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General iteration scheme for finding the common fixed points of an infinite family of nonexpansive mappings

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Abstract

The purpose of this paper is to suggest and analyze the general viscosity iteration scheme for an infinite family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$. Additionally, it proves that this iterative scheme converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$ in the framework of reflexive and smooth convex Banach space, which solves some variational inequality. Results proved in this paper improve and generalize recent known results in the literature.

Keywords: Nonexpansive mapping, general iteration scheme, contraction, smooth Banach space.
2010 MSC: 47H09, 47H10, 47J20.

1. Introduction

Let E be a real Banach space and K a nonempty closed convex subset of E. Recall that a mapping $f : K \to K$ is said to be a contraction on K, if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in K$. We use Π_K to denote the collection of all contractions on K. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ holds for all $x, y \in K$.

Recently, iterative methods for nonexpansive mappings have been applied to solve convex minimization problems. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied science. A simple algorithmic solution to the problem of minimizing a

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quadratic function over a common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation.

Let H be a real Hilbert space and A be a bounded linear operator. A is said to be a strongly positive on H [4], if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where S is a nonexpansive mapping and b is a given point in H. $F(S)$ denotes the set of fixed points of S.

Let K be a nonempty closed convex subset of H, A be a strongly positive operator and $T : K \to K$ be a nonexpansive mapping. By studying the following Ishikawa iterative algorithm:

$$\begin{cases} x_0 = x \in K \quad \text{chosen arbitrarily,} \\ z_n = \gamma_n x_n + (1-\gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \quad \forall n \geq 0. \end{cases}$$

Shang et al. [6] proved the sequence $\{x_n\}$ converges strongly to a fixed point of T under some mild conditions in a Hilbert space.

Let $\{T_n\}_{n=1}^{\infty} : K \to K$ be an infinite family of nonexpansive mappings and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 \leq \lambda_n \leq 1$ for every $i \in \mathbb{N}$ (the set of positive integers). For any $n \in \mathbb{N}$, the mapping W_n is defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I, \end{cases} \tag{1.1}$$

where I is the identity operator on E. Such a mapping W_n is called the W-mapping generated by T_n, T_{n-1}, $\ldots, T_1$ and λ_n, λ_{n-1}, $\ldots, \lambda_1$ (see [7]). Nonexpansivity of each T_i ensures the nonexpansivity of W_n. It is now one of the main tools in studying convergence of iterative methods for approaching a common fixed point of an infinite family of nonlinear mappings.

For finding approximate common fixed points of an infinite countable family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ such that the common fixed points set $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Yao et al. [10] introduced the following iterative procedure

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)W_n x_n, \quad f \in \Pi_K, \quad n \geq 0, \tag{1.2}$$

where $\gamma > 0$ is some constant and $\{\alpha_n\}$, $\{\delta_n\}$ are two sequences in $(0, 1)$. A is a strongly positive bounded linear operator on H. Under some mild conditions on the parameters, they proved that the iterative procedure (1.2) converges strongly to $p \in F$ where p is the unique solution in F of the following variational inequality

$$\langle (A - \gamma f)p, p - x^*\rangle \leq 0 \quad \text{for all} \quad x^* \in F,$$

which is the optimal condition for the minimization problem

$$\min_{x\in F}\frac{1}{2}\langle Ax,x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

Shimoji and Takahashi [7] first introduced an iterative algorithm given by an infinite family of nonexpansive mappings. Furthermore, they considered the feasibility problem of finding a solution of infinite convex inequalities and the problem of finding a common fixed point of infinite nonexpansive mappings.

Noor [5] introduced a three-step iterative sequence and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [2] applied a three-step iterative sequence for finding the approximate solution of the elastoviscoplasticity problem, eigenvalue problem and liquid crystal theory. They have shown that the three-step iterative schemes perform better than the Ishikawa type and Mann type iterative methods and proved that three-step iterations lead to highly parallelized algorithms under certain conditions.

Variational inequalities have many applications in science and engineering, such as constrained linear and nonlinear optimization, automatic control, system identification, manufacturing system design, signal and image processing and pattern recognition.

Motivated by the recent works, the purpose of this paper is to introduce a general iterative scheme

$$\begin{cases} x_0 = x\in K \quad \text{chosen arbitrary},\\ z_n=\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n A)W_n x_n,\\ y_n=(1-\beta_n)z_n+\beta_n W_n z_n,\\ x_{n+1}=(1-\gamma_n)y_n+\gamma_n W_n y_n, \quad \forall n\ge 0, \end{cases} \tag{1.3}$$

where $\gamma>0$ is some constant, $f\in\Pi_K$, A is a strongly positive operator and W_n is a mapping defined by (1.1), $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$. By using viscosity approximation methods, we establish the strong convergence of the general iterative scheme $\{x_n\}$ defined by (1.3), which solves a variational inequality. The results presented in this paper improve and extend some recent results.

Now, we consider some special cases of the iterative scheme. If $\beta_n=\gamma_n=0$ in (1.3), then (1.3) reduces to (1.2) which was considered by Yao et al. (see [10]).

2. Preliminaries

Suppose that $\{x_n\}$ is a sequence in E, then $x_n\to x$ (respectively, $x_n\rightharpoonup x$, $x_n\overset{*}{\rightharpoonup} x$) will denote strong (respectively, weak, weak*) convergence of the sequence $\{x_n\}$ to x.

A Banach space E is said to be strictly convex if,

$$\|x\|=\|y\|=1,\ x\neq y \ \text{ implies } \ \frac{\|x+y\|}{2}<1.$$

Let $S(E)=\{x\in E:\|x\|=1\}$. The space E is said to be Gâteaux differentiable (and E is said to be smooth), if $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for all $x,y\in S(E)$. For any $x,y\in E$ ($x\neq 0$), we denote this limit by (x,y). The norm is said to be uniformly Gâteaux differentiable, if for all $y\in S(E)$, the limit is attained uniformly for each $x\in S(E)$. The norm $\|\cdot\|$ of E is said to be Fréchet differentiable if for all $x\in S(E)$, the limit (x,y) exists uniformly for each $y\in S(E)$. The norm $\|\cdot\|$ of E is said to be uniformly Fréchet differentiable (or E is said to be uniformly smooth), if the limit is attained uniformly for all $x,y\in S(E)$. It is well-known that (uniformly) Fréchet differentiability of the norm E implies (uniformly) Gâteaux differentiability of norm E.

Let E^* denote the dual space of a Banach space E. Let $\varphi:[0,\infty):=R_+\to R_+$ be a continuous strictly increasing function such that $\varphi(0)=0$ and $\varphi(t)\to\infty$ as $t\to\infty$. This function φ is said to be a gauge function. The duality mapping $J_\varphi:E\to 2^{E^*}$ is defined by

$$J_\varphi(x)=\{x^*\in E^*:\langle x,x^*\rangle=\|x\|\varphi(\|x\|),\ \|x^*\|=\varphi(\|x\|)\},\quad \forall x\in E,$$

where $\langle\cdot\rangle$ denotes the generalized duality pairing. In particular, $\varphi(t) = t$, we write $J = J_2$ for J_φ is said to be normalized duality mapping, $J_q(x)(= \|x\|^{q-2}J_2(x))$ is said to be generalized duality mapping for $x \neq 0$ and $q > 1$. If E is a Hilbert space, then $J = I$ (the identity mapping). It is known that if E is said to be smooth, then the normalized duality mapping J is single-valued and norm to weak star continuous. And we know that if the norm of E is uniformly Gâteaux differentiable, then the normalized duality mapping is norm to weak star uniformly continuous on each bounded subset of E. It is also well-known that if E has a uniformly Fréchet differentiable norm, J is uniformly continuous on bounded subsets of E. Suppose that J is single-valued. Then J is said to be weakly sequentially continuous, if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, $J(x_n) \overset{*}{\rightharpoonup} J(x)$.

In a smooth Banach space, we define an operator A as strongly positive [1], if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J(x)\rangle \geq \bar{\gamma}\|x\|^2, \quad \|aI - bA\| = \sup_{\|x\|\leq 1} |\langle (aI - bA)x, J(x)\rangle|, \tag{2.1}$$

where $a \in [0, 1]$, $b \in [-1, 1]$, I is the identity mapping and J is the normalized duality mapping.

Lemma 2.1 ([1]). *Assume that A is a strongly positive linear bounded operator on a smooth Banach space E with coefficient $\bar{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Let C and D be nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is said to be retraction, if $Px = x$ for all $x \in C$. A retraction $P : C \to D$ is said to be sunny, if $P(Px + t(x - Px)) = Px$ holds for all $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also a nonexpansive mapping. In a smooth Banach space E, P is a sunny nonexpansive retraction from C onto D, if and only if the following inequality holds:

$$\langle x - Px, J(z - Px)\rangle \leq 0, \quad \forall x \in C, \quad z \in D.$$

Lemma 2.2. *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$ the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad j(x + y) \in J(x + y).$$

Concerning W_n, the next lemmas play a crucial role for proving our main results.

Lemma 2.3 ([7]). *Let K be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of K into itself such that $\bigcap_{n=1}^\infty F(T_n)$ is nonempty and $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \leq b < 1$ for any $n \geq 1$. Then, for any $x \in K$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

By using Lemma 2.3, we can define the mapping W of K into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in K.$$

Such a mapping W is said to be the W-mapping generated by $T_1, T_2, \cdots$ and $\lambda_1, \lambda_2, \cdots$. Throughout this paper, we assume that $0 < \lambda_n \leq b < 1$ for all $n \geq 1$. Nonexpansivity of each T_i ensures the non-expansivity of W_n. Since W_n is nonexpansive, then $W : K \to K$ is also nonexpansive.

Lemma 2.4 ([7]). *Let K be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be nonexpansive mappings of K into itself such that $\bigcap_{n=1}^\infty F(T_n)$ is nonempty and $\lambda_1, \lambda_2, \cdots$ be real numbers such that $0 < \lambda_n \leq b < 1$ for any $n \geq 1$. Then $F(W) = \bigcap_{n=1}^\infty F(T_n)$.*

We also need the following lemmas for the proof of our main results.

Lemma 2.5 ([8]). *Let $\{x_n\}, \{y_n\}$ be two bounded sequences in a Banach space E and $\beta_n \in [0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

A Banach space E is said to satisfy Opial's condition, if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x \in E$ implies that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$ for all $y \in E$ with $x \ne y$. The following lemma can be found in [3, p. 108].

Lemma 2.6 ([3]). *Let K be a nonempty closed convex subset of a reflexive Banach space E which satisfies Opial's condition, and suppose $T : K \to E$ is a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x$, and $x_n - Tx_n \to 0$, then $x \in F(T)$.*

Lemma 2.7 ([1, Lemma 1.9]). *Let K be a closed convex subset of a reflexive, smooth Banach space E which admits a weakly sequentially continuous duality mapping J from E to E^*. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \ne \emptyset$ and $f \in \Pi_K$, A is strongly positive linear bounded operator with coefficient $\bar{\gamma}$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Then the sequence $\{x_t\}$ defined by*

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t,$$

converges strongly as $t \to 0$ to a point $\widetilde{x}$ of $F(T)$ which solves the following variational inequality:

$$\langle (A - \gamma f)\widetilde{x}, J(\widetilde{x} - z)\rangle \le 0, \quad z \in F(T).$$

Lemma 2.8 ([9]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\rho_n)a_n + \sigma_n, \quad n \ge 0,$$

where $\{\rho_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \rho_n = \infty$;

(2) $\limsup_{n\to\infty}(\sigma_n/\rho_n) \le 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let K be a nonempty closed convex subset of a reflexive, smooth and strictly convex Banach space E which also has a weakly continuous duality mapping $J : E \to E^*$. Let T_i be a nonexpansive mapping from K into itself for $i \in \mathbb{N}$. Assume that $F = \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and $f \in \Pi_K$. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Suppose that $0 < \gamma < (\bar{\gamma}/\alpha)$, the given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$ satisfying the following conditions:*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$;

(3) $\limsup_{n\to\infty} \delta_n < 1$.

Then the general iterative scheme $\{x_n\}$ defined by (1.3) converges strongly to $P(f) \in F$, where P is a unique sunny nonexpansive retraction from Π_K onto F. If we define $P : \Pi_K \to F$ by

$$P(f) := \lim_{t\to 0} x_t, \quad f \in \Pi_K,$$

then $P(f)$ solves the variational inequality

$$\langle (\gamma f - A)P(f), J(q - P(f))\rangle \le 0, \quad \forall f \in \Pi_K, \quad q \in F.$$

Proof. We proceed with the following steps.

Step 1. We should prove that $\|x_n - p\| \le \max\{\|x_0 - p\|, \|Aq - \gamma f(q)\|/(\bar{\gamma} - \gamma\alpha)\}$ for all $n \ge 0$ and all $q \in F$ and so $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{W_n x_n\}$, $\{W_n y_n\}$ and $\{W_n z_n\}$ are bounded.

Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, with no loss of generality, that $\alpha_n < (1-\delta_n)\|A\|^{-1}$ for all n. Since A is a linear bounded operator on E, it follows from (2.1) that

$$\|A\| = \sup\{|\langle Au, Ju\rangle| : \|u\| = 1, u \in E\}.$$

Notice that

$$\begin{aligned}\langle((1-\delta_n)I - \alpha_n A)u, Ju\rangle &= 1 - \delta_n - \alpha_n\langle Au, Ju\rangle \\ &\ge 1 - \delta_n - \alpha_n\|A\| \ge 0.\end{aligned}$$

Therefore

$$\begin{aligned}\|(1-\delta_n)I - \alpha_n A\| &= \sup\{\langle((1-\delta_n)I - \alpha_n A)u, Ju\rangle : \|u\| = 1, u \in E\} \\ &= \sup\{1 - \delta_n - \alpha_n\langle Au, Ju\rangle : \|u\| = 1, u \in E\} \\ &\le 1 - \delta_n - \alpha_n\bar{\gamma}.\end{aligned}$$

Take a point $q \in F$. It follows from (1.3) that

$$\begin{aligned}\|z_n - q\| &= \|\alpha_n(\gamma f(x_n) - Aq) + \delta_n(x_n - q) + ((1-\delta_n)I - \alpha_n A)(W_n x_n - q)\| \\ &\le \alpha_n\|\gamma f(x_n) - Aq\| + \delta_n\|x_n - q\| + \|(1-\delta_n)I - \alpha_n A\|\|W_n x_n - q\| \\ &\le (1 - \delta_n - \alpha_n\bar{\gamma})\|x_n - q\| + \delta_n\|x_n - q\| + \alpha_n\|\gamma(f(x_n) - f(q))\| \\ &\quad + \alpha_n\|\gamma f(q) - Aq\| \\ &\le (1 - \alpha_n\bar{\gamma})\|x_n - q\| + \alpha_n\gamma\alpha\|x_n - q\| + \alpha_n\|\gamma f(q) - Aq\| \\ &= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - q\| + \alpha_n\|Aq - \gamma f(q)\| \\ &\le \max\{\|x_n - q\|, \quad \|Aq - \gamma f(q)\|/(\bar{\gamma} - \gamma\alpha)\},\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\|y_n - q\| &= \|(1-\beta_n)(z_n - q) + \beta_n(W_n z_n - q)\| \\ &\le (1-\beta_n)\|z_n - q\| + \beta_n\|W_n z_n - q\| \\ &\le \|z_n - q\|.\end{aligned} \tag{3.2}$$

It follows from (1.3) and (3.1) and (3.2) that

$$\begin{aligned}\|x_{n+1} - q\| &= \|(1-\gamma_n)(y_n - q) + \gamma_n(W_n y_n - q)\| \\ &\le \|y_n - q\| \le \|z_n - q\| \\ &\le \max\{\|x_n - q\|, \quad \|Aq - \gamma f(q)\|/(\bar{\gamma} - \gamma\alpha)\}.\end{aligned}$$

By the mathematical induction, we have that

$$\|x_n - q\| \le \max\{\|x_0 - q\|, \quad \|Aq - \gamma f(q)\|/(\bar{\gamma} - \gamma\alpha)\}$$

for all $n \ge 0$. Hence, $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{W_n x_n\}$, $\{W_n y_n\}$ and $\{W_n z_n\}$.

Step 2. We prove that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.3}$$

Indeed, by putting $l_n = (x_{n+1} - \delta_n x_n)/(1-\delta_n)$, we have

$$x_{n+1} = \delta_n x_n + (1-\delta_n)l_n, \quad \forall n \ge 0. \tag{3.4}$$

It follows from (3.4) and (1.3) that

$$\begin{aligned}
l_{n+1} - l_n = & \frac{(1-\gamma_{n+1})y_{n+1} + \gamma_{n+1}W_{n+1}y_{n+1} - \delta_{n+1}x_{n+1}}{1-\delta_{n+1}} \\
& - \frac{(1-\gamma_n)y_n + \gamma_n W_n y_n - \delta_n x_n}{1-\delta_n} \\
= & \frac{\gamma_{n+1}}{1-\delta_{n+1}}(W_{n+1}y_{n+1} - y_{n+1}) - \frac{\gamma_n}{1-\delta_n}(W_n y_n - y_n) \\
& + \frac{\beta_{n+1}}{1-\delta_{n+1}}(W_{n+1}z_{n+1} - z_{n+1}) - \frac{\beta_n}{1-\delta_n}(W_n z_n - z_n) \\
& + \frac{\alpha_{n+1}}{1-\delta_{n+1}}(\gamma f(x_{n+1}) - AW_{n+1}x_{n+1}) - \frac{\alpha_n}{1-\delta_n}(\gamma f(x_n) \\
& - AW_n x_n) + (W_{n+1}x_{n+1} - W_{n+1}x_n) + (W_{n+1}x_n - W_n x_n).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \le & \frac{\gamma_{n+1}}{1-\delta_{n+1}}\|W_{n+1}y_{n+1} - y_{n+1}\| + \frac{\gamma_n}{1-\delta_n}\|W_n y_n \\
& - y_n\| + \frac{\beta_{n+1}}{1-\delta_{n+1}}\|W_{n+1}z_{n+1} - z_{n+1}\| \\
& + \frac{\beta_n}{1-\delta_n}\|W_n z_n - z_n\| + \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) \\
& - AW_{n+1}x_{n+1}\| + \frac{\alpha_n}{1-\delta_n}\|\gamma f(x_n) - AW_n x_n\| \\
& + \|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_n x_n\| \\
& - \|x_{n+1} - x_n\| \\
\le & \frac{\gamma_{n+1}}{1-\delta_{n+1}}\|W_{n+1}y_{n+1} - y_{n+1}\| + \frac{\gamma_n}{1-\delta_n}\|W_n y_n \\
& - y_n\| + \frac{\beta_{n+1}}{1-\delta_{n+1}}\|W_{n+1}z_{n+1} - z_{n+1}\| \\
& + \frac{\beta_n}{1-\delta_n}\|W_n z_n - z_n\| + \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) \\
& - AW_{n+1}x_{n+1}\| + \frac{\alpha_n}{1-\delta_n}\|\gamma f(x_n) - AW_n x_n\| \\
& + \|W_{n+1}x_n - W_n x_n\|.
\end{aligned} \tag{3.5}$$

Since T_i and $U_{n,i}$ are nonexpansive, from (1.1), we have

$$\begin{aligned}
\|W_{n+1}x_n - W_n x_n\| & = \|\lambda_1 T_1 U_{n+1,2}x_n - \lambda_1 T_1 U_{n,2}x_n\| \\
& \le \lambda_1 \|U_{n+1,2}x_n - T_{n,2}x_n\| \\
& = \lambda_1 \|\lambda_2 T_2 U_{n+1,3}x_n - \lambda_2 T_2 U_{n,3}\| \\
& \le \lambda_1\lambda_2 \|U_{n+1,3}x_n - T_{n,3}x_n\| \\
& \vdots \\
& \le \lambda_1\lambda_2\cdots\lambda_n \|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
& \le M\prod_{i=1}^{n}\lambda_i,
\end{aligned} \tag{3.6}$$

where $M \ge 0$ is a constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \le M$ for all $n \ge 0$. By substituting (3.6) into (3.5), we have

$$\begin{aligned}\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\| \le {} & \frac{\gamma_{n+1}}{1-\delta_{n+1}}\|W_{n+1}y_{n+1}-y_{n+1}\|+\frac{\gamma_n}{1-\delta_n}\|W_ny_n \\ & -y_n\|+\frac{\beta_{n+1}}{1-\delta_{n+1}}\|W_{n+1}z_{n+1}-z_{n+1}\| \\ & +\frac{\beta_n}{1-\delta_n}\|W_nz_n-z_n\|+\frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) \\ & -AW_{n+1}x_{n+1}\|+\frac{\alpha_n}{1-\delta_n}\|\gamma f(x_n)-AW_nx_n\| \\ & +M\prod_{i=1}^{n}\lambda_i,\end{aligned}$$

which implies that (noting that the conditions (1)-(3) and $0<\lambda_i\le b<1$, $\forall i\ge 1$)

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|)\le 0.$$

It follows from Lemma 2.5 that $\lim_{n\to\infty}\|l_n-x_n\|=0$. Notice that (3.4), we have

$$x_{n+1}-x_n=(1-\delta_n)(l_n-x_n).$$

Therefore, we obtain that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ holds.

Step 3. We show that $\lim_{n\to\infty}\|Wz_n-z_n\|=0$. By observing that $x_{n+1}-y_n=\gamma_n(W_ny_n-y_n)$, $y_n-z_n=\beta_n(W_nz_n-z_n)$ and the condition (2), we get that

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0, \quad \lim_{n\to\infty}\|y_n-z_n\|=0. \tag{3.7}$$

On the other hand, we have

$$\|y_n-x_n\|\le\|y_n-x_{n+1}\|+\|x_{n+1}-x_n\|.$$

This together with (3.3) and (3.7) implies that

$$\lim_{n\to\infty}\|y_n-x_n\|=0.$$

It follows from (1.3) that

$$\begin{aligned}\|x_n-W_nx_n\| &\le \|x_n-y_n\|+\|y_n-z_n\|+\|z_n-W_nx_n\| \\ &\le \|x_n-y_n\|+\|y_n-z_n\|+\alpha_n\|\gamma f(x_n)-AW_nx_n\| \\ &\quad +\delta_n\|x_n-W_nx_n\|.\end{aligned}$$

This implies that

$$(1-\delta_n)\|x_n-W_nx_n\|\le\|x_n-y_n\|+\|y_n-z_n\|+\alpha_n\|\gamma f(x_n)-AW_nx_n\|.$$

Thus, we have that

$$\lim_{n\to\infty}\|x_n-W_nx_n\|=0. \tag{3.8}$$

It follows from (1.3) that $z_n-x_n=(1-\delta_n)(W_nx_n-x_n)+\alpha_n(\gamma f(x_n)-AW_nx_n)$. Then we have

$$\|z_n-x_n\|\le\|W_nx_n-x_n\|+\alpha_n(\gamma\|f(x_n)\|+\|AW_nx_n\|).$$

This together with (3.8) and the condition (1) implies $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Notice that

$$\begin{aligned}\|z_n - W_n z_n\| &\le \|z_n - x_n\| + \|x_n - W_n x_n\| + \|W_n x_n - W_n z_n\| \\ &\le 2\|z_n - x_n\| + \|x_n - W_n x_n\|,\end{aligned}$$

we have that $\lim_{n\to\infty}\|z_n - W_n z_n\| = 0$. On the other hand, we have

$$\|Wz_n - z_n\| \le \|Wz_n - W_n z_n\| + \|W_n z_n - z_n\|. \tag{3.9}$$

From [11, Remark 3.3], we have that $\|Wz_n - W_n z_n\| \to 0$ as $n \to \infty$. This together with (3.9) implies $\lim_{n\to\infty}\|Wz_n - z_n\| = 0$.

Step 4. We show that $\limsup_{n\to\infty}\langle(\gamma f - A)P(f), J(z_n - P(f))\rangle \le 0$, where $P(f) = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction mapping

$$x \mapsto t\gamma f(x) + (I - tA)Wx,$$

on K by Lemma 2.7.

Indeed, since E is a smooth Banach space, we have the sunny nonexpansive retraction $P : \Pi_K \to F$. Take a subsequence $\{z_{n_j}\} \subset \{z_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)P(f), J(z_n - P(f))\rangle = \lim_{j\to\infty}\langle(\gamma f - A)P(f), J(z_{n_j} - P(f))\rangle,$$

and $z_{n_j} \rightharpoonup q$ for some $q \in K$. Since $\lim_{j\to\infty}\|Wz_{n_j} - z_{n_j}\| = 0$, and it is well-known that a Banach space E with a weakly sequentially continuous duality mapping satisfies Opial's condition, from Lemma 2.6, we obtain $q \in F(W)$. Hence, $q \in F$. Moreover we have $z_n \rightharpoonup q$. Notice that

$$x_t - z_{n_j} = t(\gamma f(x_t) - Az_{n_j}) + (I - tA)(Wx_t - z_{n_j}).$$

It follows from Lemma 2.2 that

$$\begin{aligned}\|x_t - z_{n_j}\|^2 &\le \|(I - tA)(Wx_t - z_{n_j})\|^2 + 2t\langle\gamma f(x_t) - Az_{n_j}, J(x_t - z_{n_j})\rangle \\ &\le (1 - t\bar{\gamma})^2(\|Wx_t - Wz_{n_j}\| + \|Wz_{n_j} - z_{n_j}\|)^2 \\ &\quad + 2t\langle\gamma f(x_t) - Ax_t, J(x_t - z_{n_j})\rangle + 2t\langle Ax_t - Az_{n_j}, J(x_t - z_{n_j})\rangle \\ &\le (1 - t\bar{\gamma})^2(\|x_t - z_{n_j}\| + \|Wz_{n_j} - z_{n_j}\|)^2 \\ &\quad + 2t\langle\gamma f(x_t) - Ax_t, J(x_t - z_{n_j})\rangle + 2t\langle Ax_t - Az_{n_j}, J(x_t - z_{n_j})\rangle \\ &\le (1 - \bar{\gamma}t)^2\|x_t - z_{n_j}\|^2 + f_j(t) + 2t\langle\gamma f(x_t) - Ax_t, J(x_t - z_{n_j})\rangle \\ &\quad + 2t\langle Ax_t - Az_{n_j}, J(x_t - z_{n_j})\rangle,\end{aligned} \tag{3.10}$$

where

$$f_j(t) = (1 - \bar{\gamma}t)^2\|Wz_{n_j} - z_{n_j}\|(\|Wz_{n_j} - z_{n_j}\| + 2\|x_t - z_{n_j}\|) \to 0, \quad \text{as} \quad j \to \infty. \tag{3.11}$$

Since A is linear strong positive operator, we have

$$\langle Ax_t - Az_{n_j}, J(x_t - z_{n_j})\rangle = \langle A(x_t - z_{n_j}), J(x_t - z_{n_j})\rangle \ge \bar{\gamma}\|x_t - z_{n_j}\|^2. \tag{3.12}$$

By combining (3.10) with (3.12), we get

$$\begin{aligned}2t\langle\gamma f(x_t) - Ax_t, J(z_{n_j} - x_t)\rangle &\le (\bar{\gamma}t^2 - 2t)\bar{\gamma}\|x_t - z_{n_j}\|^2 + f_j(t) \\ &\quad + 2t\langle Ax_t - Az_{n_j}, J(x_t - z_{n_j})\rangle\end{aligned}$$

$$\begin{aligned}&\leq (\bar{\gamma}t^2-2t)\langle Ax_t-Az_{n_j}, J(x_t-z_{n_j})\rangle + f_j(t)\\&\quad + 2t\langle Ax_t-Az_{n_j}, J(x_t-z_{n_j})\rangle\\&= \bar{\gamma}t^2\langle Ax_t-Az_{n_j}, J(x_t-z_{n_j})\rangle + f_j(t).\end{aligned}$$

This implies

$$\langle \gamma f(x_t)-Ax_t, J(z_{n_j}-x_t)\rangle \leq \frac{\bar{\gamma}t}{2}\langle Ax_t-Az_{n_j}, J(x_t-z_{n_j})\rangle + \frac{1}{2t}f_j(t). \tag{3.13}$$

Let $j\to\infty$ in (3.13) and note (3.11), we have

$$\limsup_{j\to\infty}\langle \gamma f(x_t)-Ax_t, J(z_{n_j}-x_t)\rangle \leq \frac{t}{2}M, \tag{3.14}$$

where $M>0$ is a constant such that $M\geq \bar{\gamma}\langle Ax_t-Az_{n_j}, J(x_t-z_{n_j})\rangle$ for all $j>0$ and $t\in(0,1)$. By taking $t\to 0$ in (3.14) and noticing the fact the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology to the weak* topology of E^*, we have

$$\limsup_{j\to\infty}\langle(\gamma f-A)P(f), J(z_{n_j}-P(f))\rangle\leq 0.$$

Indeed, let $t\to 0$ in (3.14), we have

$$\limsup_{t\to 0}\limsup_{j\to\infty}\langle \gamma f(x_t)-Ax_t, J(z_{n_j}-x_t)\rangle\leq 0.$$

Hence, for arbitrary $\epsilon>0$, there exists a positive number δ_1 such that for any $t\in(0,\delta_1)$, we get

$$\limsup_{j\to\infty}\langle \gamma f(x_t)-Ax_t, J(z_{n_j}-x_t)\rangle < \frac{\epsilon}{2}. \tag{3.15}$$

Since $x_t\to P(f)$ as $t\to 0$, the set $\{x_t-z_{n_j}\}$ is bounded and the duality mapping J is norm-to-norm uniformly continuous on bounded subset of E, there exists $\delta_2>0$ such that, for any $t\in(0,\delta_2)$,

$$\begin{aligned}&|\langle(\gamma f-A)P(f), J(z_{n_j}-P(f))\rangle - \langle \gamma f(x_t)-Ax_t, J(z_{n_j}-x_t)\rangle|\\&= |\langle(\gamma f-A)P(f), J(z_{n_j}-P(f))-J(z_{n_j}-x_t)\rangle\\&\quad + \langle(\gamma f-A)P(f)-(\gamma f(x_t)-Ax_t), J(z_{n_j}-x_t)\rangle|\\&\leq |\langle(\gamma f-A)P(f), J(z_{n_j}-P(f))-J(z_{n_j}-x_t)\rangle|\\&\quad + \|(\gamma f-A)P(f)-(\gamma f(x_t)-Ax_t)\|\|z_{n_j}-x_t\| < \tfrac{\epsilon}{2}.\end{aligned}$$

Choose $\delta=\min\{\delta_1,\delta_2\}$, we have for all $t\in(0,\delta)$ and $j\in\mathbb{N}$,

$$\langle(\gamma f-A)P(f), J(z_{n_j}-P(f))\rangle < \langle \gamma f(x_t)-Ax_t, J(z_{n_j}-x_t)\rangle + \frac{\epsilon}{2},$$

which implies that

$$\limsup_{j\to\infty}\langle(\gamma f-A)P(f), J(z_{n_j}-P(f))\rangle \leq \limsup_{j\to\infty}\langle \gamma f(x_t)-Ax_t, J(z_{n_j}-x_t)\rangle + \frac{\epsilon}{2},$$

This together with (3.15) implies

$$\limsup_{j\to\infty}\langle(\gamma f-A)P(f), J(z_{n_j}-P(f))\rangle\leq \epsilon.$$

Since ϵ is arbitrary, we have that $\limsup_{j\to\infty}\langle(\gamma f-A)P(f), J(z_{n_j}-P(f))\rangle\leq 0$. Hence,

$$\langle(\gamma f-A)P(f), J(q-P(f))\rangle = \limsup_{n\to\infty}\langle(\gamma f-A)P(f), J(z_n-P(f))\rangle\leq 0.$$

Step 5. We prove that $x_n \to P(f)$ as $n \to \infty$. From (1.3), we have

$$\begin{aligned}\|x_{n+1} - P(f)\| &\leq \|z_n - P(f)\| \\ &= \|\alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)W_n x_n - P(f)\| \\ &= \|((1-\delta_n)I - \alpha_n A)(W_n x_n - P(f)) + \delta_n (x_n - P(f)) \\ &\quad + \alpha_n(\gamma f(x_n) - AP(f))\|.\end{aligned}$$

Hence, from Lemma 2.2 we obtain that

$$\begin{aligned}\|x_{n+1} - P(f)\|^2 &\leq \|z_n - P(f)\|^2 \\ &\leq \|((1-\delta_n)I - \alpha_n A)(W_n x_n - P(f)) + \delta_n(x_n - P(f))\|^2 \\ &\quad + 2\alpha_n \langle \gamma f(x_n) - AP(f), J(z_n - P(f))\rangle \\ &\leq (\|(1-\delta_n)I - \alpha_n A\| \|W_n x_n - P(f)\| + \delta_n \|x_n - P(f)\|)^2 \\ &\quad + 2\alpha_n \langle \gamma f(x_n) - AP(f), J(z_n - P(f))\rangle \\ &\leq ((1-\delta_n - \alpha_n \bar{\gamma})\|x_n - P(f)\| + \delta_n \|x_n - P(f)\|)^2 \\ &\quad + 2\alpha_n \langle \gamma f(x_n) - \gamma f(P(f)), J(z_n - P(f))\rangle \\ &\quad + 2\alpha_n \langle (\gamma f - A)P(f), J(z_n - P(f))\rangle \\ &\leq (1 - \bar{\gamma}\alpha_n)^2 \|x_n - P(f)\|^2 + 2\alpha\gamma\alpha_n \|x_n - P(f)\| \|z_n - P(f)\| \\ &\quad + 2\alpha_n \langle (\gamma f - A)P(f), J(z_n - P(f))\rangle \\ &\leq (1 - \bar{\gamma}\alpha_n)^2 \|x_n - P(f)\|^2 + 2\alpha\gamma\alpha_n \|x_n - P(f)\|^2 \\ &\quad + 2\alpha_n \langle (\gamma f - A)P(f), J(z_n - P(f))\rangle \\ &= (1 - 2(\bar{\gamma} - \alpha\gamma)\alpha_n + \bar{\gamma}^2 \alpha_n^2)\|x_n - P(f)\|^2 \\ &\quad + 2\alpha_n \langle (\gamma f - A)P(f), J(z_n - P(f))\rangle \\ &\leq (1 - \rho_n)\|x_n - P(f)\|^2 + \sigma_n,\end{aligned}$$

where $M_1 = \bar{\gamma}^2 \sup_{n\geq 0} \|x_n - P(f)\|^2$, $\rho_n = 2(\bar{\gamma} - \alpha\gamma)\alpha_n$, $\sigma_n = (2\alpha_n \langle (\gamma f - A)P(f), J(z_n - P(f))\rangle + M_1 \alpha_n^2)$. By (i) and Lemma 2.8, we have that $\|x_n - P(f)\| \to 0$ as $n \to \infty$. This completes the proof. □

If $f(x) = u \in K$ is a constant, then we have the following result.

Theorem 3.2. *Let K be a nonempty closed convex subset of a reflexive, smooth and strictly convex Banach space E which also has a weakly continuous duality mapping $J : E \to E^*$. Let T_i be a nonexpansive mapping from K into itself for $i \in \mathbb{N}$. Assume that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $f \in \Pi_K$. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Suppose that $0 < \gamma < (\bar{\gamma}/\alpha)$, the given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$ satisfying the following conditions:*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$;

(3) $\limsup_{n\to\infty} \delta_n < 1$.

Let $\{x_n\}$ be the iterative scheme defined by

$$\begin{cases} x_0 = x \in K \quad \text{chosen arbitrary,} \\ z_n = \alpha_n \gamma u + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)W_n x_n, \\ y_n = (1-\beta_n)z_n + \beta_n W_n z_n, \\ x_{n+1} = (1-\gamma_n)y_n + \gamma_n W_n y_n, \quad \forall n \geq 0, \end{cases}$$

where $\gamma > 0$ is some constant, A is a strongly positive operator and W_n is a mapping defined by (1.1)*. Then $\{x_n\}$ converges strongly to $z \in F$, where $z = P_F(u)$ and $P : K \to F$ is the unique sunny nonexpansive retraction from K onto F solving the variational inequality*

$$\langle \gamma u - AP(u), J(q - P(u))\rangle \leq 0, \quad u \in K, \quad q \in F.$$

Remark 3.3. Theorem 3.1 of this paper improves and extends Theorem 3.1 of [10] from a Hilbert space to a reflexive, smooth and strictly convex Banach space and from the iterative scheme (1.6) to the general iterative scheme (1.3).

4. Applications

As an application of Theorem 3.1, we can obtain the following result.

Theorem 4.1. *Let K be a nonempty closed convex subset of a Hilbert space E. Let T_i be a nonexpansive mapping from K into itself for $i \in \mathbb{N}$. Assume that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $f \in \Pi_K$. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Suppose that $0 < \gamma < (\bar{\gamma}/\alpha)$, the given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$ satisfying the following conditions:*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$;

(3) $\limsup_{n\to\infty} \delta_n < 1$.

Then the general iterative scheme $\{x_n\}$ defined by (1.3) *converges strongly to $P(f) \in F$, where P is a unique sunny nonexpansive retraction from Π_K onto F. If we define $P : \Pi_K \to F$ by*

$$P(f) := \lim_{t\to 0} x_t, \quad f \in \Pi_K,$$

then $P(f)$ solves the variational inequality

$$\langle (\gamma f - A)P(f), q - P(f)\rangle \leq 0, \quad \forall f \in \Pi_K, \quad q \in F,$$

which is the optimal condition for the minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$)

Proof. If E is a Hilbert space, then $J = I$, the identity mapping. We can conclude the desired conclusion easily from Theorem 3.1. This completes the proof. □

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