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A new modified semi-implicit midpoint rule for nonexpansive mappings and 2-generalized hybrid mappings

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Abstract

In this paper, we present a new modified semi-implicit midpoint rule with the viscosity technique for finding a common fixed point of nonexpansive mappings and 2-generalized hybrid mappings in a real Hilbert space. The proposed algorithm is based on implicit midpoint rule and viscosity approximation method. Under some mild conditions, the strong convergence of the iteration sequences generated by the proposed algorithm is derived.

Keywords: Hilbert space, fixed point, nonexpansive mapping, 2-generalized hybrid mapping, semi-implicit midpoint rule.
2010 MSC: 47H09, 47H10, 49H17.

1. Introduction

The implicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations (in particular, the stiff equations) and differential algebra equations. For related works, we refer to [2, 3, 9, 11, 13, 14, 20–22].

For the ordinary differential equation

$$x' = f(t), \quad x(0) = x_0, \tag{1.1}$$

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the implicit midpoint rule generates a sequence $\{x_n\}$ by the recursive procedure

$$x_{n+1} = x_n + hf(\frac{x_n + x_{n+1}}{2}), \tag{1.2}$$

where $h > 0$ is a stepsize. It is known that if $f : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous and sufficiently smooth, then the sequence $\{x_n\}$ generated by (1.2) converges to the exact solution of (1.1) as $h \to 0$ uniformly over $t \in [0, \bar{t})$ for any fixed $\bar{t} > 0$.

If we write the function f in the form $f(t) = g(t) - t$, then differential equation (1.1) becomes $x' = g(t) - t$. Then the equilibrium problem associated with the differential equation is the fixed point problem $t = g(t)$.

Based on the above fact, in [1] and [26], the authors presented the following semi-implicit midpoint rule for nonexpansive mappings:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(\frac{x_n + x_{n+1}}{2}), \tag{1.3}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}), \tag{1.4}$$

where f is a contraction and $T : H \to H$ is a nonexpansive mapping. They proved the weak convergence of (1.3) and strong convergence of (1.4) under some mild conditions, respectively.

Furthermore, Yao et al. [29] applied the viscosity technique to the implicit rules of nonexpansive mappings in Hilbert spaces and proved that the sequence $\{x_n\}$ defined by the following viscosity semi-implicit midpoint rule

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\frac{x_n + x_{n+1}}{2}),$$

converges strongly to the unique solution $z \in \mathrm{Fix}(T)$ of the variational inequality (VI)

$$\langle (I - f)z, x - z\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.5}$$

Motivated and inspired by the above facts, Yu and Wen [31] also proved that the sequence $\{x_n\}$ defined by the following iterative method

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\delta_n x_n + (1 - \delta_n)x_{n+1}),$$

converges strongly to the unique solution $z \in \mathrm{Fix}(T)$ of the variational inequality VI (1.5).

Remark 1.1. The usefulness of (1.4) is that it can be used to find a periodic solution of the time-dependent nonlinear evolution equation (see [26])

$$\frac{du}{dt} + A(t)u = g(t, u), \quad t \geq 0,$$

where $A(t)$ is a family of closed linear operators in a Hilbert space H and g maps $\mathbb{R}^1 \times H$ into H.

In this paper, we present a new modified semi-implicit midpoint rule with the viscosity technique for finding a common fixed point of nonexpansive mappings and 2-generalized hybrid mappings in a real Hilbert space. The proposed algorithm is based on implicit midpoint rule (see [1, 26]) and viscosity approximation method (see [19, 24, 28]). Under some mild conditions, the strong convergence of the iteration sequences generated by the proposed algorithm is derived. Our results extend, improve and develop the corresponding results in [1, 26, 29, 31].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space, C is a nonempty and closed convex subset of H. In the sequel, we denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong and weak convergences of $\{x_n\}$, respectively. Denote by $\mathrm{Fix}(T)$ the set of fixed points of a mapping $T : C \to C$. Namely,

$$\mathrm{Fix}(T) = \{x \in C : Tx = x\}.$$

For each $x, y \in H$ and $\gamma \in [0, 1]$, we have

$$\|\gamma x + (1-\gamma)y\|^2 = \gamma \|x\|^2 + (1-\gamma)\|y\|^2 - \gamma(1-\gamma)\|x-y\|^2.$$

Furthermore, we see that, for all $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.1}$$

Definition 2.1. A mapping $T : C \to H$ is said to be:

(1) a nonexpansive mapping, if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$;
(2) a nonspreading mapping, if $2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|x - Ty\|^2$, for all $x, y \in C$;
(3) a hybrid mapping, if $3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|x - Ty\|^2$, for all $x, y \in C$;
(4) a generalized hybrid mapping, if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2, \quad \forall x, y \in C;$$

(5) a 2-generalized hybrid mapping, if there exist $\delta_1, \delta_2, \epsilon_1, \epsilon_2 \in \mathbb{R}$ such that

$$\begin{aligned}\delta_1\left\|T^2x - Ty\right\|^2 + \delta_2\|Tx - Ty\|^2 + (1-\delta_1-\delta_2)\|x - Ty\|^2 \le \epsilon_1\left\|T^2x - y\right\|^2 \\ + \epsilon_2\|Tx - y\|^2 + (1-\epsilon_1-\epsilon_2)\|x - y\|^2,\end{aligned}$$

for all $x, y \in C$.

We know that the class of 2-generalized hybrid mappings contains the classes of nonexpansive mappings, nonspreading mappings, hybrid mappings and generalized hybrid mappings in a Hilbert space (see[15, 30]). We give an example for a 2-generalized hybrid mapping.

Example 2.2 ([18])**.** Let $S : [0, 2] \to \mathbb{R}$ be defined as

$$Sx = \begin{cases} 0, & x \in [0, 2); \\ 1, & x = 2. \end{cases}$$

Then S is a 2-generalized hybrid mapping and $\mathrm{Fix}(S) = \{0\}$.

In 2012, Hojo et al. [12] also gave an example for a 2-generalized hybrid mapping which is not a generalized hybrid mapping with $\mathrm{Fix}(T) = \{(0, 0)\}$ as follow.

Example 2.3. Let $A = \{x \in \mathbb{R}^2 : \|x\| \le 1\}$ and $T : A \to \mathbb{R}$ be defined as

$$Tx = \begin{cases} (0,0), & x \in A; \\ \frac{x}{\|x\|}, & x \in \mathbb{R}^2/A. \end{cases}$$

Hojo et al. [12] showed that T is a 2-generalized hybrid mapping, but T is not a generalized hybrid mapping. Note that T does not have the demiclosed property. Indeed, there exists a sequence $\{x_n\} \subset A$ such that $x_n \rightharpoonup \omega$ and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, but ω in $\mathbb{R}^2/\mathrm{Fix}(T) = \mathbb{R}^2/\{(0, 0)\}$.

Proof. Let $r_n = 1 + \frac{1}{n}$, $x_n = (r_n \cos\theta, r_n \sin\theta)$ for all $n \in \mathbb{N}$, then $x_n \to (\cos\theta, \sin\theta)$ and $Tx_n = (\cos\theta, \sin\theta)$. We also have $\|Tx_n - x_n\| = \|(r_n - 1)\cos\theta, (r_n - 1)\sin\theta)\| = (r_n - 1) \to 0$, but $(\cos\theta, \sin\theta) \ne (0, 0)$. □

To obtain our main results, we need the following lemmas.

Lemma 2.4 ([23]). *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers satisfying the property*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + b_n + \gamma_n c_n, \quad n \in \mathbb{N},$$

where $\{\gamma_n\}, \{b_n\}, \{c_n\}$ satisfy the restrictions:

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty, \quad \lim_{n\to\infty} \gamma_n = 0$;
(ii) $b_n \geq 0, \quad \sum_{n=1}^{\infty} b_n < \infty$;
(iii) $\limsup_{n\to\infty} c_n \leq 0$.

Then, $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5 ([32]). *Let H be a Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 0, 1, 2, \cdots, n$ such that $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$, the following inequality holds*

$$\left\|\sum_{i=0}^{n} \alpha_i x_i\right\|^2 \leq \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0\leq i,j\leq n} \alpha_i\alpha_j \|x_i - x_j\|^2 .$$

Lemma 2.6 ([16]). *Let $\{\alpha_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\alpha_{n_i} < \alpha_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied for all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$\alpha_{m_k} \leq \alpha_{m_k+1} \quad \text{and} \quad \alpha_k \leq \alpha_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : \alpha_j < \alpha_{j+1}\}$.

Lemma 2.7 ([17, 27]). *Let C be a closed convex subset of a real Hilbert space H. Suppose $x \in H$ and $y \in C$ are given. Then $y = P_C x$, if and only if the following inequality holds*

$$\langle x - y, z - y\rangle \leq 0,$$

for every $z \in C$.

Lemma 2.8 ([10]). (Demiclosedness principle). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping. Then, the mapping $I - T$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$, then $(I - T)x = y$.*

3. Main results

Theorem 3.1. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be a 2-generalized hybrid mapping. Moreover, let $f : C \to C$ be a contraction with coefficient $\alpha \in [0,1)$. Suppose that $\mathrm{F} := \mathrm{Fix}(T) \bigcap \mathrm{Fix}(S) \neq \emptyset$. For given $x_1 \in C$ arbitrarily, define*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T\left(\frac{\delta_n}{n}\sum_{k=0}^{n-1} S^k x_n + (1-\delta_n)x_{n+1}\right), \quad n \geq 1, \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real number sequences in $[0,1]$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\alpha_n + \beta_n + \gamma_n = 1$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$.

Then $\{x_n\}$ *converges strongly to a point* $p \in \mathrm{F}$*, where* $p = P_{\mathrm{F}} f(p)$*.*

Proof. Equation (3.1) is well-defined. As a matter of fact, for fixed $u \in C$, we can define a mapping

$$x \mapsto T_u x := \alpha f(u) + \beta u + \gamma T\left(\frac{\delta}{N}\sum_{k=0}^{N-1} S^k u + (1-\delta)x\right), \quad x \in C.$$

In light of the nonexpansiveness of T, we deduce that

$$\begin{aligned}\|T_u x - T_u y\| &= \left\|\gamma T\left(\frac{\delta}{N}\sum_{k=0}^{N-1} S^k u + (1-\delta)x\right) - \gamma T\left(\frac{\delta}{N}\sum_{k=0}^{N-1} S^k u + (1-\delta)y\right)\right\| \\ &\le \gamma(1-\delta)\|x-y\|.\end{aligned}$$

This means T_u is a contraction with coefficient $\gamma(1-\delta) \in (0,1)$. Hence the algorithm (3.1) is well-defined.

We show the sequence $\{x_n\}$ generated by (3.1) is bounded. Take any $x^* \in \mathrm{F}$ and let

$$S_n := \frac{1}{n}\sum_{k=0}^{n-1} S^k.$$

We see S_n is quasi-nonexpansive. Indeed, since S is a 2-generalized hybrid mapping, we know that S is a quasi-nonexpansive mapping, and hence

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} S^k x - x^*\right\| \le \frac{1}{n}\sum_{k=0}^{n-1}\left\|S^k x - x^*\right\| \le \frac{1}{n}\sum_{k=0}^{n-1}\|x-x^*\| = \|x-x^*\|.$$

From (3.1), we find that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\delta_n S_n x_n + (1-\delta_n)x_{n+1}) - x^*\| \\ &\le \alpha_n\|f(x_n)-f(x^*)\| + \alpha_n\|f(x^*)-x^*\| + \beta_n\|x_n-x^*\| \\ &\quad + \gamma_n\|T(\delta_n S_n x_n + (1-\delta_n)x_{n+1}) - x^*\| \\ &\le \alpha_n\alpha\|x_n-x^*\| + \alpha_n\|f(x^*)-x^*\| + \beta_n\|x_n-x^*\| \\ &\quad + \gamma_n\|\delta_n(S_n x_n - x^*) + (1-\delta_n)(x_{n+1}-x^*)\| \\ &\le \alpha_n\alpha\|x_n-x^*\| + \alpha_n\|f(x^*)-x^*\| + \beta_n\|x_n-x^*\| \\ &\quad + \gamma_n\delta_n\|S_n x_n - x^*\| + \gamma_n(1-\delta_n)\|x_{n+1}-x^*\| \\ &\le \alpha_n\|f(x^*)-x^*\| + (\alpha_n\alpha+\beta_n+\gamma_n\delta_n)\|x_n-x^*\| + \gamma_n(1-\delta_n)\|x_{n+1}-x^*\|.\end{aligned} \tag{3.2}$$

We derive from (3.2) that

$$(1-\gamma_n(1-\delta_n))\|x_{n+1}-x^*\| \le \alpha_n\|f(x^*)-x^*\| + (\alpha_n\alpha+\beta_n+\gamma_n\delta_n)\|x_n-x^*\|,$$

which implies

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \frac{\alpha_n(1-\alpha)}{1-\gamma_n(1-\delta_n)}\frac{\|f(x^*)-x^*\|}{1-\alpha} + \frac{1-\gamma_n(1-\delta_n)-\alpha_n(1-\alpha)}{1-\gamma_n(1-\delta_n)}\|x_n-x^*\| \\ &\le \max\{\frac{\|f(x^*)-x^*\|}{1-\alpha}, \|x_n-x^*\|\}.\end{aligned}$$

By the induction, we deduce

$$\|x_{n+1}-x^*\| \le \max\{\frac{\|f(x^*)-x^*\|}{1-\alpha}, \|x_1-x^*\|\}.$$

This implies that the sequence $\{x_n\}$ is bounded.

From Lemma 2.5 and (3.1), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x^*\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x^*\|^2\\ &\quad -\beta_n\gamma_n\|T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x_n\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|\delta_n S_n x_n+(1-\delta_n)x_{n+1}-x^*\|^2\\ &\quad -\beta_n\gamma_n\|T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x_n\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\delta_n\|x_n-x^*\|^2+\gamma_n(1-\delta_n)\|x_{n+1}-x^*\|^2\\ &\quad -\beta_n\gamma_n\|T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x_n\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+(\beta_n+\gamma_n\delta_n)\|x_n-x^*\|^2+\gamma_n(1-\delta_n)\|x_{n+1}-x^*\|^2\\ &\quad -\beta_n\gamma_n\|T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x_n\|^2,\end{aligned}$$

which implies

$$\begin{aligned}\beta_n\gamma_n\|T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x_n\|^2 &\le \alpha_n(\|f(x_n)-x^*\|^2-\|x_{n+1}-x^*\|^2)\\ &\quad +(\beta_n+\gamma_n\delta_n)(\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2).\end{aligned} \tag{3.3}$$

Similarly, we also have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x^*\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x^*\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|\delta_n S_n x_n+(1-\delta_n)x_{n+1}-x^*\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\delta_n\|x_n-x^*\|^2+\gamma_n(1-\delta_n)\|x_{n+1}-x^*\|^2\\ &\quad -\gamma_n\delta_n(1-\delta_n)\|S_n x_n-x_{n+1}\|^2\\ &\le \alpha_n\|f(x_n)-x^*\|^2+(\beta_n+\gamma_n\delta_n)\|x_n-x^*\|^2+\gamma_n(1-\delta_n)\|x_{n+1}-x^*\|^2\\ &\quad -\gamma_n\delta_n(1-\delta_n)\|S_n x_n-x_{n+1}\|^2,\end{aligned}$$

which yields that

$$\begin{aligned}\gamma_n\delta_n(1-\delta_n)\|S_n x_n-x_{n+1}\|^2 &\le \alpha_n(\|f(x_n)-x^*\|^2-\|x_{n+1}-x^*\|^2)\\ &\quad +(\beta_n+\gamma_n\delta_n)(\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2).\end{aligned} \tag{3.4}$$

Since F is a nonempty closed convex subset of H, we can take $p\in \mathrm{F}$ such that $p=P_{\mathrm{F}}f(p)$. By Lemma 2.7, this point p is also a unique solution of the hierarchical variational inequality

$$\langle f(p)-p, q-p\rangle \le 0, \quad \forall q\in \mathrm{F}.$$

Next we divide our proof into two possible cases.

Case1: Suppose that there exists $n_0\in\mathbb{N}$ such that

$$\|x_{n+1}-p\| \le \|x_n-p\|, \tag{3.5}$$

for all $n\ge n_0$.

Then we see that $\{\|x_n-p\|\}$ is convergent. Thus, from (i)-(iii), (3.3) and (3.5), we obtain

$$\lim_{n\to\infty}\|T(\delta_n S_n x_n+(1-\delta_n)x_{n+1})-x_n\|=0. \tag{3.6}$$

By (i), (3.1) and (3.6), we have that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|\alpha_n(f(x_n) - x_n) + \gamma_n(T(\delta_n S_n x_n + (1-\delta_n)x_{n+1}) - x_n)\| = 0. \tag{3.7}$$

Moreover, from (i)-(iv), (3.4) and (3.5) we get

$$\lim_{n\to\infty} \|S_n x_n - x_{n+1}\| = 0. \tag{3.8}$$

It follows immediately from (3.7) and (3.8) that

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{3.9}$$

By setting $y_n = \delta_n S_n x_n + (1-\delta_n)x_{n+1}$, we find that

$$\|y_n - x_n\| = \|\delta_n S_n x_n + (1-\delta_n)x_{n+1} - x_n\| \le \delta_n \|S_n x_n - x_n\| + (1-\delta_n)\|x_{n+1} - x_n\|.$$

This along with (3.7) and (3.9) implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.10}$$

Further, in light of (3.6), (3.10) and the fact $\|Ty_n - y_n\| \le \|Ty_n - x_n\| + \|y_n - x_n\|$, we deduce that

$$\lim_{n\to\infty} \|Ty_n - y_n\| = 0. \tag{3.11}$$

Next, we want to show that

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle \le 0. \tag{3.12}$$

Without loss of generality, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \omega$ for some $\omega \in C$ and

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle = \lim_{i\to\infty} \langle f(p) - p, x_{n_i} - p\rangle.$$

Since S is a 2-generalized hybrid mapping, there exist $\delta_1, \delta_2, \epsilon_1, \epsilon_2 \in \mathbb{R}$ such that

$$\begin{aligned}\delta_1 \left\|S^2x - Sy\right\|^2 + \delta_2 \|Sx - Sy\|^2 + (1-\delta_1-\delta_2)\|x - Sy\|^2 \le \epsilon_1 \left\|S^2x - y\right\|^2 + \epsilon_2 \|Sx - y\|^2 \\ + (1-\epsilon_1-\epsilon_2)\|x - y\|^2,\end{aligned}$$

for all $x, y \in C$. By replacing x by $S^k x_n$ in above inequality, we have from (2.1), for all $y \in C$ and $k = 0, 1, 2..., n-1$,

$$\begin{aligned}
&\delta_1 \left\|S^{k+2}x_n - Sy\right\|^2 + \delta_2 \left\|S^{k+1}x_n - Sy\right\|^2 + (1-\delta_1-\delta_2)\left\|S^k x_n - Sy\right\|^2 \\
&\le \epsilon_1 \left\|S^{k+2}x_n - y\right\|^2 + \epsilon_2 \left\|S^{k+1}x_n - y\right\|^2 \\
&\quad + (1-\epsilon_1-\epsilon_2)\left\|S^k x_n - y\right\|^2 \\
&\le \epsilon_1(\left\|S^{k+2}x_n - Sy\right\|^2 + \|Sy - y\|^2 \\
&\quad + 2\langle S^{k+2}x_n - Sy, Sy - y\rangle) \\
&\quad + \epsilon_2(\left\|S^{k+1}x_n - Sy\right\|^2 + \|Sy - y\|^2 \\
&\quad + 2\langle S^{k+1}x_n - Sy, Sy - y\rangle) \\
&\quad + (1-\epsilon_1-\epsilon_2)(\left\|S^k x_n - Sy\right\|^2
\end{aligned}$$

$$+ \|Sy-y\|^2 + 2\langle S^k x_n - Sy, Sy-y\rangle).$$

This implies that

$$\begin{aligned}
0 &\le (\epsilon_1-\delta_1)\left\|S^{k+2}x_n - Sy\right\|^2 + \|Sy-y\|^2 + 2\epsilon_1\langle S^{k+2}x_n - Sy, Sy-y\rangle \\
&\quad + (\epsilon_2-\delta_2)\left\|S^{k+1}x_n - Sy\right\|^2 + 2\epsilon_2\langle S^{k+1}x_n - Sy, Sy-y\rangle \\
&\quad + (\delta_1-\epsilon_1+\delta_2-\epsilon_2)\left\|S^k x_n - Sy\right\|^2 + 2(1-\epsilon_1-\epsilon_2)\langle S^k x_n - Sy, Sy-y\rangle \\
&\le (\epsilon_1-\delta_1)(\left\|S^{k+2}x_n - Sy\right\|^2 - \left\|S^k x_n - Sy\right\|^2) + (\epsilon_2-\delta_2)(\left\|S^{k+1}x_n - Sy\right\|^2 - \left\|S^k x_n - Sy\right\|^2) \\
&\quad + \|Sy-y\|^2 + 2\langle S^k x_n - Sy + \epsilon_1(S^{k+2}x_n - S^k x_n) + \epsilon_2(S^{k+1}x_n - S^k x_n), Sy-y\rangle.
\end{aligned} \tag{3.13}$$

By summing up these inequalities (3.13) with respect to $k=0$ to $k=n-1$ and dividing by n, we have

$$\begin{aligned}
0 &\le \frac{\epsilon_1-\delta_1}{n}(\left\|S^{n+1}x_n - Sy\right\|^2 + \|S^n x_n - Sy\|^2 - \|Sx_n - Sy\|^2 - \|x_n - Sy\|^2) \\
&\quad + \frac{\epsilon_2-\delta_2}{n}(\|S^n x_n - Sy\|^2 - \|x_n - Sy\|^2) + \|Sy-y\|^2 + 2\langle S_n x_n - Sy, Sy-y\rangle \\
&\quad + \frac{2}{n}\langle \epsilon_1(S^{n+1}x_n + S^n x_n - Sx_n - x_n) + \epsilon_2(S^n x_n - x_n), Sy-y\rangle.
\end{aligned} \tag{3.14}$$

Replace n by n_i and let $n_i \to \infty$. Then from (3.9) and (3.14), we have $S_{n_i}x_{n_i} \rightharpoonup \omega$ and

$$0 \le \|Sy-y\|^2 + 2\langle \omega - Sy, Sy-y\rangle.$$

By taking $y=\omega$ in the above inequality, we have

$$0 \le \|S\omega-\omega\|^2 + 2\langle \omega - S\omega, S\omega-\omega\rangle = \|S\omega-\omega\|^2 - 2\|S\omega-\omega\|^2 = -\|S\omega-\omega\|^2.$$

This implies that $w \in \mathrm{Fix}(S)$. In light of (3.10), (3.11) and Lemma 2.8, we also have that $\omega \in \mathrm{Fix}(T)$. Then it turns out that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(p)-p, x_n-p\rangle &= \lim_{i\to\infty}\langle f(p)-p, x_{n_i}-p\rangle \\
&= \langle f(p)-p, \omega-p\rangle \\
&= \langle f(p)-P_{\mathrm{F}}f(p), \omega-P_{\mathrm{F}}f(p)\rangle \\
&\le 0.
\end{aligned}$$

Finally, we prove that $x_n \to p$. Notice

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\delta_n S_n x_n + (1-\delta_n)x_{n+1}) - p, x_{n+1}-p\rangle \\
&= \alpha_n\langle f(x_n)-f(p), x_{n+1}-p\rangle + \alpha_n\langle f(p)-p, x_{n+1}-p\rangle + \beta_n\langle x_n-p, x_{n+1}-p\rangle \\
&\quad + \gamma_n\langle T(\delta_n S_n x_n + (1-\delta_n)x_{n+1}) - p, x_{n+1}-p\rangle \\
&\le \alpha_n\alpha\|x_n-p\|\|x_{n+1}-p\| + \alpha_n\langle f(p)-p, x_{n+1}-p\rangle + \beta_n\|x_n-p\|\|x_{n+1}-p\| \\
&\quad + \gamma_n\|T(\delta_n S_n x_n + (1-\delta_n)x_{n+1}) - p\|\|x_{n+1}-p\| \\
&\le \alpha_n\alpha\|x_n-p\|\|x_{n+1}-p\| + \alpha_n\langle f(p)-p, x_{n+1}-p\rangle + \beta_n\|x_n-p\|\|x_{n+1}-p\| \\
&\quad + \gamma_n\delta_n\|x_n-p\|\|x_{n+1}-p\| + \gamma_n(1-\delta_n)\|x_{n+1}-p\|^2 \\
&\le \frac{1}{2}(\alpha_n\alpha+\beta_n+\gamma_n\delta_n)\|x_n-p\|^2 + \frac{1}{2}(\alpha_n\alpha+\beta_n+\gamma_n\delta_n)\|x_{n+1}-p\|^2 \\
&\quad + \alpha_n\langle f(p)-p, x_{n+1}-p\rangle + \gamma_n(1-\delta_n)\|x_{n+1}-p\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}(\alpha_n - \frac{1}{2}\alpha_n\alpha + \frac{1}{2}\beta_n + \frac{1}{2}\gamma_n\delta_n) \|x_{n+1} - p\|^2 \leq &\left(\alpha_n - \frac{1}{2}\alpha_n\alpha + \frac{1}{2}\beta_n + \frac{1}{2}\gamma_n\delta_n - \alpha_n(1-\alpha)\right) \\ &\times \|x_n - p\|^2 + \alpha_n\langle f(p) - p, x_{n+1} - p\rangle,\end{aligned}$$

which yields

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq &\frac{\alpha_n - \frac{1}{2}\alpha_n\alpha + \frac{1}{2}\beta_n + \frac{1}{2}\gamma_n\delta_n - \alpha_n(1-\alpha)}{\alpha_n - \frac{1}{2}\alpha_n\alpha + \frac{1}{2}\beta_n + \frac{1}{2}\gamma_n\delta_n} \|x_n - p\|^2 \\ &+ \frac{\alpha_n(1-\alpha)}{\alpha_n - \frac{1}{2}\alpha_n\alpha + \frac{1}{2}\beta_n + \frac{1}{2}\gamma_n\delta_n} \frac{\langle f(p) - p, x_{n+1} - p\rangle}{1-\alpha}.\end{aligned} \tag{3.15}$$

Applying Lemma 2.4 and (3.12) to (3.15) to deduce that $x_n \to p$.

Case 2: Suppose that there exists $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - p\| < \|x_{n_i+1} - p\|$ for all $i \in \mathbb{N}$.

By Lemma 2.6, there exists a nondecreasing sequence $\{m_j\}$ in $\mathbb{N}$ such that

$$\left\|x_{m_j} - p\right\| \leq \left\|x_{m_j+1} - p\right\| \quad \text{and} \quad \|x_j - p\| \leq \left\|x_{m_j+1} - p\right\|. \tag{3.16}$$

Then by (3.3), we have

$$\begin{aligned}\beta_{m_j}\gamma_{m_j}\left\|T(\delta_{m_j}S_{m_j}x_{m_j} + (1-\delta_{m_j})x_{m_j+1}) - x_{m_j}\right\|^2 \leq &\alpha_{m_j}(\left\|f(x_{m_j}) - p\right\|^2 - \left\|x_{m_j+1} - p\right\|^2) \\ &+ (\beta_{m_j} + \gamma_{m_j}\delta_{m_j})(\left\|x_{m_j} - p\right\|^2 - \left\|x_{m_j+1} - p\right\|^2).\end{aligned} \tag{3.17}$$

And hence (i)-(iii), (3.16) and (3.17) imply that

$$\lim_{j\to\infty} \left\|T\left(\delta_{m_j}S_{m_j}x_{m_j} + (1-\delta_{m_j})x_{m_j+1}\right) - x_{m_j}\right\| = 0. \tag{3.18}$$

Moreover, from (i) and (3.18), we deduce

$$\begin{aligned}\lim_{j\to\infty}\left\|x_{m_j+1} - x_{m_j}\right\| &= \lim_{j\to\infty}\left\|\alpha_{m_j}(f(x_{m_j}) - x_{m_j}) + \gamma_{m_j}(T(\delta_{m_j}S_{m_j}x_{m_j} + (1-\delta_{m_j})x_{m_j+1}) - x_{m_j})\right\| \\ &= 0.\end{aligned} \tag{3.19}$$

Thus, like in Case 1, we derive from (3.4), (i)-(iv) and (3.16) that

$$\lim_{j\to\infty}\left\|S_{m_j}x_{m_j} - x_{m_j+1}\right\| = 0. \tag{3.20}$$

By combining (3.19) and (3.20), we find that

$$\lim_{j\to\infty}\left\|S_{m_j}x_{m_j} - x_{m_j}\right\| = 0. \tag{3.21}$$

By setting $y_{m_j} = \delta_{m_j}S_{m_j}x_{m_j} + (1-\delta_{m_j})x_{m_j+1}$, we have

$$\begin{aligned}\left\|y_{m_j} - x_{m_j}\right\| &= \left\|\delta_{m_j}S_{m_j}x_{m_j} + (1-\delta_{m_j})x_{m_j+1} - x_{m_j}\right\| \\ &\leq \delta_{m_j}\left\|S_{m_j}x_{m_j} - x_{m_j}\right\| + (1-\delta_{m_j})\left\|x_{m_j+1} - x_{m_j}\right\|.\end{aligned} \tag{3.22}$$

Thus, we have from (3.19), (3.21) and (3.22) that

$$\lim_{j\to\infty}\left\|y_{m_j} - x_{m_j}\right\| = 0. \tag{3.23}$$

Due to (3.18), (3.23) and the fact that $\left\|Ty_{m_j} - y_{m_j}\right\| \leq \left\|Ty_{m_j} - x_{m_j}\right\| + \left\|y_{m_j} - x_{m_j}\right\|$, we see

$$\lim_{j\to\infty} \left\|Ty_{m_j} - y_{m_j}\right\| = 0. \tag{3.24}$$

We want to show that

$$\limsup_{j\to\infty} \langle f(p) - p, x_{m_j} - p\rangle \leq 0, \tag{3.25}$$

where $p = P_{\mathrm{F}} f(p)$. Without loss of generality, there exists a subsequence $\{x_{m_{j_k}}\}$ of $\{x_{m_j}\}$ such that $x_{m_{j_k}} \rightharpoonup \omega$ for some $\omega \in C$ and

$$\limsup_{j\to\infty} \langle f(p) - p, x_{m_j} - p\rangle = \lim_{k\to\infty} \langle f(p) - p, x_{m_{j_k}} - p\rangle.$$

By virtue of (3.23), (3.24) and Lemma 2.8, we deduce that $\omega \in \mathrm{Fix}(T)$. By following a similar argument as in the proof of Case 1, we also have $\omega \in \mathrm{Fix}(S)$. Therefore, we have

$$\begin{aligned}\limsup_{j\to\infty} \langle f(p) - p, x_{m_j} - p\rangle &= \lim_{k\to\infty} \langle f(p) - p, x_{m_{j_k}} - p\rangle \\ &= \langle f(p) - P_{\mathrm{F}} f(p), \omega - P_{\mathrm{F}} f(p)\rangle \\ &\leq 0.\end{aligned}$$

It follows that

$$\begin{aligned}\left\|x_{m_j+1} - p\right\|^2 &= \langle \alpha_{m_j} f(x_{m_j}) + \beta_{m_j} x_{m_j} + \gamma_{m_j} T(\delta_{m_j} S_{m_j} x_{m_j} + (1-\delta_{m_j}) x_{m_j+1}) - p, x_{m_j+1} - p\rangle \\ &= \alpha_{m_j} \langle f(x_{m_j}) - f(p), x_{m_j+1} - p\rangle + \alpha_{m_j} \langle f(p) - p, x_{m_j+1} - p\rangle + \beta_{m_j} \langle x_{m_j} - p, x_{m_j+1} - p\rangle \\ &\quad + \gamma_{m_j} \langle T(\delta_{m_j} S_{m_j} x_{m_j} + (1-\delta_{m_j}) x_{m_j+1}) - p, x_{m_j+1} - p\rangle \\ &\leq \alpha_{m_j} \alpha \left\|x_{m_j} - p\right\| \left\|x_{m_j+1} - p\right\| + \alpha_{m_j} \langle f(p) - p, x_{m_j+1} - p\rangle + \beta_{m_j} \left\|x_{m_j} - p\right\| \left\|x_{m_j+1} - p\right\| \\ &\quad + \gamma_{m_j} \left\|T(\delta_{m_j} S_{m_j} x_{m_j} + (1-\delta_{m_j}) x_{m_j+1}) - p\right\| \left\|x_{m_j+1} - p\right\| \\ &\leq \alpha_{m_j} \alpha \left\|x_{m_j} - p\right\| \left\|x_{m_j+1} - p\right\| + \alpha_{m_j} \langle f(p) - p, x_{m_j+1} - p\rangle + \beta_{m_j} \left\|x_{m_j} - p\right\| \left\|x_{m_j+1} - p\right\| \\ &\quad + \gamma_{m_j} \delta_{m_j} \left\|x_{m_j} - p\right\| \left\|x_{m_j+1} - p\right\| + \gamma_{m_j} (1-\delta_{m_j}) \left\|x_{m_j+1} - p\right\|^2 \\ &\leq \frac{1}{2}(\alpha_{m_j} \alpha + \beta_{m_j} + \gamma_{m_j} \delta_{m_j}) \left\|x_{m_j} - p\right\|^2 + \frac{1}{2}(\alpha_{m_j} \alpha + \beta_{m_j} + \gamma_{m_j} \delta_{m_j}) \left\|x_{m_j+1} - p\right\|^2 \\ &\quad + \alpha_{m_j} \langle f(p) - p, x_{m_j+1} - p\rangle + \gamma_{m_j} (1-\delta_{m_j}) \left\|x_{m_j+1} - p\right\|^2,\end{aligned}$$

which yields that

$$\begin{aligned}\left(\alpha_{m_j} - \frac{1}{2}\alpha_{m_j}\alpha + \frac{1}{2}\beta_{m_j} + \frac{1}{2}\gamma_{m_j}\delta_{m_j}\right) \left\|x_{m_j+1} - p\right\|^2 &\leq \left(\alpha_{m_j} - \frac{1}{2}\alpha_{m_j}\alpha + \frac{1}{2}\beta_{m_j} + \frac{1}{2}\gamma_{m_j}\delta_{m_j} - \alpha_{m_j}(1-\alpha)\right) \\ &\quad \times \left\|x_{m_j} - p\right\|^2 \\ &\quad + \alpha_{m_j} \langle f(p) - p, x_{m_j+1} - p\rangle.\end{aligned} \tag{3.26}$$

Then we derive from (3.16) and (3.26) that $(1-\alpha) \left\|x_{m_j} - p\right\|^2 \leq \langle f(p) - p, x_{m_j+1} - p\rangle$.

By noticing (3.19) and (3.25), we have that

$$\lim_{j\to\infty} \left\|x_{m_j} - p\right\| = 0. \tag{3.27}$$

By using (3.19) and (3.27), we get $\lim_{j\to\infty} \left\|x_{m_j+1} - p\right\| = 0$, and by virtue of (3.16), we have that

$$\lim_{j\to\infty} \|x_j - p\| \leq \lim_{j\to\infty} \left\|x_{m_j+1} - p\right\| = 0.$$

This completes the proof. □

Remark 3.2. Theorem 3.1 extends, improves and develops Theorem 2.6 of Alghamdi et al. [1], Theorem 3.1 of Xu et al. [26], Theorem 4.4 of Yao et al. [29] and Theorem 3.5 of Yu and Wen [31] in the following aspects:

- Theorem 3.1 extends, improves and develops corresponding results in [1, 26, 29, 31] from the problem for finding an element of the set of $\mathrm{Fix}(T)$ to the more general and challenging problem for finding an element of the set of $\mathrm{Fix}(T)\bigcap \mathrm{Fix}(S)$.
- The algorithm (3.1) is more advantageous and more flexible than the ones given in [1, 26, 29, 31]. Therefore, the new algorithm is expected to be widely applicable.
- The proof of our Theorem 3.1 is very different from the proof of Theorem 2.6 [1], Theorem 3.1 [26], Theorem 4.4 [29] and Theorem 3.5 [31]. In Theorem 3.1, Lemma 2.6 is used to prove the result, while it was not applied in [1, 26, 29, 31].

As a direct consequence of Theorem 3.1, we obtain the following two corollaries.

Corollary 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping and let $S : C \to C$ be a generalized hybrid mapping. Moreover, let $f : C \to C$ be a contraction with coefficient $\alpha \in [0, 1)$. Suppose that $\mathrm{F} := \mathrm{Fix}(T)\bigcap \mathrm{Fix}(S) \neq \emptyset$. For given $x_1 \in C$ arbitrarily, define*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T\left(\frac{\delta_n}{n}\sum_{k=0}^{n-1} S^k x_n + (1-\delta_n)x_{n+1}\right), \quad n \geq 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real number sequences in $[0, 1]$ satisfying the conditions (i)-(iv) *in Theorem* 3.1*. Then $\{x_n\}$ converges strongly to a point $p \in \mathrm{F}$, where $p = P_{\mathrm{F}} f(p)$.*

Corollary 3.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T, S : C \to C$ be two nonexpansive mappings. Moreover, let $f : C \to C$ be a contraction with coefficient $\alpha \in [0, 1)$. Suppose that $\mathrm{F} := \mathrm{Fix}(T)\bigcap \mathrm{Fix}(S) \neq \emptyset$. For given $x_1 \in C$ arbitrarily, define*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\delta_n S x_n + (1-\delta_n)x_{n+1}), \quad n \geq 1, \tag{3.28}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real number sequences in $[0, 1]$ satisfying the conditions (i)-(iv) *in Theorem* 3.1*. Then $\{x_n\}$ converges strongly to a point $p \in \mathrm{F}$, where $p = P_{\mathrm{F}} f(p)$.*

Proof. By using the demiclosedness principle for the nonexpansive mapping S and by a similar argument as in the proof of Theorem 3.1, we can obtain the desired results immediately. □

4. Applications

In this section, we apply our main results to approximate common solutions of split feasibility problems and fixed point problems.

Let C and Q be nonempty closed convex subsets of two Hilbert spaces H_1 and H_2, respectively, and let $A : H \to H$ be a bounded linear mapping. The split feasibility problem (SFP) is the problem of finding a point with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{4.1}$$

The SFP (4.1) in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [6] for modeling inverse problems which arise in phase retrievals and in medical image reconstruction [4]. In [5, 7, 8], it has been shown that the SPF (4.1) can also be used to model the intensity-modulated radiation therapy.

The following lemma appears implicitly in Xu [25].

Lemma 4.1. *A point $x^* \in H$ solves* SFP (4.1)*, if and only if x^* is a fixed point of the operator $P_C(I - \gamma A^*(I - P_Q)A)$.*

Lemma 4.2. *For any $\gamma \in \mathbb{R}$ with $0 < \gamma < \frac{2}{\|A\|^2}$, the operator $P_C(I - \gamma A^*(I - P_Q)A)$ is nonexpansive.*

Theorem 4.3. *Let C and Q be nonempty closed convex subsets of two Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear mapping and $S : C \to C$ be a 2-generalized hybrid mapping. Moreover, let $f : C \to C$ be a contraction with coefficient $\alpha \in [0, 1)$. Denote the set of* SFP (4.1) *by Ω and assume $\Omega \cap \mathrm{Fix}(S) \neq \emptyset$. For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be the sequence generated iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n P_C(I - \gamma A^*(I - P_Q)A)\left(\frac{\delta_n}{n}\sum_{k=0}^{n-1} S^k x_n + (1 - \delta_n)x_{n+1}\right), \quad n \geq 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real number sequences in $[0, 1]$ satisfying the conditions (i)-(iv) *in Theorem* 3.1 *and γ is a positive number satisfying $\gamma \in (0, \frac{2}{\|A\|^2})$. Then $\{x_n\}$ converges strongly to a point $p \in \Omega \cap \mathrm{Fix}(S)$, where $p = P_{\Omega\cap\mathrm{Fix}}(S)f(p)$.*

Theorem 4.4. *Let C and Q be nonempty closed convex subsets of two Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear mapping and $S : C \to C$ be a nonexpansive mapping. Denote the set of* SFP (4.1) *by Ω and assume $\Omega \cap \mathrm{Fix}(S) \neq \emptyset$. For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be the sequence generated iteratively by:*

$$x_{n+1} = \beta_n x_n + \gamma_n P_C\left(I - \gamma A^*(I - P_Q)A\right)(\delta_n S x_n + (1 - \delta_n)x_{n+1}), \quad n \geq 1, \tag{4.2}$$

where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real number sequences in $[0, 1]$ and γ is a positive number satisfying the conditions

(i) $\lim_{n\to\infty}(1 - \beta_n - \gamma_n) = 0$ *and* $\sum_{n=1}^{\infty}(1 - \beta_n - \gamma_n) = \infty$*;*

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;

(iv) $\gamma \in (0, \frac{2}{\|A\|^2})$.

Then $\{x_n\}$ converges strongly to a point $p \in \Omega \cap \mathrm{Fix}(S)$. Moreover, this point p is the minimum common norm solution of the split feasibility problem (4.1) *and fixed point problem of nonexpansive mapping S.*

Proof. If we take $f = 0$ and $T = P_C\left(I - \gamma A^*(I - P_Q)A\right)$, then (3.28) reduces to (4.2). Thus, $x_n \to p$ which satisfies

$$\langle -p, q - p\rangle \leq 0, \quad \forall q \in \Omega \cap \mathrm{Fix}(S).$$

Therefore, $\|p\|^2 \leq \langle p, q\rangle \leq \|p\|\,\|q\|$, which implies $\|p\| \leq \|q\|$ for all $q \in \Omega \cap \mathrm{Fix}(S)$. This completes the proof. □

5. Numerical examples

The purpose of this section is to give two numerical examples supporting Theorem 3.1.

Example 5.1. Let $T, f : [0, 2] \to [0, 2]$ be defined by $Tx = \frac{1}{2}x$ and $f(x) = \frac{1}{3}x$, respectively. Let S be the same as Example 2.2. Let sequence $\{x_n\}$ be generated iteratively by (3.1), where $\alpha_n = \frac{1}{n+1}$, $\beta_n = \delta_n = \frac{1}{4}$ and $\gamma_n = \frac{3}{4} - \frac{1}{n+1}$. Then, sequence $\{x_n\}$ converges strongly to 0.

*S*olution: It can be observed that all the assumptions of Theorem 3.1 are satisfied. And it is also easy to check

$$\mathrm{Fix}(T) \cap \mathrm{Fix}(S) = \{0\}.$$

We rewrite (3.1) as follows

$$x_{n+1} = \frac{24n^2 + 65n - 3}{3n(23n+5)} x_n + \frac{3n-1}{23n+5} S x_n. \tag{5.1}$$

By using the algorithm (5.1) and choosing $x_1 = 2$, we see that numerical results in Table 1 and Figure 1 demonstrate Theorem 3.1.

Table 1: The values of the sequence $\{x_n\}$.

n	x_n				
1–5	2.000000000000	0.954022988506	0.917695473251	0.394138440435	0.164069177567
6–10	0.066721465543	0.026648182357	0.010491588802	0.004082723801	0.001573479503
11–15	0.000601502799	0.000228348285	0.000086169519	0.000032347354	0.000012087219
16–20	0.000004498248	0.000001667939	0.000000616450	0.000000227161	0.000000083485
21–25	0.000000030607	0.000000011196	0.000000004087	0.000000001489	0.000000000542
26–30	0.000000000197	0.000000000071	0.000000000026	0.000000000009	0.000000000003

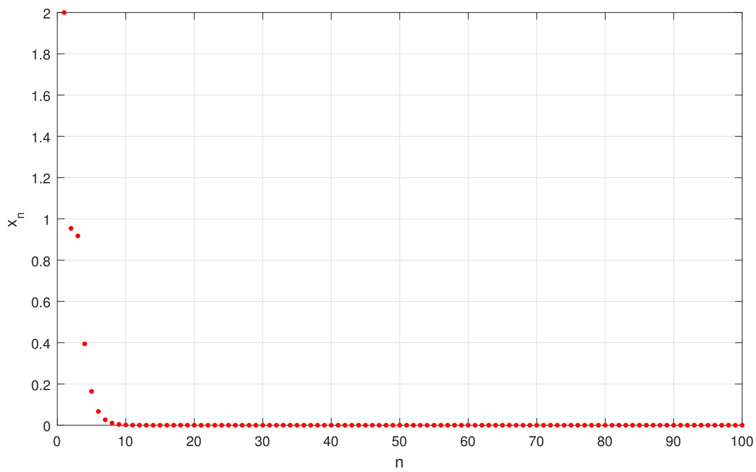


Figure 1: The convergence of $\{x_n\}$ with initial values $x_1 = 2$.

Next, we present a numerical example in $\mathbb{R}^3$ that also supports our result.

Example 5.2. Let an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$ and a usual norm $\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$ defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for all $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Let $C = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| \leq 2\}$ and $T, f : C \to C$ be defined by $T\mathbf{x} = \frac{1}{2}\mathbf{x}$ and $f(\mathbf{x}) = \frac{1}{3}\mathbf{x}$, respectively. Let $S : C \to C$ be defined as

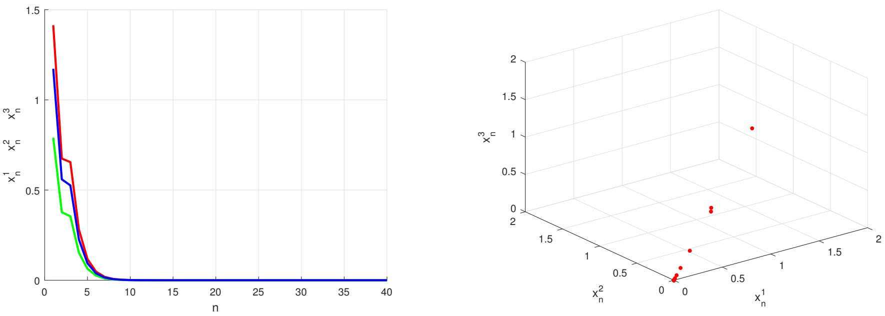
$$S\mathbf{x} = \begin{cases} (0,0,0), & \mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| < 2\}; \\ (1,0,0), & \mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 2\}. \end{cases}$$

Let sequence $\{\mathbf{x_n}\}$ be generated iteratively by (3.1), where $\alpha_n = \frac{1}{n+1}$, $\beta_n = \delta_n = \frac{1}{4}$ and $\gamma_n = \frac{3}{4} - \frac{1}{n+1}$. Then, sequence $\{\mathbf{x_n}\}$ converges strongly to (0,0,0).

*S*olution: It can be observed that all the assumptions of Theorem 3.1 are satisfied. It is also easy to check $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) = \{(0,0,0)\}$. We rewrite (3.1) as follows

$$\mathbf{x}_{n+1} = \frac{24n^2 + 65n - 3}{3n(23n+5)} \mathbf{x}_n + \frac{3n-1}{23n+5} S\mathbf{x}_n. \tag{5.2}$$

By utilizing the algorithm (5.2) and choosing $\mathbf{x}_1 = (\sqrt{2}, \sqrt{\frac{5}{8}}, \sqrt{\frac{11}{8}})$, we report the numerical results in Table 2. In addition, Figure 2 also demonstrates Theorem 3.1.



(a) 2D (b) 3D

Figure 2: The convergence of $\{\mathbf{x}_n\}$ with initial values $\mathbf{x}_1 = (\sqrt{2}, \sqrt{\frac{5}{8}}, \sqrt{\frac{11}{8}})$.

Table 2: The values of the sequence $\{\mathbf{x}_n\}$.

n	$\mathbf{x}_n^1$	$\mathbf{x}_n^2$	$\mathbf{x}_n^3$
1	1.414213562373095	0.790569415042095	12.000000000000000
2	0.674596124580269	0.377110697979850	2.000000000000000
3	0.654935301640607	0.354617556541516	1.482539682539682
4	0.281286315448209	0.152303694155651	0.947402160129433
5	0.117091888792877	0.063399910480541	0.614609008256618
⋮	⋮	⋮	⋮
10	0.001122951243861	0.000608026816108	0.000901849510747
⋮	⋮	⋮	⋮
15	$8.626332460955843\times10^{-6}$	$4.670765084056447\times10^{-6}$	$6.927864189991733\times10^{-6}$
⋮	⋮	⋮	⋮
20	$5.958106411228669\times10^{-8}$	$3.226042529501134\times10^{-8}$	$4.784994345318513\times10^{-8}$
⋮	⋮	⋮	⋮
25	$3.865693355959523\times10^{-10}$	$2.093096415470645\times10^{-10}$	$3.104563693951749\times10^{-10}$
⋮	⋮	⋮	⋮
30	$2.406823664130327\times10^{-12}$	$1.303185100363625\times10^{-12}$	$1.932935873944464\times10^{-12}$
⋮	⋮	⋮	⋮
36	$0.522463890075058\times10^{-14}$	$0.282890336824844\times10^{-14}$	$0.419594177594879\times10^{-14}$
37	$0.187408074983236\times10^{-14}$	$0.101472914134002\times10^{-14}$	$0.150508654456355\times10^{-14}$
38	$0.067171073725230\times10^{-14}$	$0.036370068883204\times10^{-14}$	$0.053945529965437\times10^{-14}$
39	$0.024057745143698\times10^{-14}$	$0.013026170336804\times10^{-14}$	$0.019320932948891\times10^{-14}$
40	$0.008610366797317\times10^{-14}$	$0.004662120406309\times10^{-14}$	$0.006915042060785\times10^{-14}$

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