(X\setminus f^{-1}[V]\big)\neq\emptyset) \\ &\Rightarrow (x\in f^{-1}[V])(x\in e^*\text{-}cl(X\setminus f^{-1}[V])=X\setminus e^*\text{-}int(f^{-1}[V])) \\ &\Rightarrow x\in e^*\text{-}Fr(f^{-1}[V]). \end{aligned}$$

Then we have

$$A\subset\bigcup\big\{e^*\text{-}Fr(f^{-1}[V])|V\in RO(Y,f(x))\big\}\ldots, \tag{4.1}$$

$$\left.\begin{array}{r} x \notin A \Rightarrow f \text{ is a.}e^*\text{.c. at } x \\ V \in RO(Y, f(x)) \end{array}\right\} \Rightarrow (\exists U \in e^*O(X, x))(U \subset f^{-1}[V])$$
$$\Rightarrow x \in e^*\text{-}int(f^{-1}[V])$$
$$\Rightarrow x \notin e^*\text{-}Fr(f^{-1}[V])$$
$$\Rightarrow x \notin \bigcup \left\{ e^*\text{-}Fr(f^{-1}[V]) | V \in RO(Y, f(x)) \right\}.$$

Then we have

$$\bigcup \left\{ e^*\text{-}Fr(f^{-1}[V]) | V \in RO(Y, f(x)) \right\} \subset A \ldots, \tag{4.2}$$

$$(4.1), (4.2) \Rightarrow A = \bigcup \left\{ e^*\text{-}Fr(f^{-1}[V]) | V \in RO(Y, f(x)) \right\}.$$

□

Definition 4.15. A function $f : X \to Y$ is said to be

(1) complementary almost e^*-continuous (briefly c.a.e^*.c.) if for each regular open set V of Y, $f^{-1}[Fr(V)]$ is e^*-closed in X, where $Fr(V)$ denotes the frontier of V.
(2) weakly a-continuous (briefly w.a.c.) if for each point $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in aO(X, x)$ such that $f[U] \subset cl(V)$.

Remark 4.16. We have the following diagram from Definitions 3.1, 4.1, 4.15 and Theorem 4.3:

$$\begin{array}{ccccccc} e^*\text{-continuity} & \longrightarrow & \text{almost } e^*\text{-continuity} & \longrightarrow & \text{weakly } e^*\text{-continuity} & \longrightarrow & \text{faintly } e^*\text{-continuity} \\ & & & & \uparrow & & \\ & & & & \text{weakly } a\text{-continuity} & & \end{array}$$

These implications above are not reversible as shown by the following examples.

Example 4.17. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f = \{(a, a), (b, a), (c, b)\}$. Then f is f.e^*.c., but it is not w.e^*.c. at point c of X.

Example 4.18. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f = \{(a, b), (b, c), (c, a)\}$. Then f is w.e^*.c., but it is not w.a.c. at point c of X.

Example 4.19. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f = \{(a, c), (b, b), (c, a)\}$. Then f is both w.e^*.c. and c.a.e^*.c., but it is not a.e^*.c. at point c of X.

Example 4.20. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Define the function $f : (X, \tau) \to (X, \tau)$ by $f = \{(a, d), (b, d), (c, c), (d, a)\}$. Then f is a.e^*.c., but it is not e^*.c. at point d of X.

Lemma 4.21. *Let X be a topological space and $A, B \subset X$. If A is an a-open set and B is an e^*-open set, then $A \cap B$ is an e^*-open set in X.*

Proof. Let $A \in aO(X)$ and $B \in e^*O(X)$.

$$\left.\begin{array}{r} A \in aO(X) \Rightarrow A \subset int(cl(int_\delta(A))) \\ B \in e^*O(X) \Rightarrow B \subset cl(int(cl_\delta(B))) \end{array}\right\} \Rightarrow A \cap B \subset int(cl(int_\delta(A))) \cap cl(int(cl_\delta(B)))$$
$$\Rightarrow A \cap B \subset cl[int(cl(int_\delta(A))) \cap int(cl_\delta(B))]$$

$$
\begin{aligned}
&= cl[int[cl(int_\delta(A)) \cap int(cl_\delta(B))]] \\
&= cl[int[cl(int_\delta(A)) \cap int_\delta(cl_\delta(B))]] \\
&= cl[int[cl(int_\delta(int_\delta(A))) \cap int_\delta(cl_\delta(B))]] \\
&= cl(int(cl(int_\delta(A)) \cap cl_\delta(B))) \\
&\subset cl(int(cl(cl_\delta(int_\delta(A) \cap B)))) \\
&\subset cl(int(cl_\delta(cl_\delta(A \cap B)))) \\
&\subset cl(int(cl_\delta(A \cap B))).
\end{aligned}
$$

Then we have $A \cap B$ is an e^*-open set in X. □

Theorem 4.22. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly a-continuous and complementary almost e*-continuous, then f is almost e*-continuous.*

Proof. Let $x \in X$ and let V be a regular open set of Y containing $f(x)$.

$$
\left.
\begin{array}{r}
\left.\begin{array}{r} V \in RO(Y, f(x)) \\ f \text{ is c.a.}e^*\text{.c.} \end{array}\right\} \Rightarrow f^{-1}[Fr(V)] \in e^*C(X) \Rightarrow X \setminus f^{-1}[Fr(V)] \in e^*O(X, x) \\
\left.\begin{array}{r} (x \in X)(V \in O(Y, f(x)) \\ f \text{ is w.}a\text{.c.} \end{array}\right\} \Rightarrow (\exists G \in aO(X, x))(f[G] \subset cl(V))
\end{array}
\right\}
$$

$$
\begin{aligned}
&\overset{\text{Lemma 4.21}}{\Rightarrow} (U := G \cap (X \setminus f^{-1}[Fr(V)]) \in e^*O(X, x)) \\
&\qquad (f[U] \subset f[G \cap (X \setminus Fr(V))] \subset cl(V) \cap (Y \setminus Fr(V)) = V) \\
&\Rightarrow (U \in e^*O(X, x))(f[U] \subset f[G] \cap f[X \setminus f^{-1}[Fr(V)]] \subset cl(V) \cap (Y \setminus Fr(V)) = V).
\end{aligned}
$$

□

Theorem 4.23. *If $f : (X, \tau) \to (Y, \sigma)$ is almost e*-continuous, $g : (X, \tau) \to (Y, \sigma)$ is weakly a-continuous and Y is Hausdorff, then the set $\{x | f(x) = g(x)\}$ is e*-closed in X.*

Proof. Let $A = \{x | f(x) = g(x)\}$.

$$
\begin{aligned}
&\left.\begin{array}{r} x \notin A \Rightarrow f(x) \neq g(x) \\ Y \text{ is Hausdorff} \end{array}\right\} \Rightarrow (\exists U \in O(Y, f(x)))(\exists V \in O(Y, g(x)))(U \cap V = \emptyset) \\
&\left.\begin{array}{r} \Rightarrow (\exists U \in O(Y, f(x)))(\exists V \in O(Y, g(x)))(int(cl(U)) \cap cl(V) = \emptyset) \\ f \text{ is a.}e^*\text{.c. and } g \text{ is w.}a\text{.c.} \end{array}\right\} \\
&\left.\begin{array}{r} \Rightarrow (\exists G \in e^*O(X, x))(f[G] \subset int(cl(U)))(\exists H \in aO(X, x))(g[H] \subset cl(V)) \\ W := G \cap H \end{array}\right\} \\
&\Rightarrow (\exists W \in e^*O(X, x))(f[W] \cap g[W] \subset int(cl(U)) \cap cl(V) = \emptyset) \\
&\Rightarrow (\exists W \in e^*O(X, x))(W \cap A = \emptyset) \\
&\Rightarrow x \in e^*\text{-}int(X \setminus A) = X \setminus e^*\text{-}cl(A) \\
&\Rightarrow x \notin e^*\text{-}cl(A).
\end{aligned}
$$

□

Lemma 4.24. *Let X and Y be topological spaces. Then the following properties hold:*

(a) *If $G \subset X$ and $H \subset Y$, then $cl_\delta(G) \times cl_\delta(H) \subset cl_\delta(G \times H)$,*
(b) *If $G \in e^*O(X)$ and $H \in e^*O(Y)$, then $G \times H \in e^*O(X \times Y)$.*

Proof. (a) It is straightforward.

(b) Let $G \in e^*O(X)$ and $H \in e^*O(Y)$.

$$\left.\begin{array}{r} G \in e^*O(X) \Rightarrow G \subset cl(int(cl_\delta(G))) \\ H \in e^*O(Y) \Rightarrow H \subset cl(int(cl_\delta(H))) \end{array}\right\} \Rightarrow G \times H \subset cl(int(cl_\delta(G))) \times cl(int(cl_\delta(H)))$$

$$= cl(int[cl_\delta(G) \times cl_\delta(H)]) \overset{(a)}{\subset} cl(int(cl_\delta(G \times H))).$$

□

Theorem 4.25. *If $f : (X,\tau) \to (Z,\eta)$ is weakly a-continuous, $g : (Y,\sigma) \to (Z,\eta)$ is almost e^*-continuous and Z is Hausdorff, then the set $\{(x,y)|f(x) = g(y)\}$ is e^*-closed in $X \times Y$.*

Proof. Let $A = \{(x,y)|f(x) = g(y)\}$.

$$\left.\begin{array}{r} (x,y) \notin A \Rightarrow f(x) \neq g(y) \\ Z \text{ is Hausdorff} \end{array}\right\} \Rightarrow (\exists U \in O(Z,f(x)))(\exists V \in O(Z,g(y)))(U \cap V = \emptyset)$$

$$\left.\begin{array}{r} \Rightarrow (\exists U \in O(Z,f(x)))(\exists V \in O(Z,g(y)))(cl(U) \cap int(cl(V)) = \emptyset) \\ f \text{ is w.}a\text{.c. and } g \text{ is a.}e^*\text{.c.} \end{array}\right\}$$

$$\Rightarrow (\exists G \in e^*O(X,x))(f[G] \subset cl(U))(\exists H \in e^*O(Y,y))(g[H] \subset int(cl(V)))(cl(U) \cap int(cl(V)) = \emptyset)$$

$$\overset{\text{Lemma 4.24 } (b)}{\Rightarrow} (G \times H \in e^*O(X \times Y))((x,y) \in G \times H \subset (X \times Y) \setminus A)$$

$$\Rightarrow (x,y) \notin e^*\text{-}cl(A).$$

□

Definition 4.26. A space X is said to be e^*-T_2 [6] if for any distinct points x, y of X, there exist disjoint e^*-open sets U, V of X such that $x \in U$ and $y \in V$.

Theorem 4.27. *If for each pair of distinct points x and y in a space X, there exists a function f of X into a Hausdorff space Y such that $f(x) \neq f(y)$, f is weakly a-continuous at x and almost e^*-continuous at y, then X is e^*-T_2.*

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r} (x,y \in X)(f(x) \neq f(y)) \\ Y \text{ is Hausdorff} \end{array}\right\} \Rightarrow (\exists V_1 \in O(Y,f(x)))(\exists V_2 \in O(Y,f(y)))(V_1 \cap V_2 = \emptyset)$$

$$\left.\begin{array}{r} \Rightarrow (\exists V_1 \in O(Y,f(x)))(\exists V_2 \in O(Y,f(y)))(cl(V_1) \cap int(cl(V_2)) = \emptyset) \\ f \text{ is w.a.c. at } x \text{ and } f \text{ is a.}e^*\text{.c. at } y \end{array}\right\}$$

$$\Rightarrow (\exists U_1 \in e^*O(X,x))(f[U_1] \subset cl(V_1))(\exists U_2 \in e^*O(X,y))(f[U_2] \subset int(cl(V_2)))(cl(V_1) \cap int(cl(V_2)) = \emptyset)$$

$$\Rightarrow (\exists U_1 \in e^*O(X,x))(\exists U_2 \in e^*O(X,y))(f[U_1 \cap U_2] \subset f[U_1] \cap f[U_2] = \emptyset)$$

$$\Rightarrow (\exists U_1 \in e^*O(X,x))(\exists U_2 \in e^*O(X,y))(U_1 \cap U_2 = \emptyset).$$

□

Definition 4.28. A function $f : X \to Y$ has an e^*-closed graph if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in e^*O(X,x)$ and an open set V of Y containing y such that $[U \times cl(V)] \cap G(f) = \emptyset$.

Lemma 4.29. *A function $f : X \to Y$ has an e^*-closed graph if and only if for each $(x,y) \in (X \times Y) \setminus G(f)$, there exist $U \in e^*O(X,x)$ and an open set V of Y containing y such that $f[U] \cap cl(V) = \emptyset$.*

Proof. Necessity. Let $(x, y) \in (X \times Y) \setminus G(f)$.

$$\left.\begin{array}{r}(x,y) \in (X \times Y) \setminus G(f)\\ f \text{ has an } e^*\text{-closed graph}\end{array}\right\} \Rightarrow (\exists U \in e^*O(X,x))(\exists V \in O(Y,y))([U \times cl(V)] \cap G(f) = \emptyset)$$
$$\Rightarrow (\exists U \in e^*O(X,x))(\exists V \in O(Y,y))(f[U] \cap cl(V) = \emptyset).$$

Sufficiency. Let $(x, y) \in (X \times Y) \setminus G(f)$.

$$\left.\begin{array}{r}(x,y) \notin G(f)\\ \text{Hypothesis}\end{array}\right\} \Rightarrow (\exists U \in e^*O(X,x))(\exists V \in O(Y,y))(f[U] \cap cl(V) = \emptyset)$$
$$\Rightarrow (\exists U \in e^*O(X,x))(\exists V \in O(Y,y))([U \times cl(V)] \cap G(f) = \emptyset).$$

□

Theorem 4.30. *If $f : (X, \tau) \to (Y, \sigma)$ is an almost e^*-continuous function and Y is Hausdorff, then f has an e^*-closed graph.*

Proof. Let $(x, y) \notin G(f)$.

$$\left.\begin{array}{r}(x,y) \notin G(f) \Rightarrow y \neq f(x)\\ Y \text{ is Hausdorff}\end{array}\right\} \Rightarrow (\exists V \in O(Y, f(x)))(\exists W \in O(Y,y))(V \cap W = \emptyset)$$
$$\Rightarrow (\exists V \in O(Y, f(x)))(\exists W \in O(Y,y))(V \cap cl(W) = \emptyset)$$
$$\left.\begin{array}{r}\Rightarrow f(x) \in Y \setminus cl(W) \in RO(Y)\\ f \text{ is a.}e^*\text{.c.}\end{array}\right\}$$
$$\Rightarrow (\exists U \in e^*O(X,x))(f[U] \subset Y \setminus cl(W))$$
$$\Rightarrow (\exists U \in e^*O(X,x))(f[U] \cap cl(W) = \emptyset).$$

This shows that f has an e^*-closed graph. □

Corollary 4.31. *If $f : X \to Y$ is an e^*-continuous function and Y is Hausdorff, then f has an e^*-closed graph.*

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