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General viscosity iterative method for a sequence of quasi-nonexpansive mappings

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Abstract

In this paper, we study a general viscosity iterative method due to Aoyama and Kohsaka for the fixed point problem of quasi-nonexpansive mappings in Hilbert space. First, we obtain a strong convergence theorem for a sequence of quasi-nonexpansive mappings. Then we give two applications about variational inequality problem to encourage our main theorem. Moreover, we give a numerical example to illustrate our main theorem.

Keywords: Quasi-nonexpansive mapping, variational inequality, fixed point, viscosity iterative method.
2010 MSC: 47H10, 47J20.

1. Introduction

Throughout the present paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $T : C \to C$ be a mapping. In this paper, we denote the fixed-point set of T by $Fix(T)$. A mapping T is said to be quasi-nonexpansive, if $Fix(T) \neq \emptyset$ and $\| Tx - p \| \leq \| x - p \|$ for all $x \in C$ and $p \in Fix(T)$. We know that if $T : C \to C$ is quasi-nonexpansive, then $Fix(T)$ is closed and convex (see [3] for more general results). A mapping T is said to be nonexpansive, if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$. A mapping T is called demiclosed at 0, if any sequence $\{x_n\}$ weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

The viscosity iterative method was proposed by Moudafi [11] firstly. Choose an arbitrary initial $x_0 \in H$, the sequence $\{x_n\}$ is constructed by:

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} Tx_n, \quad \forall n \geq 0,$$

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where T is a nonexpansive mapping and f is a contraction with a coefficient $\alpha \in [0,1)$ on H, the sequence $\{\varepsilon_n\}$ is in $(0,1)$, such that:

(i) $\lim_{n\to\infty} \varepsilon_n = 0$;

(ii) $\sum_{n=0}^{\infty} \varepsilon_n = \infty$;

(iii) $\lim_{n\to\infty}(\frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n+1}}) = 0$.

Then $\lim_{n\to\infty} x_n = x^*$, where $x^* \in C(C = Fix(T))$ is the unique solution of the variational inequality

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \forall x \in Fix(T). \tag{1.1}$$

Maingé considered the viscosity iterative method for quasi-nonexpansive mappings in Hilbert space in [9]. His focus was on the following algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_\omega x_n,$$

where $\{\alpha_n\}$ is a slow vanishing sequence, and $\omega \in (0,1]$, $T_\omega := (1-\omega)I + \omega T$, T has two main conditions:

(i) T is quasi-nonexpansive;

(ii) $I - T$ is demiclosed at 0.

He proved the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality (1.1).

Tian and Jin considered the following iterative process in [13]:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T_\omega x_n, \quad \forall n \geq 0,$$

where the sequence $\{\alpha_n\}$ satisfies certain conditions, $\omega \in (0, \frac{1}{2})$, $T_\omega = (1-\omega)I + \omega T$, and T is also satisfied the same conditions in Maingé [9] . Then they proved that $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in Fix(T).$$

Recently, Aoyama and Kohsaka considered the following general iterative method in [1]:

$$x_{n+1} = \alpha_n f_n(x_n) + (1-\alpha_n) S_n x_n,$$

where f_n is a θ-contraction with respect to $\Omega = \cap_{n=1}^{\infty} Fix(S_n)$ and $\{f_n\}$ is stable on Ω, and $\{S_n\}$ is a sequence of strongly quasi-nonexpansive mappings of C into C. That is to say, S_n is quasi-nonexpansive and $S_n x_n - x_n \to 0$ whenever $\{x_n\}$ is a bounded sequence in C and $\| x_n - p \| - \| S_n x_n - p \| \to 0$ for some point $p \in \Omega$. Then they proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, $\{x_n\}$ converges strongly to the unique fixed point of a contraction $P_\Omega \circ f_1$.

Many various iterative algorithms have been studied and extended by many authors, especially about quasi-nonexpansive mappings (see [1, 4, 6–13, 15]).

Motivated by the above results, we extend the iterative method to quasi-nonexpansive mappings. We consider the following iterative process:

$$x_{n+1} = \alpha_n f_n(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) S_i^{\lambda_n} x_n, \tag{1.2}$$

where $S_i^{\lambda_n} = (1-\lambda_n)I + \lambda_n S_i$, and $\{S_i\}_{i=1}^{\infty}$ is a sequence of quasi-nonexpansive mappings. Under the appropriate conditions, we establish the strong convergence of the sequence $\{x_n\}$ generated by (1.2).

2. Preliminaries

We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $f : C \to C$ be a mapping, Ω is a nonempty subset of C, and θ is a real number in $[0, 1)$. A mapping f is said to be a θ-contraction with respect to Ω, if

$$\| f(x) - f(z) \| \leq \theta \| x - z \|, \quad \forall x \in C,\, z \in \Omega.$$

f is said to be a θ-contraction, if f is a θ-contraction with respect to C. The following lemmas are useful for our main result.

Lemma 2.1 ([1]). *Let Ω be a nonempty subset of C and $f : C \to C$ a θ-contraction with respect to Ω, where $0 \leq \theta < 1$. If Ω is closed and convex, then $P_\Omega \circ f$ is a θ-contraction on Ω, where P_Ω is the metric projection of H onto Ω.*

Lemma 2.2 ([1]). *Let $f : C \to C$ be a θ-contraction, where $0 \leq \theta < 1$ and $T : C \to C$ a quasi-nonexpansive mapping. Then $f \circ T$ is a θ-contraction with respect to $Fix(T)$.*

Let D be a nonempty subset of C. A sequence $\{f_n\}$ of mappings of C into H is said to be stable on D, if $\{f_n(z) : n \in \mathbb{N}\}$ is a singleton for every $z \in D$. It is clear that if $\{f_n\}$ is stable on D, then $f_n(z) = f_1(z)$ for all $n \in \mathbb{N}$ and $z \in D$.

Lemma 2.3 ([9]). *Let $T_\omega := (1 - \omega)I + \omega T$, with T be a quasi-nonexpansive mapping on H, $Fix(T) \neq \phi$, and $\omega \in (0, 1]$, $q \in Fix(T)$. Then the following statements are reached:*

(i) *$Fix(T) = Fix(T_\omega)$;*

(ii) *T_ω is a quasi-nonexpansive mapping;*

(iii) *$\| T_\omega x - q \|^2 \leq \| x - q \|^2 - \omega(1 - \omega) \| Tx - x \|^2$ for all $x \in H$.*

Lemma 2.4 ([5]). *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \beta_n)s_n + \beta_n \delta_n, \quad n \geq 0,$$
$$s_{n+1} \leq s_n - \eta_n + t_n, \quad n \geq 0,$$

where $\{\beta_n\}$ is a sequence in $(0, 1)$, η_n is a sequence of nonnegative real numbers, and $\{\delta_n\}$ and $\{t_n\}$ are two sequences in $\mathbb{R}$ such that:

(i) *$\sum_{n=0}^{\infty} \beta_n = \infty$;*

(ii) *$\lim_{n\to\infty} t_n = 0$;*

(iii) *$\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ for any subsequence $\{n_k\} \subset \{n\}$.*

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 ([10]). *Assume A is a strongly positive linear bounded operator on Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \| A \|^{-1}$. Then $\| I - \rho A \| \leq 1 - \rho\bar{\gamma}$.*

3. Main results

In this section, we prove the following strong convergence theorem.

Theorem 3.1. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, $\{S_n\}$ a sequence of quasi-nonexpansive mappings of C into C such that $\Omega = \cap_{i=1}^{\infty} Fix(S_i)$ is nonempty, and $I - S_i$ is demiclosed at 0. Assume that $\{f_n\}$ is a sequence of mappings of C into C such that each f_n is a θ-contraction with* *respect to* Ω *and* $\{f_n\}$ *is stable on* Ω, *where* $0 \le \theta < 1$. *Let* $\{x_n\}$ *be a sequence defined by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f_n(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) S_i^{\lambda_n} x_n,$$

for $n \in \mathbb{N}$, *where* $S_i^{\lambda_n} = (1-\lambda_n)I + \lambda_n S_i$, $\lambda_n \in (0, 1]$ *and* $\{\lambda_n\}$ *satisfies* $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1$. *Suppose that* $\{\alpha_n\}$ *is a sequence in* $(0, 1]$ *such that* $\alpha_0 = 1$, $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\{\alpha_n\}$ *is strictly decreasing. Then* $\{x_n\}$ *converges to* $\omega \in \Omega$, *where* ω *is the unique fixed point of a contraction* $P_\Omega \circ f_1$.

First, we show some lemmas, then we prove Theorem 3.1. In the rest of this section, we set

$$\beta_n = \alpha_n(1 + (1 - 2\theta)(1 - \alpha_n)),$$

and

$$\gamma_n = \alpha_n^2 \parallel f_n(x_n) - \omega \parallel^2 + 2\alpha_n \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle S_i^{\lambda_n} x_n - \omega, f_1(\omega) - \omega\rangle.$$

Lemma 3.2. $\{x_n\}$, $\{S_i x_n\}$ *and* $\{f_n(x_n)\}$ *are bounded, and moreover,*

$$\parallel x_{n+1} - \omega \parallel \le \alpha_n \parallel f_n(x_n) - \omega \parallel + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) \parallel S_i^{\lambda_n} x_n - \omega \parallel, \tag{3.1}$$

and

$$\parallel x_{n+1} - \omega \parallel^2 \le (1 - \beta_n) \parallel x_n - \omega \parallel^2 + \gamma_n,$$

hold for every $n \in \mathbb{N}$.

Proof. From Lemma 2.3, we know $S_i^{\lambda_n}$ is quasi-nonexpansive and $Fix(S_i) = Fix(S_i^{\lambda_n})$ for all $i \in \mathbb{N}$. Since f_n is a θ-contraction with respect to Ω, $S_i^{\lambda_n}$ is quasi-nonexpansive, $\omega \in \Omega \subset Fix(S_i) = Fix(S_i^{\lambda_n})$, and $\{f_n\}$ is stable on Ω, it follows that

$$\begin{aligned}
\parallel x_{n+1} - \omega \parallel &= \parallel \alpha_n f_n(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) S_i^{\lambda_n} x_n - \omega \parallel \\
&\le \alpha_n(\parallel f_n(x_n) - f_n(\omega) \parallel + \parallel f_n(\omega) - \omega \parallel) \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) \parallel S_i^{\lambda_n} x_n - \omega \parallel \\
&\le \alpha_n \theta \parallel x_n - \omega \parallel + \alpha_n \parallel f_1(\omega) - \omega \parallel + (1 - \alpha_n) \parallel x_n - \omega \parallel \\
&= (1 - \alpha_n(1 - \theta)) \parallel x_n - \omega \parallel + \alpha_n(1 - \theta)\frac{\parallel f_1(\omega) - \omega \parallel}{1 - \theta}
\end{aligned} \tag{3.2}$$

for every $n \in \mathbb{N}$. Thus, by the induction on n, for every $i \in \mathbb{N}$, we have

$$\parallel S_i x_n - \omega \parallel \le \parallel x_n - \omega \parallel \le \max\{\parallel x_1 - \omega \parallel, \frac{\parallel f_1(\omega) - \omega \parallel}{1 - \theta}\}.$$

Therefore, it turns out that $\{x_n\}$ and $\{S_i x_n\}$ are bounded, and moreover, $\{f_n(x_n)\}$ is also bounded. Equation (3.1) follows from (3.2).

By assumption, for every $i \in \mathbb{N}$, it follows that

$$\begin{aligned}
\langle S_i^{\lambda_n} x_n - \omega, f_n(x_n) - \omega\rangle &\le \parallel S_i^{\lambda_n} x_n - \omega \parallel \cdot \parallel f_n(x_n) - f_n(\omega) \parallel \\
&\quad + \langle S_i^{\lambda_n} x_n - \omega, f_n(\omega) - \omega\rangle \\
&\le \theta \parallel x_n - \omega \parallel^2 + \langle S_i^{\lambda_n} x_n - \omega, f_1(\omega) - \omega\rangle,
\end{aligned} \tag{3.3}$$

and thus

$$
\begin{aligned}
\| x_{n+1} - \omega \|^2 &= \| \alpha_n(f_n(x_n) - \omega) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(S_i^{\lambda_n} x_n - \omega) \|^2 \\
&= \alpha_n^2 \| f_n(x_n) - \omega \|^2 + \| \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(S_i^{\lambda_n} x_n - \omega) \|^2 \\
&\quad + 2\alpha_n \langle \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(S_i^{\lambda_n} x_n - \omega), f_n(x_n) - \omega \rangle \\
&\leq \alpha_n^2 \| f_n(x_n) - \omega \|^2 + (1-\alpha_n)^2 \| x_n - \omega \|^2 \\
&\quad + 2\alpha_n \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle S_i^{\lambda_n} x_n - \omega, f_n(x_n) - \omega \rangle \\
&\leq \alpha_n^2 \| f_n(x_n) - \omega \|^2 + (1-\alpha_n)^2 \| x_n - \omega \|^2 + 2\alpha_n(1-\alpha_n)\theta \| x_n - \omega \|^2 \\
&\quad + 2\alpha_n \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle S_i^{\lambda_n} x_n - \omega, f_1(\omega) - \omega \rangle \\
&= (1-\beta_n) \| x_n - \omega \|^2 + \gamma_n
\end{aligned}
$$

for every $n \in \mathbb{N}$. □

Lemma 3.3. *The following hold:*

- $0 < \beta_n \leq 1$ *for every* $n \in \mathbb{N}$*;*
- $2\alpha_n(1-\alpha_n)/\beta_n \to 1/(1-\theta)$ *and* $2\alpha_n/\beta_n \to 1/(1-\theta)$*;*
- $\alpha_n^2 \| f_n(x_n) - \omega \|^2 / \beta_n \to 0$*;*
- $\sum_{n=1}^{\infty} \beta_n = \infty$.

Proof. Since $0 < \alpha_n \leq 1$ and $-1 < 1 - 2\theta \leq 1$, we know that

$$0 < \alpha_n^2 = \alpha_n(1 + (-1)(1-\alpha_n)) < \beta_n \leq \alpha_n(1 + (1-\alpha_n)) = \alpha_n(2-\alpha_n) \leq 1.$$

From $\alpha_n \to 0$ we have $2\alpha_n(1-\alpha_n)/\beta_n \to 1/(1-\theta)$ and $2\alpha_n/\beta_n \to 1/(1-\theta)$. Since $\{f_n(x_n)\}$ is bounded and

$$\frac{\alpha_n^2}{\beta_n} = \frac{\alpha_n}{1 + (1-2\theta)(1-\alpha_n)} \to 0,$$

it follows that $\alpha_n^2 \| f_n(x_n) - \omega \|^2 / \beta_n \to 0$.

Finally, we prove $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that $1 - 2\theta \geq 0$. Then it follows that $\beta_n \geq \alpha_n$ for every $n \in \mathbb{N}$. Thus, $\sum_{n=1}^{\infty} \beta_n = \infty$. Next, we suppose that $1 - 2\theta < 0$. Then $\beta_n > 2(1-\theta)\alpha_n$ for every $n \in \mathbb{N}$. Thus, $\sum_{n=1}^{\infty} \beta_n \geq 2(1-\theta)\sum_{n=1}^{\infty} \alpha_n = \infty$. This completes the proof. □

Proof of Theorem 3.1. By Lemma 2.1, it implies that $P_\Omega \circ f_1$ is a θ-contraction on Ω and hence it has a unique fixed point on Ω.

From Lemma 3.2, we know that

$$
\begin{aligned}
\| x_{n+1} - \omega \|^2 \leq (1-\beta_n) \| x_n - \omega \|^2 &+ \alpha_n^2 \| f_n(x_n) - \omega \|^2 \\
&+ 2\alpha_n \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle S_i^{\lambda_n} x_n - \omega, f_1(\omega) - \omega \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\beta_n) \parallel x_n - \omega \parallel^2 + \alpha_n^2 \parallel f_n(x_n) - \omega \parallel^2 \\
&\quad + 2\alpha_n \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \langle \lambda_n (S_i x_n - x_n), f_1(\omega) - \omega \rangle \\
&\quad + 2\alpha_n \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \langle x_n - \omega, f_1(\omega) - \omega \rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\parallel x_{n+1} - \omega \parallel^2 \le (1-\beta_n) \parallel x_n - \omega \parallel^2 + \beta_n \Big[& \frac{\alpha_n^2 \parallel f_n(x_n) - \omega \parallel^2}{\beta_n} \\
& + \frac{2\alpha_n}{\beta_n} \lambda_n \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \parallel x_n - S_i x_n \parallel \cdot \parallel f_1(\omega) - \omega \parallel \\
& + \frac{2\alpha_n}{\beta_n} (1-\alpha_n) \langle x_n - \omega, f_1(\omega) - \omega \rangle \Big].
\end{aligned}
\tag{3.4}
$$

On the other hand, we obtain from Lemma 2.3 (iii) that

$$
\begin{aligned}
\parallel x_{n+1} - \omega \parallel^2 &= \parallel \alpha_n (f_n(x_n) - \omega) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(S_i^{\lambda_n} x_n - \omega) \parallel^2 \\
&= \alpha_n^2 \parallel f_n(x_n) - \omega \parallel^2 + \parallel \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(S_i^{\lambda_n} x_n - \omega) \parallel^2 \\
&\quad + 2\alpha_n \langle \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) S_i^{\lambda_n} x_n - \omega, f_n(x_n) - \omega \rangle \\
&\le \alpha_n^2 \parallel f_n(x_n) - \omega \parallel^2 + (1-\alpha_n)^2 \parallel x_n - \omega \parallel^2 \\
&\quad - (1-\alpha_n)\lambda_n(1-\lambda_n) \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \parallel S_i x_n - x_n \parallel^2 \\
&\quad + 2\alpha_n \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \langle S_i^{\lambda_n} x_n - \omega, f_n(x_n) - \omega \rangle.
\end{aligned}
\tag{3.5}
$$

By using (3.3), we have

$$
\begin{aligned}
&(1-\alpha_n)^2 \parallel x_n - \omega \parallel^2 + 2\alpha_n \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \langle S_i^{\lambda_n} x_n - \omega, f_n(x_n) - \omega \rangle \\
&\quad \le (1-\alpha_n)^2 \parallel x_n - \omega \parallel^2 + 2\alpha_n (1-\alpha_n)\theta \parallel x_n - \omega \parallel^2 \\
&\qquad + 2\alpha_n \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \langle S_i^{\lambda_n} x_n - \omega, f_1(\omega) - \omega \rangle) \\
&\quad \le (1-\beta_n) \parallel x_n - \omega \parallel^2 + 2\alpha_n (1-\alpha_n) \parallel x_n - \omega \parallel \cdot \parallel f_1(\omega) - \omega \parallel .
\end{aligned}
\tag{3.6}
$$

Since $S_i^{\lambda_n}$ is quasi-nonexpansive, from (3.5) and (3.6), it follows that

$$
\begin{aligned}
\parallel x_{n+1} - \omega \parallel^2 \le& \parallel x_n - \omega \parallel^2 + \alpha_n^2 \parallel f_n(x_n) - \omega \parallel^2 + 2\alpha_n (1-\alpha_n) \parallel x_n - \omega \parallel \cdot \parallel f_1(\omega) - \omega \parallel \\
& - (1-\alpha_n)\lambda_n(1-\lambda_n) \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \parallel S_i x_n - x_n \parallel^2 .
\end{aligned}
$$

Suppose that M is a positive constant such that

$$
M \ge \sup\{\alpha_n \parallel f_n(x_n) - \omega \parallel^2 + 2(1-\alpha_n) \parallel x_n - \omega \parallel \cdot \parallel f_1(\omega) - \omega \parallel, n \in \mathbb{N}\}.
$$

So we have

$$\| x_{n+1} - \omega \|^2 \leq \| x_n - \omega \|^2 + \alpha_n M - (1-\alpha_n)\lambda_n(1-\lambda_n) \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \| S_i x_n - x_n \|^2 . \tag{3.7}$$

Set

$$\begin{aligned} s_n &= \| x_n - \omega \|^2, \ t_n = \alpha_n M, \\ \delta_n &= \frac{\alpha_n^2 \| f_n(x_n) - \omega \|^2}{\beta_n} + \frac{2\alpha_n}{\beta_n}\lambda_n \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \| x_n - S_i x_n \| \cdot \| f_1(\omega) - \omega \| \\ &\quad + \frac{2\alpha_n}{\beta_n}(1-\alpha_n)\langle x_n - \omega, f_1(\omega) - \omega \rangle, \\ \eta_n &= (1-\alpha_n)\lambda_n(1-\lambda_n) \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \| S_i x_n - x_n \|^2 . \end{aligned}$$

Then (3.4) and (3.7) can be rewritten as the following forms, respectively,

$$s_{n+1} \leq (1-\beta_n)s_n + \beta_n \delta_n, \qquad s_{n+1} \leq s_n - \eta_n + t_n.$$

Finally, we observe that the condition $\lim_{n\to\infty} \alpha_n = 0$ and Lemma 3.3 imply $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$, respectively. In order to complete the proof by using Lemma 2.4, it suffices to verify that

$$\lim_{k\to\infty} \eta_{n_k} = 0,$$

implies

$$\limsup_{k\to\infty} \delta_{n_k} \leq 0,$$

for any subsequence $\{n_k\} \subset \{n\}$.

In fact, for every $i \in \mathbb{N}$, if $\eta_{n_k} \to 0$ as $k \to \infty$, then

$$(1-\alpha_{n_k})\lambda_{n_k}(1-\lambda_{n_k}) \sum_{i=1}^{n_k} (\alpha_{i-1} - \alpha_i) \| S_i x_{n_k} - x_{n_k} \|^2 \to 0 .$$

And since $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$, there exist $\underline{\lambda} > 0$ and $\overline{\lambda} > 0$, such that $0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda} < 1$. Since $\lim_{n\to\infty} \alpha_n = 0$, there exist some positive integer n_0 and $\overline{\alpha} < 1$, such that $\alpha_n < \overline{\alpha}$, when $n > n_0$, then

$$\begin{aligned} (1-\overline{\alpha})\underline{\lambda}(1-\overline{\lambda})(\alpha_{i-1} - \alpha_i) \| S_i x_{n_k} - x_{n_k} \|^2 &\leq (1-\overline{\alpha})\underline{\lambda}(1-\overline{\lambda}) \sum_{i=1}^{n_k} (\alpha_{i-1} - \alpha_i) \| S_i x_{n_k} - x_{n_k} \|^2 \\ &\leq (1-\alpha_{n_k})\lambda_{n_k}(1-\lambda_{n_k}) \sum_{i=1}^{n_k} (\alpha_{i-1} - \alpha_i) \| S_i x_{n_k} - x_{n_k} \|^2 \to 0 . \end{aligned}$$

Therefore, since $\{\alpha_n\}$ is strictly decreasing, it follows that

$$\| S_i x_{n_k} - x_{n_k} \| \to 0 \text{ and } \sum_{i=1}^{n_k} (\alpha_{i-1} - \alpha_i) \| S_i x_{n_k} - x_{n_k} \|^2 \to 0$$

for every $i \in \mathbb{N}$.

By using the condition that $I - S_i$ is demiclosed at 0, we obtain $\omega_w(x_{n_k}) \subset F = \cap_{i=1}^{\infty} Fix(S_i)$. From Lemma 3.3, it turns out that

$$\begin{aligned} \limsup_{k\to\infty} \frac{2\alpha_{n_k}(1-\alpha_{n_k})}{\beta_{n_k}} \langle x_{n_k} - \omega, f_1(\omega) - \omega \rangle &= \frac{1}{1-\theta} \limsup_{k\to\infty} \langle x_{n_k} - \omega, f_1(\omega) - \omega \rangle \\ &= \frac{1}{1-\theta} \sup_{z \in \omega_w(x_{n_k})} \langle z - \omega, f_1(\omega) - \omega \rangle \leq 0 . \end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i) \parallel S_i x_{n_k} - x_{n_k} \parallel^2 \to 0$ and $\{f_n(x_n)\}, \{S_i x_n\}$ are bounded, it is easy to see that $\limsup_{k\to\infty}\delta_{n_k} \le 0$. From Lemma 2.4, we conclude that $x_n \to \omega$. □

Remark 3.4. When $S_n = S$, we can remove the following conditions: $\alpha_0 = 1$ and $\{\alpha_n\}$ is strictly decreasing. In fact, the above conditions guarantee the coefficients $\alpha_{i-1} - \alpha_i$ greater than 0 for every $i \in \mathbb{N}$.

The following corollary is the direct consequence of Theorem 3.1.

Corollary 3.5. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, $S : C \to C$ a quasi-nonexpansive mapping, such that $Fix(S) \neq \emptyset$ and $I - S$ is demiclosed at 0. Assume that $\alpha_n \to 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and f_n satisfies the same conditions of Theorem 3.1. Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f_n(x_n) + (1-\alpha_n)S^{\lambda_n}x_n \tag{3.8}$$

for $n \in \mathbb{N}$, where $S^{\lambda_n} = (1-\lambda_n)I + \lambda_n S$, and $\{\lambda_n\}$ also satisfies the same conditions of Theorem 3.1. Then $\{x_n\}$ converges to $\omega \in \Omega$, where ω is the unique fixed point of a contraction $P_\Omega \circ f_1$.

Remark 3.6. If $f_n = f$ and $\lambda_n = \lambda$ for all $n \in \mathbb{N}$, (3.8) becomes the viscosity approximation process which is introduced by Maingé (see [9]).

4. Application to variational inequality problem

In this section, by applying Theorem 3.1 and Corollary 3.5, first we study the following variational inequality problem, which is to find a point $x^* \in \Omega$, such that

$$\langle F(x^*), x - x^*\rangle \ge 0, \quad \forall x \in \Omega, \tag{4.1}$$

where Ω is a nonempty closed convex subset of a real Hilbert space H, and $F : H \to H$ is a nonlinear operator.

The problem (4.1) is denoted by $VI(\Omega, F)$. It is well-known that $VI(\Omega, F)$ is equivalent to the fixed point problem (see, [7]). If the solution set of $VI(\Omega, F)$ is denoted by Γ, we know that $\Gamma = Fix(P_\Omega(I - \lambda F))$, where $\lambda > 0$ is an arbitrary constant, P_Ω is the metric projection onto Ω, and I is the identity operator on H.

Assume that, F is η-strongly monotone and L-Lipschitzian continuous, that is, F satisfies the conditions

$$\langle Fx - Fy, x - y\rangle \ge \eta \parallel x - y \parallel^2, \quad \forall x, y \in \Omega,$$

$$\parallel Fx - Fy \parallel \le L \parallel x - y \parallel, \quad \forall x, y \in \Omega.$$

By using Corollary 3.5, we obtain the following convergence theorem for solving the problem $VI(\Omega, F)$.

Theorem 4.1. *Let F be η-strongly monotone and L-Lipschitzian continuous with $\eta > 0$, $L > 0$. Assume that S is a quasi-nonexpansive operator with $\Omega = Fix(S) \neq \emptyset$, and $I - S$ is demiclosed at 0. And $\{\alpha_n\}$ is a sequence in $(0, 1]$ such that $\alpha_n \to 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$. Let $\{x_n\}$ be a sequence defined by $x_1 \in H$ and*

$$x_{n+1} = (I - \mu\alpha_n F)S^{\lambda_n}x_n, \tag{4.2}$$

where $S^{\lambda_n} = (1-\lambda_n)I + \lambda_n S$, $\lambda_n \in (0, 1]$, $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 1$, and $0 < \mu < \frac{2\eta}{L^2}$. Then $\{x_n\}$ converges strongly to the unique solution of $VI(\Omega, F)$.

Proof. Set $f_n = (I - \mu F)S^{\lambda_n}$ for $n \in \mathbb{N}$ and $\theta = \sqrt{1 - 2\mu\eta + \mu^2 L^2}$. Note that

$$\begin{aligned} \parallel (I-\mu F)x - (I-\mu F)y \parallel^2 &= \parallel x - y \parallel^2 - 2\mu\langle x - y, Fx - Fy\rangle + \mu^2 \parallel Fx - Fy \parallel^2 \\ &\le \parallel x - y \parallel^2 - 2\mu\eta \parallel x - y \parallel^2 + \mu^2 L^2 \parallel x - y \parallel^2 \\ &= (1 - \mu(2\eta - \mu L^2)) \parallel x - y \parallel^2 . \end{aligned}$$

From $0 < \mu < \frac{2\eta}{L^2}$, we obtain that $I - \mu F$ is a θ-contraction. Since S is quasi-nonexpansive, from Lemma 2.3, S^{λ_n} is quasi-nonexpansive. By Lemma 2.2, f_n is a θ-contraction with respect to $Fix(S)$, and it is stable on Ω. Moreover, it follows from (4.2) that

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n)S^{\lambda_n} x_n$$

for $n \in \mathbb{N}$. Thus from Corollary 3.5, we have that $\{x_n\}$ converges strongly to $\omega = P_{Fix(S)} \circ f_1(\omega) = P_{Fix(S)}(I - \mu F)\omega$, which is the unique solution of $VI(\Omega, F)$. □

Remark 4.2. The iteration (4.2) is called the hybrid steepest descent method, (see[2, 14] for more details).

Finally, we study the following variational inequality problem, which is to find a point $x^* \in Fix(S)$, such that

$$\langle (\gamma f - A)x^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(S), \tag{4.3}$$

where f is a α-contraction and A is strongly positive, that is, there exists a constant $\bar{\gamma} > 0$ such that $\langle Ax, x \rangle \geq \bar{\gamma} \parallel x \parallel^2$ for all $x \in H$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. The problem (4.3) is denoted by VIP, where x^* is the unique solution of VIP, and we have $x^* = P_{Fix(S)}(I - A + \gamma f)x^*$.

Theorem 4.3. *Assume that $S : H \to H$ is a quasi-nonexpansive operator with $\Omega = Fix(S) \neq \emptyset$, and $I - S$ is demiclosed at* 0. *Let $\{x_n\}$ be a sequence defined by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma t f(x_n) + (I - \alpha_n t A)S^{\lambda_n} x_n, \quad \forall n \geq 0, \tag{4.4}$$

where $S^{\lambda_n} = (1 - \lambda_n)I + \lambda_n S$, and $0 < t < \frac{1}{\|A\|}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the same conditions of Theorem 4.1. *Then $\{x_n\}$ converges strongly to the unique solution of the VIP.*

Proof. Set $f_n = t\gamma f + (I - tA)S^{\lambda_n}$. By using Lemma 2.5, note that

$$\begin{aligned} \parallel f_n(x) - f_n(p) \parallel =& \parallel (t\gamma f + (I - tA)S^{\lambda_n})x - (t\gamma f + (I - tA)S^{\lambda_n})p \parallel \\ \leq& t\gamma\alpha \parallel x - p \parallel + (1 - t\gamma) \parallel x - p \parallel \\ =& (1 - t(\bar{\gamma} - \gamma\alpha)) \parallel x - p \parallel . \end{aligned}$$

From $0 < \gamma < \bar{\gamma}/\alpha$, we obtain that f_n is a θ-contraction with respect to $Fix(S)$, and it is stable on $Fix(S)$. Moreover, it follows from (4.4) that

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n)S^{\lambda_n} x_n$$

for $n \in \mathbb{N}$. Thus from Corollary 3.5, we have that $\{x_n\}$ converges strongly to the unique solution of VIP. □

Remark 4.4. Let $\xi_n = \alpha_n t$, since $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we have $\xi_n \to 0$ and $\sum_{n=1}^{\infty} \xi_n = \infty$, then (4.4) become that

$$x_{n+1} = \xi_n \gamma f(x_n) + (I - \xi_n A)S^{\lambda_n} x_n,$$

which is introduced by Tian and Jin (see [13]).

5. Numerical example

In this section, we give an example to support Theorem 3.1.

Example 5.1. In Theorem 3.1, we assume that $H = R$. Take $f_n(x) = \frac{x}{n}$, $S_i x = x \cos\frac{x}{i}$, where $x \in [-\pi, \pi]$. Given the parameter $\lambda_n = \frac{3+2n}{6n}$ for every $n \in \mathbb{N}$.

By the definitions of S_i, we have $\cap_{i=1}^{n} Fix(S_i) = \{0\}$. S_i is a quasi-nonexpansive mapping since, if $x \in [-\pi, \pi]$ and $q = 0$, then

$$\parallel S_i x - q \parallel = \parallel S_i x - 0 \parallel = | x | \cdot | \cos\frac{x}{i} | \leq | x | = | x - q | .$$

From Theorem 3.1, we can conclude that the sequence $\{x_n\}$ converges strongly to 0, as $n \to \infty$. We can rewrite (1.2) as follows

$$x_{n+1} = \frac{1}{n}\alpha_n x_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\frac{4n-3}{6n}x_n + \frac{3+2n}{6n}x_n \cos\frac{x_n}{i}). \tag{5.1}$$

Next, we give the parameter α_n has three different expressions in (5.1), that is to say, we set $\alpha_n^{(1)} = \frac{1}{n+1}$, $\alpha_n^{(2)} = \frac{1}{2n+1}$, $\alpha_n^{(3)} = \frac{1}{\sqrt{n}+1}$. Then, through taking a distinct initial guess $x_1 = 3$, by using software Matlab, we obtain the numerical experiment results in Table 1, where n is the iterative number, and the expression of error we take $\frac{|x_{n+1}-x_n|}{|x_n|}$.

Table 1: The values of $\{x_n\}$.

n	$\alpha_n^{(1)}$		$\alpha_n^{(2)}$		$\alpha_n^{(3)}$	
	x_n	error	x_n	error	x_n	error
50	0.0313	1.97×10^{-2}	-0.0699	1.04×10^{-2}	0.0001	1.38×10^{-1}
100	0.0159	9.90×10^{-3}	-0.0488	5.20×10^{-3}	0.0000	9.89×10^{-2}
500	0.0032	2.00×10^{-3}	-0.0210	1.10×10^{-3}		
1000	0.0016	9.99×10^{-4}	-0.0146	5.24×10^{-4}		
5000	0.0003	1.99×10^{-4}	-0.0063	1.04×10^{-4}		
10000	0.0002	9.99×10^{-5}	-0.0044	5.22×10^{-5}		

From Table 1, we can easily see that with iterative number increases, $\{x_n\}$ approaches to the unique fixed point 0 and the errors gradually approach to zero. And with the change of α_n, the convergent speed of the sequence $\{x_n\}$ will be changed, when $\alpha_n = \alpha_n^{(3)}$, the speed of the sequence $\{x_n\}$ is more faster than others, and when $\alpha_n = \alpha_n^{(2)}$ the convergent speed of the sequence $\{x_n\}$ become slower. Through this example, we can conclude that our algorithm is feasible.

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