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Generation of discrete integrable systems and some algebro-geometric properties of related discrete lattice equations

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Abstract

With the help of infinite-dimensional Lie algebras and the Tu scheme, we address a discrete integrable hierarchy to reduce the generalized relativistic Toda lattice (GRTL) system containing the relativistic Toda lattice equation and its generalized lattice equation. Meanwhile, the Riemann theta functions are utilized to present its algebro-geometric solutions. Besides, a reduced spectral problem is given to find an integrable discrete hierarchy obtained via R-matrix theory, which can be reduced to the Toda lattice equation and a generalized Toda lattice (GTL) system. The Lax pair and the infinite conservation laws of the GTL system are also derived. Finally, the Hamiltonian structure of the GTL system is generated by the Poisson tensor.

Keywords: Spectral problem, algebro-geometric solution, R-matrix, Hamiltonian structure.
2010 MSC: 35P30, 35Q51, 37K05, 37K10, 37K15, 37K30.

1. Introduction

As it is known, the nonlinear lattice equations have lots of applications in statistical and quantum physics. For example, the relativistic Toda lattice equation governs a system of unit masses connected by

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nonlinear springs [15]. It is interesting that we search for the corresponding integrable lattice hierarchy where the Toda lattice equation lies in. For the sake of that, Tu [16] made use of Lie algebras and Lax pairs to derive the resulting Toda hierarchy whose Hamiltonian structure was also generated which was called the Tu scheme. It follows that Picking et al. [10, 21, 22] generated some lattice hierarchies. Fan, et al. [4, 18] also obtained some discrete integrable hierarchies by using the Tu scheme. Some properties of discrete integrable systems, such as Hamiltonian structures, Darboux transformations, exact solutions, etc. were discussed in [8, 9, 11, 20]. Based on the above mentioned consequences, we adopt a Lie algebra consisting of 2×2 matrices to introduce a Lax pair so that the Tu scheme is devoted to generate a discrete integrable hierarchy which can be reduced to the well-known relativistic Toda lattice equation and a generalized relativistic Toda lattice (GRTL) system. Then we again apply the scheme for generating algebro-geometric solutions proposed by Geng et al. [6] to discover the algebro-geometric solutions of the GRTL system via the Albel-Jacobi coordinates and the Riemann theta functions as well as theory on hyperelliptic cures. Specially, we construct two sets of Baker functions for straightening out of the discrete flows for the GRTL system. In addition, we reduce the isospectral problem introduced in the paper to two shift operators. Then, by applying the R-matrix formalism [1–3], we give rise to a new form of the Toda lattice hierarchy which can be reduced to the Toda lattice equation and its series of the generalized discrete integrable systems. Again, utilizing the Casimir functions of the Lie-Poisson bracket presented in the paper and the Novikov equation, we obtain the representation of the Lax pair of the GTL system derived from the Toda lattice hierarchy. Finally, we obtain the Hamiltonian structure of the GTL system via the Poisson tensor linked to the Casimir function and the discrete expanding integrable model of the Toda lattice system is generated by constructing a new integrable hierarchy.

2. A discrete integrable hierarchy and its reductions

The simplest subalgebra of the Lie algebras A_1 are given by

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The corresponding loop algebras can be defined as

$$h_i(n) = h_i\lambda^n, e(n) = e\lambda^n, f(n) = f\lambda^n, n \in \mathbf{Z}.$$

Based on the above, we introduce the following discrete spectral problems

$$\psi_{n+1} = \psi(n+1) = U_n\psi(n) \equiv U_n\psi, U_n = h_1(1) + p_nh_1(0) + q_ne(0) + \alpha f(1) + s_nh_2(0), \tag{2.1}$$

where α is an arbitrary constant.

$$\psi(n)_t = A(h_1(0) - h_2(0)) + Be(0) + Cf(0),$$

where

$$A = \sum_{j\geq 0} a_j(n)\lambda^{-j}, B = \sum_{j\geq 0} b_j(n)\lambda^{-j}, C = \sum_{j\geq 0} c_j(n)\lambda^{-j}. \tag{2.2}$$

According to the Tu scheme, we first solve the following stationary discrete zero curvature equation for V_n :

$$(EV_n)U_n = U_nV_n \tag{2.3}$$

to give rise to

$$\begin{cases} (\lambda + p_n)(EA - A) + \alpha\lambda EB - q_nC = 0, \\ (\lambda + p_n)EC - \alpha\lambda(EA + A) - s_nC = 0, \\ q_nEA + s_nEB - (\lambda + p_n)B + q_nA = 0, \\ q_nEC - s_nEA - \alpha\lambda B + s_nA = 0. \end{cases} \tag{2.4}$$

Substituting (2.2) into (2.4) yields

$$\begin{cases} \Delta a_{j+1} + p_n \Delta a_j + \alpha E b_{j+1} - q_n c_j = 0, \\ q_n E a_j + s_n E b_j - b_{j+1} - p_n b_j + q_n a_j = 0, \\ E c_{j+1} + p_n E c_j - \alpha(E a_{j+1} + a_{j+1}) - s_n c_j = 0, \\ q_n E c_j - s_n E a_j - \alpha b_{j+1} + s_n a_j = 0, \end{cases} \tag{2.5}$$

which leads to

$$(\alpha q_n E + \alpha q_n + s_n E - s_n) a_j + (\alpha s_n E - \alpha p_n) b_j - q_n E c_j = 0,$$

where $\Delta = E - 1$. Taking

$$b_0 = 0, E c_0 - \alpha(E+1) a_0 = 0, \Delta a_0 + \alpha E b_0 = 0,$$

then it is easy to see that we can set $a_0 = 1$ and $c_0 = 2\alpha$. Hence, from Eq. (2.5) we get that

$$\begin{aligned} b_1 &= 2q_n, a_1 = -2\alpha q_n, c_1 = -2\alpha p_{n-1} - 2\alpha^2(q_n + q_{n-1}) + 2\alpha s_{n-1}, \\ b_2 &= -2\alpha q_n q_{n+1} + 2 s_n q_{n+1} - 2 p_n q_n - 2\alpha q_n^2, \\ a_2 &= 2\alpha q_n q_{n-1} + 2\alpha p_n q_n + 2\alpha^2 q_n^2 - 2\alpha s_n q_{n+1} - 2\alpha q_n s_{n-1} + 2\alpha^2(q_n q_{n+1} + q_n q_{n-1}), \\ c_2 &= 2\alpha {p_{n-1}}^2 + 4\alpha^2 p_{n-1} q_n + 4\alpha^2 p_{n-1} q_{n-1} - 2\alpha p_{n-1} s_{n-1} + 2\alpha^2 p_n q_n + 2\alpha^3 q_n^2 - 2\alpha^2 s_n q_{n-1} \\ &\quad - 4\alpha^2 q_n s_{n-1} + 2\alpha^3 q_n q_{n+1} + 4\alpha^3 q_n q_{n-1} - 2\alpha s_{n-1} p_{n-2} - 2\alpha^2 s_{n+1} q_{n-1} - 2\alpha^2 s_{n-1} q_{n-2} + 2\alpha s_{n-1} s_{n-2} \\ &\quad + 2\alpha^2 q_{n-1} p_{n-2} + 2\alpha^3 {q_{n-1}}^2 - 2\alpha^2 q_{n-1} s_{n-2} + 2\alpha^3 q_{n-1} q_{n-2}. \end{aligned}$$

Note

$$V_{n,+} = \sum_{i=0}^{m} [a_i(h_1(m-i) - h_2(m-i)) + b_i e(m-i) + c_i f(m-i)] = \lambda^m V - V_{n,-},$$

then Eq. (2.3) can be decomposed into the following

$$(E V_{n,+}) U_n - U_n(V_{n,+}) = U_n(V_{n,-}) - (E V_{n,-}) U_n. \tag{2.6}$$

It is easy to find that the degree of powers of λ in the left-hand side of Eq. (2.6) is more than zero, while the right-hand side is less than zero. Hence, the degree of both sides of Eq. (2.6) is zero. Therefore, one infers that

$$(E V_{n,+}) U_n - U_n(V_{n,+}) = -\Delta a_{m+1} h_1(0) + b_{m+1} e(0) - (E c_{m+1} - \alpha E a_{m+1} - \alpha a_{m+1}) f(0) + \alpha b_{m+1} h_2(0).$$

Assume that

$$V_m = V_{n,+} + \Delta_m, \Delta_m = k_1 h_2(0) + k_2 f(0).$$

Thus,

$$\begin{aligned} k_1 &= \frac{1}{\alpha} c_m, k_2 = -c_m, \\ (E V_m) U_n - U_n V_m &= (-\Delta a_{m+1} + q_n c_m) h_1(0) + (b_{m+1} - \frac{1}{\alpha} q_n c_m) e(0) \\ &\quad + (\alpha b_{m+1} + \frac{1}{\alpha} s_n E c_m - q_n E c_m - \frac{1}{\alpha} s_n c_m) h_2(0). \end{aligned}$$

Thus, the discrete zero curvature equation

$$\frac{dU_n}{dt_m} - (E V_m) U_n + U_n V_m = 0$$

admits a discrete integrable hierarchy of the form:

$$\begin{pmatrix} p_n \\ q_n \\ s_n \end{pmatrix}_{t_m} = \begin{pmatrix} p_n\Delta a_m \\ b_{m+1} - \frac{1}{\alpha}q_n c_m = b_{m+1} + q_n c_m \\ \alpha b_{m+1} + \frac{1}{\alpha}s_n\Delta c_m - q_n E c_m = -s_n E a_m + s_n a_m + \frac{1}{\alpha}s_n\Delta c_m \end{pmatrix}. \tag{2.7}$$

Taking $s_n = 0$ and $\alpha = -1$, we find that Eq. (2.7) becomes

$$\begin{cases} p_{t_m} = p_n\Delta a_m, \\ q_{t_m} = b_{m+1} + q_n c_m. \end{cases} \tag{2.8}$$

When $m = 1$, we get the relativistic Toda lattice system:

$$\begin{cases} p_{t_1} = 2p_n(q_{n+1} - q_n), \\ q_{t_1} = 2q_n q_{n+1} - 2q_n q_{n-1} - 2p_n q_n + 2q_n p_{n-1}. \end{cases} \tag{2.9}$$

In fact, Eq. (2.8) is the relativistic Toda lattice hierarchy. For $m = 1$, Eq. (2.8) can be reduced to

$$\begin{cases} p_{n,t} = -2\alpha p_n(q_{n+1} - q_n), \\ q_{n,t} = 2s_n q_{n+1} - 2\alpha q_n q_{n+1} - (2\alpha + 2\alpha^2)q_n^2 - 2\alpha q_n p_{n-1} - 2\alpha^2 q_n q_{n-1} + 2\alpha q_n s_{n-1} - 2p_n q_n, \\ s_{n,t} = s_n(-2p_n + 2p_{n-1} - 2\alpha q_n + 2\alpha q_{n-1} + 2s_n - 2s_{n-1}). \end{cases} \tag{2.10}$$

Obviously, when $s_n = 0$ and $\alpha = -1$, (2.10) is reduced to Eq. (2.9). Hence, Eq. (2.10) becomes a generalized relativistic Toda lattice (GRTL) system. It is easy to find that the Lax pair of Eq. (2.10) is given by

$$\psi(n+1) = U_n\psi(n), \psi(n)_t = V_n^{(1)}\psi(n),$$

where

$$\begin{aligned} V_n^{(1)} = \begin{pmatrix} V_{11}^{(1)} & V_{12}^{(1)} \\ V_{21}^{(1)} & V_{22}^{(1)} \end{pmatrix} &= h_1(1) - 2\alpha q_n h_1(0) + 2q_n e(0) + 2\alpha f(1) - h_2(1) \\ &+ [2\alpha q_n - 2\alpha q_{n-1} - 2\alpha p_{n-1} - 2\alpha^2(q_n + q_{n-1}) + 2\alpha s_{n-1}]f(0). \end{aligned}$$

3. Algebro-geometric solutions to the GRTL system

In this section, we shall discuss the algebro-geometric solutions of the GRTL system by following the way proposed by Geng et al. [6]. However, we shall present two sets of Baker functions for straightening out of the discrete flows of the GRTL system. To make use of the scheme given by Geng et al., we want to express the relativistic Toda lattice hierarchy (2.8) by the Lenard's gradient sequences $S_j(n), 0 \le j \in \mathbf{Z}$:

$$J_n S_{j+1}(n) = K_n S_j(n), J_n S_{-1}(n) = 0, j \ge 0, \tag{3.1}$$

where

$$K_n = \begin{pmatrix} 0 & s_nE - p_n & q_nE + q_n \\ q_n & 0 & -p_n\Delta \\ -q_nE & \alpha(s_nE - p_n) & \alpha q_nE + \alpha q_n + s_n\Delta \end{pmatrix},$$

$$J_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha E & \Delta \\ -q_nE & \alpha(s_nE - p_n) & \alpha q_nE + \alpha q_nE + \alpha q_n + s_n\Delta \end{pmatrix},$$

$$S_j(n) = \begin{pmatrix} s_j^{(1)}(n) \\ s_j^{(2)}(n) \\ s_j^{(3)}(n) \end{pmatrix}.$$

From the equation $J_n S_{(-1)}(n) = 0$, one infers that

$$S_{(-1)}(n) = \begin{pmatrix} 2\alpha \\ 0 \\ 1 \end{pmatrix} \, ker \, J_n\{cS_0(n) : c \in R\}.$$

Thus, starting from (3.1) engenders that

$$S_0(n) = \begin{pmatrix} -2\alpha p_{n-1} - 2\alpha^2(q_n + q_{n-1}) + 2\alpha s_{n-1} \\ 2q_n \\ -2\alpha q_n \end{pmatrix}.$$

Eq. (3.1) implies that

$$\begin{cases} \Delta S_j^{(3)}(n+1) + p_n \Delta S_j^{(3)}(n) + \alpha E S_j^{(2)}(n+1) - q_n S_j^{(1)}(n) = 0, \\ q_n E S_j^{(3)}(n) + s_n E S_j^{(2)}(n) - S_j^{(2)}(n+1) - p_n S_j^{(2)}(n) + q_n S_j^{(3)}(n) = 0, \\ E S_j^{(1)}(n+1) + p_n E S_j^{(1)}(n) - \alpha(E S_j^{(3)}(n+1) + S_j^{(3)}(n+1)) - s_n S_j^{(1)}(n) = 0, \\ q_n E S_j^{(1)}(n) - s_n E S_j^{(3)}(n) - \alpha S_j^{(2)}(n+1) + s_n S_j^{(3)}(n) = 0. \end{cases}$$

Let

$$\psi(n+1) = U_n \psi(n), \psi(N)_{t_m} = V_n^{(m)} \psi(n), \tag{3.2}$$

where

$$V_n^{(m)} = \begin{pmatrix} A_n^{(m)} & B_n^{(m)} \\ C_n^{(m)} & -A_n^{(m)} \end{pmatrix},$$

and

$$A_n^{(m)} = \sum_{j=0}^{m} S_{j-1}^{(1)}(n)\lambda^{m-j}, B_n^{(m)} = \sum_{j=0}^{m} S_{j-1}^{(2)}(n)\lambda^{m+1-j}, C_n^{(m)} = \sum_{j=0}^{m} S_{j-1}^{(1)}(n)\lambda^{m-j}.$$

The compatibility condition of the discrete Lax pair (3.2) reads that

$$\begin{pmatrix} p_n \\ q_n \\ s_n \end{pmatrix}_{t_m} = \begin{pmatrix} p_n \Delta S_m^{(3)}(n) \\ S_{m+1}^{(2)}(n) + q_n S_m^{(1)}(n) \\ -s_n E S_m^{(3)}(n) + s_n S_m^{(3)}(n) + \frac{1}{\alpha} s_n \Delta S_m^{(1)}(n) \end{pmatrix}. \tag{3.3}$$

3.1. Decomposition of the GRTL system

Suppose Eq. (3.2) has two basic solutions $\psi(n) = (\psi^{(1)}(n), \psi^{(2)}(n))^T, \phi(n) = (\phi^{(1)}(n), \phi^{(2)})^T$, we define a Lax matrix W_n as follows:

$$W_n = \begin{pmatrix} f(n) & g(n) \\ h(n) & -f(n) \end{pmatrix}, \tag{3.4}$$

and require W_n satisfying the following equations

$$W_{n+1}U_n - U_n W_n = 0, W_{n,t_m} = [V_n^{(m)}, W_n],$$

which are equivalent to

$$\begin{cases} (\lambda - p_n)\Delta f(n) + \alpha\lambda g(n+1) - q_n h(n) = 0, \\ q_n f(n+1) + s_n g(n+1) - (\lambda + p_n) g(n) + q_n f(n) = 0, \\ (\lambda + p_n) h(n+1) - \alpha\lambda f(N+1) - \alpha\lambda f(n) - s_n h(n) = 0, \\ q_n h(n+1) - s_n f(n+1) - \alpha\lambda g(n) + s_n f(n) = 0, \end{cases} \tag{3.5}$$

and

$$\begin{cases} f(n)_{t_m} = B_n^{(m)} h(n) - C_n^{(m)} g(n), \\ g(n)_{t_m} = 2A_n^{(m)} g(n) - 2B_n^{(m)} f(n), \\ h(n)_{t_m} = 2C_n^{(m)} f(n) - 2A_n^{(m)} h(n), \end{cases}$$

where

$$f(n) = \sum_{j=0}^{N} f_{j-1}(n)\lambda^{N+1-j}, g(n) = \sum_{j=0}^{N} g_{j-1}(n)\lambda^{N-j}, h(n) = \sum_{j=0}^{N} h_{j-1}(n)\lambda^{N-j}. \tag{3.6}$$

Substituting (3.6) into (3.5) yields

$$J_n G_{j+1}(n) = K_n G_j(n), J_n G_{-1}(n) = 0, j \geq 0, \tag{3.7}$$

where $G_j(n) = (h_j(n), g_j(n), f_j(n))^T$. Equation $J_n G_{-1}(n) = 0$ has the general solutions

$$G_{-1}(n) = \alpha_0 S_{-1}(n), \tag{3.8}$$

where α_0 is a constant. Acting with $J^{-1}K_n$ and $K_n^{-1}J_n$ respectively on Eq. (3.8) yields

$$G_0(n) = \alpha_0 S_0(n) + \alpha_1 S_{-1}(n).$$

According to (3.7), we have

$$\begin{cases} h_0(n) = -2\alpha p_{n-1} - 2\alpha^2(q_n + q_{n-1}) + 2\alpha s_{n-1} + 2\alpha\alpha_1, \\ g_0(n) = 2q_n, f_0(n) = -2\alpha q_n + \alpha_1, \cdots . \end{cases}$$

We apply $g(n)$ and $h(n)$ to be polynomials of λ to define the elliptic coordinates $\{\mu_j(n)\}$ and $\{\nu_j(n)\}$:

$$g(n) = \lambda g_0(n) \prod_{i=1}^{N} (\lambda - \mu_j(n)), h(n) = \lambda h_0(n) \prod_{j=1}^{N} \nu_j(n). \tag{3.9}$$

Comparing the coefficients of the same power for λ in (3.9), one gets

$$g_1(n) = -g_0(n) \sum_{j=1}^{N} \mu_j(n), h_0(n) = -h_{-1}(n) \sum_{j=1}^{N} \nu_j(n),$$

which can be rewritten as:

$$\begin{cases} \alpha q_{n+1} - s_n \dfrac{q_{n+1}}{q_n} + p_n + \alpha q_n = \displaystyle\sum_{j=1}^{N} \mu_j(n) + \alpha_1, \\ p_{n-1} + \alpha(q_n + q_{n-1}) - s_{n-1} = \displaystyle\sum_{j=1}^{N} \nu_j(n) + \alpha_1. \end{cases}$$

From (3.4) we have

$$\det W_n = f^2(n) + g(n)h(n) = \prod_{j=1}^{2N} (\lambda - \lambda_j) \equiv R(\lambda). \tag{3.10}$$

Substituting (3.6) into (3.10) and comparing coefficients of related powers of λ give rise to

$$\alpha_1 = -\frac{1}{2} \sum_{j=1}^{2N} \lambda_j, \tag{3.11}$$

$$f(n)|_{\lambda=\mu_k(n)} = \sqrt{R(\mu_k(n))}, f(n)|_{\lambda=\nu_k(n)} = \sqrt{R(\nu_k(n))}.$$

Taking $m = 1$, and starting from (3.2), one infers

$$g(n)_{t_1} = 2A_n^{(1)} g(n) - 2B_n^{(1)} f(n) = 2(\lambda - 2\alpha q_n) g(n) - 4q_n f(n).$$

For $t_1 = t$, it is easy to find that

$$g(n)_t - \lambda\mu_{k,t}(n) \prod_{i\neq k i=1}^{N} (\mu_k(n) - \mu_j(n)) 2q_n = -2q_n\mu_{k,t}(n) \prod_{i\neq k i=1}^{N} (\mu_k(n) - \mu_i(n)).$$

Hence, it follows that

$$g(n)_t|_{\lambda=\mu_k(n)} = \mu_k(n)\mu_k(n)_t \prod_{i\neq k, i=1}^{N} (\mu_k(n) - \mu_i(n)) = 2\sqrt{R(\mu_k(n))}.$$

Thus, we have

$$\mu_{kt} = \frac{2\sqrt{R(\mu_k(n))}}{\prod_{i\neq k,i=1}^{N} (\mu_k(n) - \mu_i(n))}.$$

Similarly, one infers that

$$h_t|_{\lambda=\nu_k(n)} = 4\alpha\nu_k(n)\sqrt{R(\nu_k(n))}. \tag{3.12}$$

Besides, we know that

$$\begin{aligned} h(n)_t &= -\lambda h_{-1}(n)\nu_k(n) \prod_{i\neq k,i=1}^{N} (\nu_k(n) - \nu_i(n)), \\ h(n)_t|_{\lambda=\nu_k(n)} &= -\nu_k(n)\nu(n)_{k,t} \prod_{i\neq k,i=1}^{N} (\nu_k(n) - \nu_i(n)). \end{aligned} \tag{3.13}$$

Eqs. (3.12) and (3.13) imply that

$$\nu(n)_{k,t} = \frac{2\sqrt{R(\nu_k(n))}}{\prod_{i\neq k,i=1}^{N} (\nu_k(n) - \nu_i(n))}.$$

3.2. Straightening out of the continuous flows

We first introduce the Riemann surface Γ of the hyperelliptic curve $\xi^2 = R(\xi) = \prod_{j=1}^{2N} (\xi - \xi_j)$. On Γ there are two infinite points ∞_1 and ∞_2 which are not branch points of Γ. We fix a set of regular cycle paths: $a_1, a_1, \ldots, a_N; b_1, b_2, \ldots, B_n$ which are independent and have the intersection numbers in the following:

$$a_k \cdot a_j = b_k \cdot b_j = 0, a_k \cdot b_j = \delta_{kj}, k, j = 1, 2, \ldots, N.$$

We choose holomorphic differential on Γ :

$$\tilde{\omega}_l = \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, 1 \leq l \leq N,$$

and let

$$A_{kj} = \int_{a_j} \tilde{\omega}_k, B_{kj} = \int_{b_j} \tilde{\omega}_k.$$

Then $A = (A_{ij}),\ B = (B_{ij})$ are $N \times N$ invertible matrices. Thus, we define the matrices C and τ by

$$C = (c_{kj}) = A^{-1}, \tau = (\tau_{kj}) = A^{-1}B,$$

then it can be verified that τ is symmetric and has a positive defined imaginary part. We normalize $\tilde{\omega}_j$ into the new basis ω_j :

$$\omega_j = \sum_{l=1}^{N} c_{jl}\tilde{\omega}_l, l = 1, 2, \ldots, N,$$

which satisfies

$$\int_{a_k} \omega_j = \sum_{l=1}^{N} c_{jl} \int_{a_k} \tilde{\omega}_l = \sum_{l=1}^{N} c_{jl} A_{lk} = \delta_{jk}, \int_{b_k} \omega_j = \tau_{jk}.$$

For a fixed point p_0, we introduce the Abel-Jacobi coordinates

$$\rho_m = (\rho_m^{(1)}, \ldots, \rho_m^{(N)})^T, m = 1, 2,$$

whose components present that

$$\rho_1(j)(n) = \sum_{k=1}^{N} \int_{p_0}^{p(\mu_k)} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} \int_{\lambda(p_0)}^{\mu_k(n)} c_{jl} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}},$$

$$\rho_2(j)(n) = \sum_{k=1}^{N} \int_{p_0}^{p(\nu_k)} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} \int_{\lambda(p_0)}^{\nu_k(n)} c_{jl} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}},$$

where

$$p(\mu_k(n)) = (\lambda = \mu_k(n), \xi = \sqrt{R(\mu_k(n))}), p(\nu_k(n)) = (\lambda = \nu_k(n), \xi = \sqrt{R(\nu_k(n))}) \in \Gamma,$$

here $\lambda(p_0)$ is the local coordinate of p_0.

As a result, we obtain that

$$\partial_t \rho_1^{(j)}(n) = 2c_{jN} \equiv \Omega_j^{(1)}, \partial_t \rho_2^{(j)}(n) = \Omega_j^{(1)},$$

with the help of the equalities

$$\sum_{k=1}^{N} \frac{\mu_k^{l-1}(n)}{\prod_{i\neq k, i=1}(\mu_k(n) - \mu_i(n))} = \delta_{lN}, 1 \leq l \leq N.$$

3.3. Straightening out of the discrete flow

Assume the fundamental solution matrix of (3.2) to be of the form

$$Q_n = (\phi(n), \hat{\phi}(n)) = \begin{pmatrix} \phi^{(1)} & \hat{\phi}^{(1)}(n) \\ \phi^{(2)} & \hat{\phi}^{(2)}(n) \end{pmatrix}, Q_0 = I.$$

It is easy to see that

$$Q_{n+1} = U_n U_{n-1} \ldots U_0$$

from which we have

$$\begin{aligned} &\phi^{(1)}(1) = \lambda + p_0, \phi^{(2)}(1) = \alpha\lambda, \hat{\phi}^{(1)}(1) = q_0, \hat{\phi}^{92)}(1) = s_0, \\ &\phi^{(1)}(2) = \lambda^2 + (p_1 + p_0 + \alpha q_1)\lambda, \phi^{(2)}(2) = \alpha\lambda(\lambda + p_0 + s_1), \\ &\hat{\phi}^{(1)}(2) = \lambda q_0 + p_1 q_0 + q_1 s_0, \hat{\phi}^{(2)} = \alpha\lambda q_0 + s_1 s_0, \ldots. \end{aligned}$$

Suppose δ is the eigenvalue of matrix W_n in the solution space of the spectral problem $\psi(n+1) = U_n\psi(n)$, which is invariant under the action of W_n due to $(EW_n)U_n = U_nW_n$. The corresponding eigenfunction is $\psi(n)$ which is called the Baker function satisfying

$$\psi(n+1) = U_n\psi(n), W_n\psi(n) = \delta\psi(n).$$

It is easy to see that

$$\det|\delta I - W_n| = \delta^2 - f^2(n) - g(n)h(n) = 0,$$

which has two eigenvalues $\delta^{\pm} = \pm\delta$, where

$$\delta = \sqrt{f^2(n) + g(n)h(n)} = \frac{1}{2}\sqrt{R(\lambda)}.$$

The corresponding Baker functions can be taken as

$$\phi^{\pm}(n) = \phi(n) + \alpha\hat{\phi}(n), \hat{\phi}^{\pm}(n) = \hat{\phi}(n) + \beta\phi(n),$$

where

$$\alpha = \frac{\pm\delta - f(0)}{q(0)}, \text{ or } \quad \alpha = \frac{h(0)}{\pm\delta + f(0)}, \tag{3.14}$$

$$\beta = \frac{\pm\delta + f(0)}{h(0)}, \text{ or } \quad \beta = \frac{g(0)}{\pm\delta - f(0)}. \tag{3.15}$$

A brief proof is described as follows:
Since $\phi^{\pm}(n)$ satisfies equation

$$(\delta I - W_n)\phi^{\pm}(n) = 0, \tag{3.16}$$

we assume that $\phi^{\pm}(n) = \phi(n) + \alpha\hat{\phi}(n)$, then (3.16) becomes

$$(\delta I - W_n)\begin{pmatrix} \phi^{(1)}(n) + \alpha\hat{\phi}^{(1)}(n) \\ \phi^{(2)}(n) + \alpha\hat{\phi}^{(2)} \end{pmatrix} = 0,$$

that is,

$$\begin{pmatrix} \delta - f & -g \\ -h & \delta + f \end{pmatrix}\begin{pmatrix} \phi^{(1)}(n) + \alpha\hat{\phi}^{(1)}(n) \\ \phi^{(2)}(n) + \alpha\hat{\phi}^{(2)} \end{pmatrix} = 0. \tag{3.17}$$

Specially, let $n = 0$, Eq. (3.17) still holds. Thus, we have

$$\begin{cases} (\delta - f(0))(\phi^{(1)}(0) + \alpha\hat{\phi}^{(1)}(0)) - g(0)(\phi^{(2)}(0) + \alpha\hat{\phi}^{(2)}(0)) = 0, \\ -h(0)(\phi^{(1)}(0) + \alpha\hat{\phi}^{(1)}(0)) + (\delta + f(0))(\phi^{(2)}(0) + \alpha\hat{\phi}^{(2)}(0)) = 0. \end{cases} \tag{3.18}$$

Substituting $\phi^{(1)}(0) = 1, \phi^{(2)}(0) = 0, \hat{\phi}^{(1)}(0) = 0, \hat{\phi}^{(2)}(0) = 1$ into (3.18) yields (3.14). Similarly, we can get (3.15).

Theorem 3.1. *Let $p^{\pm}(n,\lambda), q^{\pm}(n,\lambda)$ be the first component and the second one, respectively, of the Baker functions $\phi^{\pm}(n)$ and $\hat{\phi}^{\pm}(n)$. Then we have*

$$\begin{cases} p^{+}(n,\lambda)p^{-}(n,\lambda) = \dfrac{q_0}{q_n}\displaystyle\prod_{j=1}^{N-1}\frac{\lambda - \mu_j(n)}{\lambda - \mu_j(0)}, \\ q^{+}(n,\lambda)q^{-}(n,\lambda) = \dfrac{\alpha q_0 + \alpha_1}{-p_{n-1} - \alpha(E+1)q_{n-1} + s_{n-1} + \alpha_1}\displaystyle\prod_{j=1}^{N-1}\frac{\lambda - \nu_j(n)}{\lambda - \nu_j(0)}. \end{cases} \tag{3.19}$$

Actually, it is easy to see that starting from

$$Q_{n+1} = U_nQ_n, W_nQ_n = W_nU_{n-1}Q_{n-1} = \ldots = U_{n-1}U_{n-2}\ldots = U_0W_0Q_0 = Q_nW_0Q_0 = Q_nW_0,$$

we can infer (3.19).

Similar to the way presented in [6], we have the following:

Theorem 3.2 (Straightening out of the discrete flow)**.**

$$\Delta\rho^{(1)}(n) = \rho^{(1)}(n+1) - \rho^{(1)}(n) = \Omega^{(0)}(\mathrm{mod}\mathcal{J}),$$
$$\Delta\rho^{(2)}(n) = \rho^{(2)}(n+1) - \rho^{(2)}(n) = \Omega^{(0)}(\mathrm{mod}\mathcal{J}),$$

where $\Omega^{(0)} = \int_{\infty_1}^{\infty_2}\omega$.

3.4. Algebro-geometric solutions

According to the Riemann theorem [6], there exist constant vectors $M^{(1)}$ and $M^{(2)}$ such that the theta function $\theta(\mathcal{A}(p(\lambda)) - \rho^{(i)}(n) - M^{(i)})$ has N zeros at $\mu_1(n), \dots, \mu_N(n)$ for $i = 1$ or $\nu_1(n), \dots, \nu_N(n)$ for $i = 2$. We have the inversion formula

$$\sum_{j=1}^{N}\mu_j^k(n) = I_k(\Gamma) - \sum_{s=1}^{2}\mathrm{res}_{\lambda=\infty_s} d \in \theta(\mathcal{A}(p) - \rho^{(1)}(n) - M^{(1)}),$$

$$\sum_{j=1}^{N}\nu_j^k(n) = I_k(\Gamma) - \sum_{s=1}^{2}\mathrm{res}_{\lambda=\infty_s} d \in \theta(\mathcal{A}(p) - \rho^{(2)}(n) - M^{(2)}),$$

with the constant $I_k(\Gamma) = \sum\limits_{j=1}^{N}\int_{\alpha_j}\lambda^k\omega_j$. As stated in [6], we get

$$\sum_{j=1}^{N}\mu_j(n) = I_1(\Gamma) + \partial_t \ln\frac{\theta(\rho^{(1)}(n) + M^{(1)} + \pi^{(1)})}{\theta(\rho^{(1)}(n) + M^{(1)} + \pi^{(2)})} \equiv I_1(\gamma) + \partial_t \ln Y_1,$$

$$\sum_{j=1}^{N}\nu_j(n) = I_1(\Gamma) + \partial_t \ln\frac{\theta(\rho^{(2)}(n) + M^{(2)} + \pi^{(2)})}{\theta(\rho^{(2)}(n) + M^{(2)} + \pi^{(1)})} \equiv I_1(\gamma) + \partial_t \ln Y_2,$$

where $\pi^{(i)} = \int_{\infty_i}^{p_0}\omega, i = 1, 2$.
From (2.10), (3.6), (3.9), and (3.11), one infers that

$$\frac{q_{n,t}}{q_n} = -2\sum_{j=1}^{N}\mu_j(n) - 2\alpha\sum_{j=1}^{N}\nu_j(n) - 2(1+\alpha)\alpha_1 \equiv Q - \partial_t(Y_1Y_2^{\alpha})^2,$$

where $Q = -2I_1(\gamma) - 2\alpha I_2(\gamma) + (1+\alpha)\sum\limits_{j=1}^{2N}\lambda_j$. Thus, we have

$$q_n = e^{Qt}(Y_1Y_2^{\alpha})^{-2}. \tag{3.20}$$

Again we have from (2.10), (3.6), and (3.9) that

$$\partial_t \ln p_n = -2\alpha(E-1)q_n = -2\alpha\Delta q_n, \tag{3.21}$$

$$\partial_t \ln s_n = -2\sum_{j=1}^{N}\Delta\nu_j(n) + 2\alpha\Delta q_{n-1}. \tag{3.22}$$

Substituting (3.20) into (3.21) and (3.22) yields the algebro-geometric solutions of the GRTL system.

Remark 3.3. As for study of algebro-geometric solutions of differential equations and discrete lattice equations, Geng et al. [5, 7, 12–14, 17, 19] have worked out a series of interesting works. By following these consequences, algebro-geometric solutions of the discrete lattice equations obtained by (3.3) could be generated, here we do not go on discussing the question.

4. Reductions of spectral problems and the corresponding discrete integrable hierarchies, Hamiltonian structures

In the section, we reduce the spectral problem (2.1) and apply the R-matrix theory to generate a discrete integrable hierarchy whose Hamiltonian structures are obtained by Poisson tensors associated with Lie brackets of Lie algebras. At first, we simply recall basic notations and some related propositions based on [1–3].

Definition 4.1. Let g be a Lie algebra. A linear map $r : g \to g$ is called the r-matrix if the induced bracket

$$[a, b]_r = [r(a), b] + [a, r(b)], a, b \in g$$

is again a Lie bracket.

Definition 4.2. Let g be a Lie algebra with Lie bracket $[a, b], a, b \in g$. The Lie Poisson bracket on g^* which is a dual Lie algebra of g is the Poisson bracket $\{,\} : C^\infty(g^*) \times C^\infty(g^*) \to C^\infty(g^*)$ given by

$$\{f_1, f_2\}(L) =< L, [df_1, df_2] >, L \in g^*, f_1, f_2 \in C^\infty(g^*),$$

where $< a, b >$ is an invariant metric on $g :< a, b >= \mathrm{tr}(ab) = \mathrm{tr}(ba), df$ is the differential of f. It can be verified that the following formula

$$\{f_1, f_2\}(L) =< [L, df_1], r(df_2) > - < [L, df_2], r(df_1) > \tag{4.1}$$

is a Lie-Poisson bracket associated with $[,]_r$.

A Poisson tensor related to the bracket (4.1) is formulated by:

$$\{f_1, f_2\}(L) =< df_2, P(L)df_1 > .$$

It is easy to compute that

$$P(L)df = [r(df), L] + r^*(df, L]), \tag{4.2}$$

where r^* is the adjoint of r with respect to the trace duality.

Proposition 4.3. *The Hamiltonian flow to a Casimir function H is the Lax equation*

$$L_t = P(L)dH = [L, r(dH)]. \tag{4.3}$$

A natural set of Casimir functions on g is given by

$$H_q(L) = \frac{1}{q+1}\mathrm{tr}(L^{q+1}), dH_q = L^q.$$

Thus, we have the Hamiltonian structure based on (4.3):

$$L_{t_q} = [L, r(L^q)] = P(L)dH_{q+1}. \tag{4.4}$$

Assume the Lie algebra g is decomposed into two proper subalgebras g_+, g_-, i.e., $g = g_+ \cdot g_-$, then the difference of the projection operators onto these subalgebras

$$r = P_+ - P_-$$

is an r-matrix. Therefore, Eq. (4.4) can be written as

$$L_{t_q} = [P_{\geq k}(L^q), L], k = 0, 1, 2, \cdots . \tag{4.5}$$

If we choose a Lie algebra g of the shift operators as follows

$$g = \{L = \sum_{i<\infty} u_i(n)E^i\}, \tag{4.6}$$

where $E^i u(n) = (E^i u(n))E^i$, then we see only two kinds of admissible reductions of (4.6) in the following

$$\begin{aligned} &k = 0 : L = E^{\alpha+n} + u_{\alpha+n-1}E^{\alpha+n-1} + \ldots + u_\alpha E^\alpha, u_{\alpha+n} = 1, \\ &k = 1 : \bar{L} = \bar{u}_{\alpha+n}E^{\alpha+n} + \bar{u}_{\alpha+n-1}E^{\alpha+n-1} + \ldots + \bar{u}_{\alpha+1}E^{\alpha+1} + E^\alpha, \bar{u}_\alpha = 1, \end{aligned} \tag{4.7}$$

where $-n \leq \alpha \leq -1$. Therefore, Eq. (4.5) exactly becomes

$$L_{t_q} = 2[P_{\geq k}(L^q), L] = [A_q, L], k = 0, 1, \tag{4.8}$$

where $A_q = P_{\geq k}(\nabla C_q)$ and ΔC_q satisfies Novikov equation $[\nabla C_q, L] = 0$. Assume $\psi(n) = (\psi_1(n), \psi_2(n))^T$, Eq. (2.1) is decomposed into a set of discrete equations

$$\begin{cases} \psi_1(n+1) = (\lambda + p_n)\psi_1(n) + q_n\psi_2(n), \\ \psi_2(n+1) = \alpha\lambda\psi_1(n) + s_n\psi_2(n), \end{cases}$$

from which we have

$$\begin{cases} \psi_2(n+2) - s_{n+1}\psi_2(n+1) = (\lambda + p_n)\psi_2(n+1) - (\lambda + p_n)s_n\psi_2(n) + \alpha\lambda q_n\psi_2(n), \\ \psi_2(n+2) - s_{n+1}\psi_2(n+1) = [\psi_2(n+1) - s_n\psi_2(n) + \alpha q_n\psi_2(n)]\lambda + p_n\psi_2(n+1) - p_n s_n\psi_2(n). \end{cases} \tag{4.9}$$

If $\Psi(n) = \psi(2(n) - s_n\psi_2(n)$ and $q_n = 0$, then (4.9) reduces to

$$\Psi(n+1) - p_n\Psi(n) = \Psi(n)\lambda$$

from which we have a shift operator defined by:

$$L_1 = E - p_n. \tag{4.10}$$

If we set $s_n = 0$ and $\alpha = \frac{1}{\lambda}$, $E\psi_2(n) = \alpha\lambda\psi_1(n)$, then we have

$$\psi_1(n+1) - p_n\psi_1(n) - q_n E^{-1}\psi_1(n) = \lambda\psi_1(n),$$

from which a reduced operator from (4.9) is given by

$$L_2 = E - p_n - q_n E^{-1}. \tag{4.11}$$

Obviously, (4.10) is a special case of (4.11). Operator (4.11) is again a special one of (4.7), which appears in [2]. For $q_n = 1$ in (4.8), operator (4.11) is very useful because the Toda lattice equation can be given by:

$$\begin{cases} q_{n,t_1} = v_n(p_n - p_{n-1}), \\ p_{n,t_1} = q_{n+1} - q_n. \end{cases}$$

When $q_n = 2$, Eq. (4.8) reduces to

$$\begin{cases} q_{n,t_2} = q_n(p_n^2 - p_{n-1}^2 - q_{n+1} + q_{n-1}), \\ p_{n,t_2} = -q_n(p_n + p_{n+1}) + q_n(p_n + p_{n-1}), \end{cases} \tag{4.12}$$

which is a generalized Toda lattice (GTL) system. In what follows, we discover the Lax pair of the GTL system. At first, assuming that

$$\nabla C_2 = a_{-1}(n)E^{-2} + a_{-1}(n)E^{-1} + a_0(n) + a_1(n)E + a_2(n)E^2$$

and substituting it into the Novikov equation

$$[L, \nabla C_2] = 0,$$

we have

$$\begin{cases} a_1(n+1) - p_n a_2(n) - a_1(n) + a_2(n)p_{n+2} = 0, \\ a_0(n+1) - p_n a_1(n) - q_n a_2(n-1) - a_0(n) + a_1(n)p_{n+1} + a_2(n)q_{n+2} = 0, \\ a_{-1}(n+1) - p_n a_0(n) - q_n a_1(n+1) - a_{-1}(n) + a_0(n)p_n + a_1(n)q_{n+1} = 0, \\ -p_n a_{-1}(n) - q_n a_0(n-1) + a_{-1}(n)p_{n-1} + a_0(n)q_n + a_2(n+1) - a_2(n) = 0, \\ -q_n a_{-1}(n-1) + a_{-1}(n)q_{n-1} - p_n a_{-2}(n) + a_{-2}(n)p_{n-2} = 0, \\ a_{-2}(n)q_{n-2} - q_n a_{-2}(n-1) = 0. \end{cases}$$

Solving the above equations yields

$$\nabla C_2 = E^2 - (p_n + p_{n+1})E + p_n^2 - q_n - q_{n+1} + (p_{n-1}q_n + p_n q_n)E^{-1} + q_n q_{n-1}E^{-2}.$$

Therefore, we get

$$A_2 = P_{\geq 0}\nabla C_2 = E^2 - (p_n + p_{n+1})E + p_n^2 - q_n - q_{n+1}.$$

Since

$$L_2\psi(n) = E\psi(n) - p_n\psi(n) - q_n\psi_1(n) = \lambda\psi, \psi_1(n) = E^{-1}\psi(n),$$

we have

$$E\begin{pmatrix} \psi_1(n) \\ \psi(n) \end{pmatrix} = U\begin{pmatrix} \psi_1(n) \\ \psi(n) \end{pmatrix}, \tag{4.13}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ q_n & \lambda + p_n \end{pmatrix}.$$

It is easy to have that

$$\begin{aligned} A_2\psi_1(n) &= E\psi(n) - (p_n - p_{n-1})\psi(n) + (p_{n-1}^2 - q_n - q_{n-1})\psi_1, \\ A_2\psi(n) &= (\lambda^2 - q_n)\psi(n) + (\lambda - p_n)q_n\psi_1(n), \end{aligned}$$

from which we get

$$\psi(n)_{t_2} = V_2\psi(n),$$

where

$$V_2 = \begin{pmatrix} p_{n-1}^2 - q_n - q_{n-1} & E - p_n + p_{n-1} \\ (\lambda - p_n)q_n & \lambda^2 - q_n \end{pmatrix}.$$

In what follows, we want to seek for infinite conservation laws of (4.12). From (4.13), we have

$$\frac{\psi(n+1)}{\psi(n)} = \frac{\psi_{n+1}}{\psi_n} = \lambda + p_n + \frac{q_n\psi_{n-1}}{\psi_n}. \tag{4.14}$$

Note $\Gamma_n = \frac{\psi_{n+1}}{\psi_n}$, then (4.14) is written as

$$\Gamma_n = \lambda + p_n + \frac{q_n}{\Gamma_{n-1}},$$

from which one infers that

$$\Gamma_n\Gamma_{n-1}(\Gamma_{n+1} - \Gamma_n) - \Gamma_n\Gamma_{n-1}(p_{n+1} - p_n) + q_{n+1}\Gamma_{n-1} - q_n\Gamma_n = 0. \tag{4.15}$$

Assuming

$$\Gamma_n = \sum_{j=1}^{\infty} \lambda^{-j} w_n^{(j)},$$

and substituting it into (4.15), we have

$$\begin{cases} w_n^{(1)} = \gamma q_{n+1}, w_n^{(2)} = -\gamma^2 p_{n+1} q_{n+1} \Delta^{-1}(q_{n+1}q_{n+2}), \\ \sum\limits_{j+i+k=m} w_n^{(j)} w_{n-1}^{(i)} w_{n+1}^{(k)} - \sum\limits_{i+j+k=m} w_n^{(i)} w_{n-1}^{(j)} w_n^{(k)} - (p_{n+1} - p_n) \sum\limits_{i+j=m} w_n^{(i)} w_{n-1}^{(j)} \\ \quad + q_{n+1} w_{n-1}^{(m)} - q_n w_n^{(m)} = 0, m \geq 3, \end{cases}$$

where γ is a constant. It is easy to calculate that

$$\psi_{n,t} = V_2 \psi_n = E^2 \psi(n) - p_{n+1} E \psi(n) - q_{n+1} E \psi_1(n) - p_n \psi(n) + 2p_n^2 - q_n \psi(n).$$

Hence,

$$\begin{aligned} \frac{\psi_{n,t}}{\psi_n} &= \frac{\psi_{n+2}}{\psi_n} - p_{n+1}\Gamma_n - q_{n+1} - q_n - p_n \Gamma_n + 2p_n^2, \\ \partial_t \ln \Gamma_n &= (E-1)(\Gamma_{n+1}\Gamma_n - p_{n+1}\Gamma_n - q_{n+1} - q_n - p_n\Gamma_n + 2p_n^2). \end{aligned} \tag{4.16}$$

Again we have

$$\frac{\partial}{\partial t}\Gamma_n = (E-1)\frac{\partial}{\partial t}\ln \psi_n. \tag{4.17}$$

Denoting

$$M_n = \Gamma_{n+1}\Gamma_n - p_{n+1}\Gamma_n - q_{n+1} - q_n - p_n\Gamma_n + 2p_n^2, N_n = \ln \Gamma_n,$$

then we get from (4.16) and (4.17) that

$$\frac{\partial}{\partial t} N_n^{(j)} = (E-1) M_n^{(j)},\ j = 0, 1, 2, \cdots. \tag{4.18}$$

Besides, we see that

$$\frac{\partial}{\partial t}\ln \Gamma_n = \frac{\partial}{\partial t}\ln w_n^{(1)} + \frac{\partial}{\partial_t}\Big(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\lambda^{-k}}{k}\Omega^k\Big), \tag{4.19}$$

where $\Omega = \sum\limits_{j=0}^{\infty} \lambda^{-j} \frac{w_n^{(j+2)}}{w_n^{(1)}}$. Accordingly, one infers from (4.18) and (4.19) that

$$\begin{aligned} N_n^{(0)} &= \ln(\gamma q - n + 1),\ N_n^{(1)} = -\gamma p_{n+1}\Delta^{-1}(q_{n+1}q_{n+2}), \ldots, \\ M_n^{(0)} &= 2p_n^2 - q_{n+1} - q_n, M_n^{(1)} = -\gamma(p_n + p_{n+1})q_{n+1}, \ldots, \\ M_n^{(m)} &= \sum_{i+j=m} w_{n+1}^{(i)} w_n^{(j)} - p_{n+1} w_n^{(m)} - p_n w_n^{(m)}, m \geq 2. \end{aligned}$$

In what follows, we want to seek the Hamiltonian structure of the discrete integrable system (4.12) by applying new approach called variational formulas for discrete vector fields which is different from the way presented in [1–3]. The gradient of the functional $H : g \to R$ presents that

$$dH = \sum_i E^{-i} \frac{\delta H}{\delta u_i},$$

where $\frac{\delta H}{\delta u_i}$ denotes the variation with respect to discrete vector fields u_i. For the discrete lattice system (4.12), according to the shift operator (4.11), we have

$$dH_2 = E\frac{\delta H}{\delta_{-1}} + \frac{\delta H}{\delta u_0} = E\frac{\delta H}{\delta_{-q_n}} + \frac{\delta H}{\delta(-p_n)} \equiv H_{-1}(n+1)E + H_0(n). \tag{4.20}$$

Substituting (4.20) into (4.2), that is, into

$$P(L_2)dH_2 = 2[P_{\geq 0}(dH_2), L_2] - 2P_{>0}[dH_2, L_2]$$

yields that

$$P(L_2)dH_2 = -2q_{n+1}H_{-1}(n+1) + 2q_nH_{-1}(n) + [2q_nH_0(n-1) - 2H_0(n)q_n]E^{-1}.$$

Thus, we have

$$\begin{pmatrix} -q_n \\ -p_n \end{pmatrix}_{t_2} = \begin{pmatrix} -2H_0(n)q_n + 2q_nH_0(n-1) \\ -2q_{n+1}H_{-1}(n+1) + 2q_nH_{-1}(n) \end{pmatrix} = \begin{pmatrix} 0 & -2q_n(1-E^{-1}) \\ -2(E-1)q_n & 0 \end{pmatrix} \begin{pmatrix} H_{-1}(n) \\ H_0(n) \end{pmatrix}.$$

Therefore, the Poisson tensor presents

$$P(L) = \begin{pmatrix} 0 & 2q_n(1-E^{-1}) \\ 2(E-1)q_n & 0 \end{pmatrix},$$

which is consistent to that presented in [1]. The Hamiltonian structure of (4.12) is given by

$$\begin{pmatrix} q_n \\ p_n \end{pmatrix}_{t_2} = P(L)\frac{\delta H}{\delta u}.$$

Remark 4.4. According to the approach, we could derive Hamiltonian structures of other higher-order discrete lattice systems generated from Eq. (4.8).

Finally, we discover a kind of discrete expanding integrable model of the Toda lattice equation. We first construct a new integrable hierarchy

$$\bar{L}_{t_q} = [P_{\geq k}(L^q), \bar{L}], \tag{4.21}$$

where $\bar{L}$ represents an enlarging shit Lie algebra of the Lie subalgebra L presented in (4.7), which can be expressed by $\bar{L} = L + \tilde{L}$, where $\tilde{L}$ is the enlarging part of L. For the shift operator (4.11), we replace $-p_n$ by p_n, $-q_n$ by v_n, then (4.11) can be written as

$$L = E + p_n + v_nE^{-1}.$$

Now we take the following shift operator

$$\bar{L} = v_nE^{-1} + p_n + E + q_nE^2 + w_nE^3 + r_nE^4 =: L_1 + L_2,$$

where $L_1 = v_nE^{-1} + p_n + E, L_2 = q_nE^2 + w_nE^3 + r_nE^4$.

By making use of the modified Lax representation (4.21), we can generate the following discrete integrable system

$$\begin{cases} v_{n,t_1} = p_nv_n - v_np_{n-1}, \\ p_{n,t_1} = v_{n+1} - v_n, \\ q_{n,t_1} = p_nq_n - q_np_{n+2}, \\ w_{n,t_1} = p_nw_n + q_{n+1} - w_np_{n+1}, \\ r_{n,t_1} = p_nr_n - w_n - r_np_{n+4}, \\ r_{n+1} - r_n = 0. \end{cases} \tag{4.22}$$

Taking $r_n = 1$, we find that Eq. (4.22) reduces to

$$\begin{cases} v_{n,t_1} = v_n(p_n - p_{n-1}), \\ p_{n,t_1} = v_{n+1} - v_n, \\ q_{n,t_1} = q_n(p_n - p_{n+2}), \\ (p_n - p_{n+2})_{t_1} = (p_{n+4} - p_n)(p_{n+1} - p_n) + q_{n+1}. \end{cases} \tag{4.23}$$

Taking $q_n = p_n = 0$, we notice that Eq. (4.23) reduces to the Toda lattice system.

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