Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5661–5671
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Some coupled fixed point results on cone metric spaces over Banach algebras and applications

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Communicated by Z. Kadelburg

Abstract

Our purpose in this work is to present several coupled fixed point results for different contraction mappings on cone metric spaces over Banach algebras by virtue of the properties of spectral radiuses. Also as an application, we give a simple example at the end of the paper.

Keywords: Cone metric spaces over Banach algebras, coupled fixed points, contractions, spectral radiuses.
2010 MSC: 47H10, 47H09.

1. Introduction

The coupled fixed point theorems of contractions as well as the iterative technique are useful and are applicable to many situations. For example, Gnana Bhaskar and Lakshmikantham [4], by using a weak contractivity type of assumption, proved a fixed point theorem for a mixed monotone mapping in a metric space endowed with a partial order, moreover they used the obtained results to verify the existence and uniqueness of solution for a periodic boundary value problem. In order to extend and generalize some recent fixed point results of mixed monotone mappings, Lakshmikantham and Ćirić [6] introduced the concept of the mixed g-monotone mappings and presented several coupled coincidence and coupled common fixed point theorems for the mixed g-monotone mappings under certain contractive condition in partially ordered complete metric spaces. Recently, there is a trend to study the existence of fixed point of contractions on cone metric spaces (see [1–3, 8–12, 14, 15]). In particular in the past three years, some researchers started to study the existence problems of (coupled) fixed points for some contractions in cone metric spaces over

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Received 2016-09-06

Banach algebras (see [7, 16]). It is of interest and signification to determine if it remains possible to establish the existence of a unique (coupled) fixed point of a mixed monotone mapping in such a space since as pointed out in [7], it can be proved that cone metric spaces over Banach algebras are not equivalent to metric spaces in terms of the existence of the fixed points of some nonlinear mappings. In this paper, we follow the trend to study the existence problem of coupled fixed points for different contraction mappings on cone metric spaces over Banach algebras. And for the sake of showing the availability of our results, an example is also given.

More precisely, we prove the existence of $(x, y) \in X \times X$ for the mappings $F, G : X \times X \to X$ and $g : X \to X$, such that $F(x, y) = G(x, y) = gx$ and $F(y, x) = G(y, x) = gy$, where X is a partially cone metric space over Banach algebras. Furthermore, we present other coupled fixed point results and coincidence point results under some natural and weak assumptions.

2. Preliminaries

In this section, we mainly recall some necessary conceptions and notations.

We assume always that the Banach algebra $\mathcal{A}$ has a unit e, i.e., $ex = xe = x$ for all $x \in \mathcal{A}$. $x \in \mathcal{A}$ is said to be invertible if there is $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. We refer the reader to [7] for more details.

A non-empty closed convex subset P of the Banach algebra $\mathcal{A}$ is called a cone if

(i) $\{\theta, e\} \subset P$;
(ii) $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
(iii) $P^2 = PP \subset P$;
(iv) $P \cap (-P) = \theta$,

where θ denotes the null of the Banach algebra $\mathcal{A}$.

Given a cone $P \subset \mathcal{A}$, we define a partial ordering " $\preceq$ " with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ and $x \neq y$. $x \ll y$ stands for $y - x \in \text{int}P$, where $\text{int}P$ is the interior of P. P is called a solid cone if $\text{int}P \neq \emptyset$.

Definition 2.1 ([5])**.** Let X be a nonempty set and $\mathcal{A}$ be a real Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

(d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space over the Banach algebra $\mathcal{A}$.

In the following, we always assume that (X, d) is a cone metric space over the Banach algebra $\mathcal{A}$.

Definition 2.2 ([14])**.** An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Note that if (x, y) is a coupled fixed point of F, then (y, x) is also a coupled fixed point of F.

For the following concepts, we refer the reader to [6],[8] and [15].

An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in P$ with $\theta \ll c$, there is $N \in \mathbb{N}$ (the set of non-negative integers) such that $d(x_n, x) \ll c$ for all $n \geq N$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x. Meanwhile x is called the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

If for every $c \in P$ with $\theta \ll c$ there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$, then $\{x_n\}$ is called a Cauchy sequence in X. (X, d) is complete if every Cauchy sequence in X is convergent.

Let f, g be self-mappings on X. Then $x \in X$ is called a coincidence point of pair (f, g) if $fx = gx$, and $z \in X$ is called a point of coincidence of pair (f, g) if $fx = gx = z$.

Next we review several lemmas which are indispensable for our proofs.

Lemma 2.3 ([16]). *Let x, y be vectors in $\mathcal{A}$. If x and y commute, then the spectral radius r satisfies the following properties:*

(i) $r(xy) \leq r(x)r(y)$;
(ii) $r(x + y) \leq r(x) + r(y)$;
(iii) $|r(x) - r(y)| \leq r(x - y)$;
(iv) *If* $0 \leq r(x) < 1$, *then* $e - x$ *is invertible and* $r((e - x)^{-1}) \leq (1 - r(x))^{-1}$.

Lemma 2.4 ([10]). *Let P be a solid cone in the Banach algebra $\mathcal{A}$ and if $\|x_n\| \to 0(n \to \infty)$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $x_n \ll c$.*

Lemma 2.5 ([10]). *Let P be a solid cone in the Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a Cauchy sequence in P, then $\{ku_n\}$ is a Cauchy sequence too.*

3. Coupled coincidence point results for contractions on cone metric spaces over Banach algebras

In the rest of the paper, we always assume that (X, d) is a complete cone metric space over the Banach algebra $\mathcal{A}$ and P is solid cone of $\mathcal{A}$. We establish in this section a coupled coincidence point result for two mappings $F, G : X \times X \to X$ satisfying certain contractive condition given by a fixed mapping g defined on X. Let $a \in \mathcal{A}$, we use $r(a)$ to denote the spectral radius of a.

Theorem 3.1. *Let the mappings $F, G : X \times X \to X$ and $g : X \to X$ satisfy*

$$\begin{aligned} d(F(x,y), G(u,v)) \preceq & a_1 d(gx, gu) + a_2 d(gy, gv) + a_3(d(F(x,y), gx) + d(G(u,v), gu)) \\ & + a_4(d(F(x,y), gu) + d(G(u,v), gx)) \end{aligned}$$

for all $x, y, u, v \in X$, where a_1, a_2, a_3 and $a_4 \in P$ with $r(a_1) + r(a_2) + 2r(a_3) + 2r(a_4) < 1$. Moreover assume that F, G, and g fulfill the following conditions:

1. $F(X \times X) \subseteq g(X)$,
2. $G(X \times X) \subseteq g(X)$, *and*
3. $g(X)$ *is a complete subspace of* X.

Then F, G and g have a common coupled coincidence point.

Proof. Let x_0, y_0 be two arbitrary elements in X. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again noting $G(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = G(x_1, y_1)$ and $gy_2 = G(y_1, x_1)$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{2n+1} = F(x_{2n}, y_{2n})$, $gy_{2n+1} = F(y_{2n}, x_{2n})$, $gx_{2n+2} = G(x_{2n+1}, y_{2n+1})$ and $gy_{2n+2} = G(y_{2n+1}, x_{2n+1})$.

For each $n \in \mathbb{N}$, by the given conditions, we have

$$\begin{aligned} d(gx_{2n+1}, gx_{2n+2}) =& d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \\ & \preceq a_1 d(gx_{2n}, gx_{2n+1}) + a_2 d(gy_{2n}, gy_{2n+1}) + a_3(d(gx_{2n+1}, gx_{2n}) + d(gx_{2n+2}, gx_{2n+1})) \\ & \quad + a_4(d(gx_{2n+1}, gx_{2n+1}) + d(gx_{2n+2}, gx_{2n})). \end{aligned}$$

So

$$(e - a_3 - a_4) d(gx_{2n+1}, gx_{2n+2}) \preceq (a_1 + a_3 + a_4) d(gx_{2n}, gx_{2n+1}) + a_2 d(gy_{2n}, gy_{2n+1}). \tag{3.1}$$

Similarly, we get

$$(e - a_3 - a_4)d(gy_{2n+1}, gy_{2n+2}) \preceq (a_1 + a_3 + a_4)d(gy_{2n}, gy_{2n+1}) + a_2 d(gx_{2n}, gx_{2n+1}). \tag{3.2}$$

By Lemma 2.3 and the given conditions, $e - a_3 - a_4$ is invertible. Let

$$\lambda = (a_1 + a_2 + a_3 + a_4)(e - a_3 - a_4)^{-1}.$$

From the inequalities (3.1) and (3.2), we obtain that

$$d(gx_{2n+1}, gx_{2n+2}) + d(gy_{2n+1}, gy_{2n+2}) \preceq \lambda(d(gx_{2n}, gx_{2n+1}) + d(gy_{2n}, gy_{2n+1})). \tag{3.3}$$

On the other hand, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} d(gx_{2n+1}, gx_{2n}) =& d(F(x_{2n}, y_{2n}), G(x_{2n-1}, y_{2n-1})) \\ \preceq& a_1 d(gx_{2n}, gx_{2n-1}) + a_2 d(gy_{2n}, gy_{2n-1}) + a_3(d(gx_{2n+1}, gx_{2n}) + d(gx_{2n}, gx_{2n-1})) \\ &+ a_4(d(gx_{2n+1}, gx_{2n-1}) + d(gx_{2n}, gx_{2n})), \end{aligned}$$

which implies that

$$(e - a_3 - a_4)d(gx_{2n}, gx_{2n+1}) \preceq (a_1 + a_3 + a_4)d(gx_{2n}, gx_{2n-1}) + a_2 d(gy_{2n}, gy_{2n-1}). \tag{3.4}$$

By the similar arguments as above, we can get

$$(e - a_3 - a_4)d(gy_{2n}, gy_{2n+1}) \preceq (a_1 + a_3 + a_4)d(gy_{2n}, gy_{2n-1}) + a_2 d(gx_{2n}, gx_{2n-1}). \tag{3.5}$$

Adding the inequalities (3.4) and (3.5), we get

$$d(gx_{2n}, gx_{2n+1}) + d(gy_{2n}, gy_{2n+1}) \preceq \lambda(d(gx_{2n}, gx_{2n-1}) + d(gy_{2n}, gy_{2n-1})). \tag{3.6}$$

Then the inequality (3.3) together with (3.6) implies that

$$\begin{aligned} d(gx_{2n+1}, gx_{2n+2}) + d(gy_{2n+1}, gy_{2n+2}) &\preceq \lambda(d(gx_{2n}, gx_{2n+1}) + d(gy_{2n}, gy_{2n+1})) \\ &\preceq \lambda^2(d(gx_{2n}, gx_{2n-1}) + d(gy_{2n}, gy_{2n-1})) \\ &\vdots \\ &\preceq \lambda^{2n+1}(d(gx_0, gx_1) + d(gy_0, gy_1)). \end{aligned}$$

Let $\{w_n\}_{n=0}^{\infty} = (gx_0, gx_1, gx_2, \ldots)$ and $\{z_n\}_{n=0}^{\infty} = (gy_0, gy_1, gy_2, \ldots)$. Then for $n \in \mathbb{N}$, we have

$$d(w_n, w_{n+1}) + d(z_n, z_{n+1}) \preceq \lambda^n(d(w_0, w_1) + d(z_0, z_1)). \tag{3.7}$$

We need only to consider the following two cases:

Case 1: $d(w_0, w_1) + d(z_0, z_1) = \theta$. This case yields that $w_0 = w_1$ and $z_0 = z_1$. By the formula (3.7), we get that $w_0 = w_n$ and $z_0 = z_n$ for each $n \in \mathbb{N}$. Hence $gx_0 = gx_1 = F(x_0, y_0)$ and $gy_0 = gy_1 = F(y_0, x_0)$. Now we show that $G(x_0, y_0) = gx_0$ and $G(y_0, x_0) = gy_0$. For that, we have

$$\begin{aligned} d(gx_0, G(x_0, y_0)) =& d(F(x_0, y_0), G(x_0, y_0)) \\ \preceq& a_1 d(gx_0, gx_0) + a_2 d(gy_0, gy_0) + a_3(d(gx_0, gx_0) + d(G(x_0, y_0), gx_0)) \\ &+ a_4(d(gx_0, gx_0) + d(G(x_0, y_0), gx_0)). \end{aligned}$$

Hence

$$d(gx_0, G(x_0, y_0)) \preceq (a_3 + a_4)d(gx_0, G(x_0, y_0)).$$

Since $r(a_3) + r(a_4) < 1$, $d(gx_0, G(x_0, y_0)) = \theta$ and $gx_0 = G(x_0, y_0)$. Similarly, we can show that $gy_0 = G(y_0, x_0)$. Therefore we get that (x_0, y_0) is a common coupled coincidence point of F, G and g.

Case 2: $d(w_0, w_1) + d(z_0, z_1) \neq \theta$. Indeed let $m > n$, then

$$d(w_n, w_m) \preceq d(w_n, w_{n+1}) + \cdots + d(w_{m-1}, w_m)$$

and

$$d(z_n, z_m) \preceq d(z_n, z_{n+1}) + \cdots + d(z_{m-1}, z_m).$$

In order to prove the following conclusion, we firstly verify the fact that $r(\lambda) < 1$.

In fact since $r(a_1) + r(a_2) + 2r(a_3) + 2r(a_4) < 1$, then $r(a_3) + r(a_4) < 1$ which together with Lemma 2.3 implies that $(e - a_3 - a_4)^{-1}$ is existent. Then from Lemma 2.3 again,

$$\begin{aligned} r(\lambda) &= r((a_1 + a_2 + a_3 + a_4)(e - a_3 - a_4)^{-1}) \\ &\leq r(e - (a_3 + a_4))^{-1} r(a_1 + a_2 + a_3 + a_4) \\ &\leq \frac{r(a_1) + r(a_2) + r(a_3) + r(a_4)}{1 - r(a_3) - r(a_4)} \\ &< 1. \end{aligned}$$

By (3.7) and the fact of $r(\lambda) < 1$, we have

$$d(w_n, w_m) + d(z_n, z_m) \preceq \lambda^n (e - \lambda)^{-1} (d(w_0, w_1) + d(z_0, z_1)).$$

Thus for each $c \in P$ with $\theta \ll c$, we can find a sufficient $k \in \mathbb{N}$ such that

$$\frac{\lambda^n}{e - \lambda} (d(w_0, w_1) + d(z_0, z_1)) \ll c,$$

which gives, for all $n \geq k$,

$$d(w_n, w_m) + d(z_n, z_m) \ll c.$$

So $\{w_n\}$ and $\{z_n\}$ are Cauchy sequences in $g(X)$. As $g(X)$ is complete, there exist x, y in X such that $w_n = gx_n \to gx$ and $z_n = gy_n \to gy$ as $n \to +\infty$. These give that $gx_{2n+1} \to gx$, $gx_{2n} \to gx$, $gy_{2n+1} \to gy$, and $gy_{2n} \to gy$ as $n \to \infty$. Now we prove that $F(x, y) = G(x, y) = gx$ and $F(y, x) = G(y, x) = gy$. Clearly,

$$d(F(x, y), gx) \preceq d(F(x, y), gx_{2n+2}) + d(gx_{2n+2}, gx). \tag{3.8}$$

So the given conditions yield that

$$\begin{aligned} d(F(x, y), gx_{2n+2}) =& d(F(x, y), G(x_{2n+1}, y_{2n+1})) \\ \preceq& a_1 d(gx, gx_{2n+1}) + a_2 d(gy, gy_{2n+1}) + a_3 (d(F(x, y), gx) + d(gx_{2n+2}, gx_{2n+1})) \\ &+ a_4 (d(F(x, y), gx_{2n+1}) + d(gx_{2n+2}, gx)). \end{aligned}$$

Then the formula (3.8) turns to

$$d(F(x, y), gx) \preceq \lambda_1 d(gx_{2n+2}, gx) + \lambda_2 d(gx, gx_{2n+1}) + \lambda_3 d(gy, gy_{2n+1}),$$

where

$$\lambda_1 = (e + a_3 + a_4)(e - a_3 - a_4)^{-1},\ \lambda_2 = (a_1 + a_3 + a_4)(e - a_3 - a_4)^{-1} \text{ and } \lambda_3 = a_2 (e - a_3 - a_4)^{-1}.$$

Since $gx_{2n+1} \to gx, gy_{2n+1} \to gy$ and $gx_{2n+2} \to gx$ as $n \to +\infty$, then for $c \gg \theta$ there is $N_0 \in \mathbb{N}$ such that

$$d(gx_{2n+2}, gx) \ll \frac{\lambda_1^{-1}}{3} c, \quad d(gx_{2n+1}, gx) \ll \frac{\lambda_2^{-1}}{3} c, \quad \text{and} \quad d(gy_{2n+2}, gy) \ll \frac{\lambda_3^{-1}}{3} c$$

for all $n \geq N_0$. So $d(F(x, y), gx) \ll c$, that is, $F(x, y) = gx$. By the similar arguments as above and the following inequality

$$d(gx, G(x, y)) \preceq d(gx, gx_{2n+1}) + d(gx_{2n+1}, G(x, y)) = d(gx, gx_{2n+1}) + d(F(x_{2n}, y_{2n}), G(x, y)),$$

we get $G(x, y) = gx$. Hence $F(x, y) = G(x, y) = gx$. Similarly we can get $F(y, x) = G(y, x) = gy$. Therefore (x, y) is a common coupled coincidence point of F, G and g. $\square$

4. Coupled fixed point results for T-contractions on cone metric spaces over Banach algebras

At first, we introduce the conception of T-contractions which is the main object considered in this section.

Definition 4.1. Let $T, F : X \to X$ be two mappings. F is said to be a T-contraction if there exists $k \in P$ with $r(k) < 1$, such that

$$d(TFx, TFy) \preceq kd(Tx, Ty), \quad \forall x, y \in X.$$

Remark 4.2. A contraction is T-contractive since it suffices to take $T = I$, where I is the identity mapping on X.

Example 4.1. Let $\mathcal{A} = \mathbb{R}^2$. For each $x = (x_1, x_2) \in \mathcal{A}$, let $\|x\| = |x_1| + |x_2|$. The multiplication is defined by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2).$$

Then $\mathcal{A}$ is a Banach algebra with unit $e = (1, 1)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0\}$ and $Y = \mathbb{R}^2$. A metric d on Y is defined by

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, |x_2 - y_2|) \in P.$$

Then (Y, d) is a complete cone metric space over the Banach algebra $\mathcal{A}$.

Now define the mappings $F : Y \to Y$ by

$$F(x, y) = \left(\ln(e^{x-2} + 1), \tan\left(\frac{2}{\pi}\arctan(y + 1)\right)\right)$$

and $T : Y \to Y$ by $T(x, y) = (-e^x - x, -\arctan(y + 1) - y)$. Then F is a T-contraction.

In fact, let $k = (\frac{2}{e^2}, \frac{2}{\pi}) \in P$, clearly $k \in P$ and $0 < r(k) < 1$ and it is not difficult to verify that, for each pair $x, y \in Y$,

$$d(TFx, TFy) \preceq kd(Tx, Ty).$$

Theorem 4.3. *Suppose that $T : X \to X$ is a surjective and one to one mapping. Furthermore, if the mapping $F : X \times X \to X$ satisfies*

$$d(TF(x, y), TF(u, v)) \preceq \alpha d(Tx, Tu) + \beta d(Ty, Tv) \tag{4.1}$$

for all $x, y, u, v \in X$, where $\alpha, \beta \in P$ with $r(\alpha + \beta) < 1$, then there exist unique $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a unique coupled fixed point.

Proof. Take $x_0, y_0 \in X$ and we denote

$$x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0), \qquad y_{n+1} = F(y_n, x_n) = F^{n+1}(y_0, x_0)$$

for all $n \in \mathbb{N}$. Now according to (4.1), we have

$$d(Tx_n, Tx_{n+1}) = d(TF(x_{n-1}, y_{n-1}), \quad TF(x_n, y_n)) \preceq \alpha d(Tx_{n-1}, Tx_n) + \beta d(Ty_{n-1}, Ty_n) \tag{4.2}$$

and

$$d(Ty_n, Ty_{n+1}) = d(TF(y_{n-1}, x_{n-1}), \quad TF(y_n, x_n)) \preceq \alpha d(Ty_{n-1}, Ty_n) + \beta d(Tx_{n-1}, Tx_n). \tag{4.3}$$

Let $d_n = d(Tx_n, Tx_{n+1}) + d(Ty_n, Ty_{n+1})$. From (4.2) and (4.3), we obtain

$$d_n \preceq (\alpha + \beta)(d(Tx_{n-1}, Tx_n) + d(Ty_{n-1}, Ty_n)) = \lambda d_{n-1},$$

where $\lambda = \alpha + \beta$, $r(\lambda) < 1$. Thus, for all n,

$$\theta \preceq d_n \preceq \lambda d_{n-1} \preceq \lambda^2 d_{n-2} \preceq \cdots \preceq \lambda^n d_0. \tag{4.4}$$

Without loss of generality, we assume that $d_0 > \theta$. Otherwise (x_0, y_0) is a coupled fixed point of F. If $m > n$, then we have

$$d(Tx_n, Tx_m) \preceq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m) \tag{4.5}$$

and similarly,

$$d(Ty_n, Ty_m) \preceq d(Ty_n, Ty_{n+1}) + d(Ty_{n+1}, Ty_{n+2}) + \cdots + d(Ty_{m-1}, Ty_m). \tag{4.6}$$

By (4.5), (4.6), and (4.4), we have

$$\begin{aligned} d(Tx_n, Tx_m) + d(Ty_n, Ty_m) &\preceq d_n + d_{n+1} + \cdots + d_{m-1} \\ &\preceq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})d_0 \\ &\preceq \lambda^n (e - \lambda)^{-1} d_0. \end{aligned}$$

Since $r(\lambda) = r(\alpha + \beta) < 1$, by Remark 2.1 in [16], we get $\|\lambda^n\| \to 0$, which together with Lemmas 2.4 and 2.5, implies that for every $c \in \operatorname{int} P$, there exists $N \in \mathbb{N}$ such that $d(Tx_n, Tx_m) + d(Ty_n, Ty_m) \ll c$ for every $m > n > N$. So $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in X. The completeness of X gives that there exist $x^*, y^* \in X$ such that

$$\lim_{n\to+\infty} TF^n(x_0, y_0) = x^*, \qquad \lim_{n\to+\infty} TF^n(y_0, x_0) = y^*.$$

Noting (4.1), we can easily verify that

$$\begin{aligned} d(TF(x^*, y^*), Tx^*) &\preceq d(TF(x^*, y^*), TF(x_n, y_n)) + d(TF(x_n, y_n), Tx^*) \\ &\preceq \alpha d(Tx^*, Tx_n) + \beta d(Ty^*, Ty_n) + d(Tx_{n+1}, Tx^*). \end{aligned}$$

From the surjective property of T and Lemma 2.5, it follows that $d(TF(x^*, y^*), Tx^*) = \theta$, that is, $TF(x^*, y^*) = Tx^*$. Since T is one-to-one, then $F(x^*, y^*) = x^*$. Similarly, we can get $F(y^*, x^*) = y^*$. Therefore, (x^*, y^*) is a coupled fixed point of F. Now if (x', y') is another coupled fixed point of F, then

$$d(Tx^*, Tx') = d(TF(x^*, y^*), \quad TF(x', y')) \preceq \alpha d(Tx^*, Tx') + \beta d(Ty^*, Ty') \tag{4.7}$$

and

$$d(Ty^*, Ty') = d(TF(y^*, x^*), \quad TF(y', x')) \preceq \alpha d(Ty^*, Ty') + \beta d(Tx^*, Tx'). \tag{4.8}$$

From (4.7) and (4.8), we have

$$d(Tx^*, Tx') + d(Ty^*, Ty') \preceq \lambda[d(Tx^*, Tx') + d(Ty^*, Ty')]. \tag{4.9}$$

Since $r(\lambda) = r(\alpha + \beta) < 1$, it follows from (4.9) that $d(Tx^*, Tx') + d(Ty^*, Ty') = \theta$. Hence

$$d(Tx^*, Tx') = d(Ty^*, Ty') = \theta.$$

That is $Tx^* = Tx'$ and $Ty^* = Ty'$. As T is one to one, we have $(x^*, y^*) = (x', y')$. Thus F has a unique coupled fixed point. □

Corollary 4.4. *Suppose that (X, d) is a complete cone metric space over the Banach algebra $\mathcal{A}$, P is a solid cone of $\mathcal{A}$, and $T : X \to X$ is a surjective and one to one mapping. Then any T-contraction on X has a unique fixed point.*

5. Coincidence point results for contractions on cone metric spaces over Banach algebras

In order to present the next result, we firstly introduce some necessary conditions.

Let $\phi : P \to P$ be a mapping satisfying:

(1) if $a, b \in P$ with $a \preceq b$, then there exists $k \in \mathcal{A}$ with $r(k) < 1$ for which $\phi(a) \preceq k\phi(b)$;

(2) $\phi(a+b) \preceq \phi(a)+\phi(b)$ for all $a, b \in P$;

(3) ϕ is sequentially continuous, i.e., if $a_n, a \in P$ and $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} \phi(a_n) = \phi(a)$;

(4) if $\phi(a_n) \to \theta$, then $a_n \to \theta$.

It is clear that $\phi(a) = \theta$ if and only if $a = \theta$ provided ϕ satisfies all above properties.

Theorem 5.1. *Suppose that f, g, h are self-mappings on X satisfying*

$$\phi(d(fx, gy)) \preceq a\phi(d(hx, hy)) + b\phi(d(hx, fx)) + c\phi(d(hy, gy)), \tag{5.1}$$

where $a, b, c \in P$ with $0 < r(a) + r(b) + r(c) < 1$. Moreover if $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then $f, g,$ and h have a unique point of coincidence in X.

Proof. Let $x_0 \in X$. Since $f(X) \cup g(X) \subset h(X)$, starting with x_0 we define a sequence $\{y_n\}$ such that

$$y_{2n} = fx_{2n} = hx_{2n+1} \qquad \text{and} \qquad y_{2n+1} = gx_{2n+1} = hx_{2n+2}$$

for all $n \geq 0$. We shall prove that $\{y_n\}$ is a Cauchy sequence in X. If $y_n = y_{n+1}$ for some n, e.g., if $y_{2n} = y_{2n+1}$, then from (5.1) we obtain

$$\begin{aligned}
\phi(d(y_{2n+2}, y_{2n+1})) &= \phi(d(fx_{2n+2}, gx_{2n+1})) \\
&\preceq a\phi(d(hx_{2n+2}, hx_{2n+1})) + b\phi(d(hx_{2n+2}, fx_{2n+2})) + c\phi(d(hx_{2n+1}, gx_{2n+1})) \\
&= a\phi(d(y_{2n+1}, y_{2n})) + b\phi(d(y_{2n+1}, y_{2n+2})) + c\phi(d(y_{2n}, y_{2n+1})).
\end{aligned}$$

Since $y_{2n} = y_{2n+1}$ it follows from the above inequality that

$$\phi(d(y_{2n+2}, y_{2n+1})) \preceq b\phi(d(y_{2n+1}, y_{2n+2})).$$

As $r(b) < 1$, $\phi(d(y_{2n+2}, y_{2n+1})) = \theta$ which gives $d(y_{2n+2}, y_{2n+1}) = \theta$, i.e., $y_{2n+2} = y_{2n+1}$. Similarly we obtain that

$$y_{2n} = y_{2n+1} = y_{2n+2} = \cdots = v.$$

So $\{y_n\}$ is a Cauchy sequence. Suppose $y_n \neq y_{n+1}$ for all n. Then from (5.1) it follows that

$$\begin{aligned}
\phi(d(y_{2n}, y_{2n+1})) &= \phi(d(fx_{2n}, gx_{2n+1})) \\
&\preceq a\phi(d(hx_{2n}, hx_{2n+1})) + b\phi(d(hx_{2n}, fx_{2n})) + c\phi(d(hx_{2n+1}, gx_{2n+1})) \\
&= a\phi(d(y_{2n-1}, y_{2n})) + b\phi(d(y_{2n-1}, y_{2n})) + c\phi(d(y_{2n}, y_{2n+1})) \\
&= (a+b)\phi(d(y_{2n-1}, y_{2n})) + c\phi(d(y_{2n}, y_{2n+1})),
\end{aligned}$$

i.e.,

$$\phi(d(y_{2n}, y_{2n+1})) \preceq (a+b)(e-c)^{-1}\phi(d(y_{2n-1}, y_{2n})) = \lambda\phi(d(y_{2n-1}, y_{2n})),$$

where

$$r(\lambda) \leq r((a+b)(e-c)^{-1}) \leq \frac{r(a)+r(b)}{1-r(c)} < 1.$$

Writing $d_n = \phi(d(y_n, y_{n+1}))$, we obtain

$$d_{2n} \preceq \lambda d_{2n-1}. \tag{5.2}$$

Again

$$\begin{aligned}
\phi(d(y_{2n+2}, y_{2n+1})) &= \phi(d(fx_{2n+2}, gx_{2n+1})) \\
&\preceq a\phi(d(hx_{2n+2}, hx_{2n+1})) + b\phi(d(hx_{2n+2}, fx_{2n+2})) + c\phi(d(hx_{2n+1}, gx_{2n+1})) \\
&= a\phi(d(y_{2n+1}, y_{2n})) + b\phi(d(y_{2n+1}, y_{2n+2})) + c\phi(d(y_{2n}, y_{2n+1})) \\
&= (a+c)\phi(d(y_{2n+1}, y_{2n})) + b\phi(d(y_{2n+1}, y_{2n+2})),
\end{aligned}$$

i.e.,

$$\phi(d(y_{2n+2}, y_{2n+1})) \preceq (a+c)(e-b)^{-1}\phi(d(y_{2n+1}, y_{2n})) = \phi(d(y_{2n+1}, y_{2n})).$$

Let $\mu = (a+c)(e-b)^{-1}$, then $r(\mu) = r((a+c)(e-b)^{-1}) < 1$. Therefore

$$d_{2n+1} \preceq \mu d_{2n}. \tag{5.3}$$

From (5.2) and (5.3) we get

$$d_{2n} \preceq \lambda d_{2n-1} \preceq \lambda\mu d_{2n-2} \preceq \cdots \preceq \lambda^n \mu^n d_0,$$

and

$$d_{2n+1} \preceq \mu d_{2n} \preceq \lambda\mu d_{2n-1} \preceq \cdots \preceq \lambda^n \mu^{n+1} d_0.$$

Thus

$$d_{2n} + d_{2n+1} \preceq \lambda^n \mu^n (e+\mu) d_0 \tag{5.4}$$

and

$$d_{2n+1} + d_{2n+2} \preceq \lambda^n \mu^{n+1} (e+\lambda) d_0. \tag{5.5}$$

Let $n, m \in \mathbb{N}$, then for the sequence $\{y_n\}$, we consider $\phi(d(y_n, y_m))$ in two cases.

(i) If n is even and $m > n$, then using (5.4) we obtain

$$\begin{aligned}\phi(d(y_n, y_m)) &\preceq k\phi(d(y_n, y_{n+1})) + k\phi(d(y_{n+1}, y_{n+2})) + \cdots + k\phi(d(y_{m-1}, y_m))\\ &\preceq k(d_n + d_{n+1} + d_{n+2} + d_{n+3} + \cdots)\\ &\preceq k(\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}(e+\mu)d_0 + \lambda^{\frac{n+2}{2}}\mu^{\frac{n+2}{2}}(e+\mu)d_0 + \cdots).\end{aligned}$$

So

$$\phi(d(y_n, y_m)) \preceq k(\lambda\mu)^{\frac{n}{2}}(e+\mu)(e-\lambda\mu)^{-1}d_0.$$

(ii) If n is odd and $m > n$, then using (5.5) we obtain

$$\begin{aligned}\phi(d(y_n, y_m)) &\preceq k\phi(d(y_n, y_{n+1})) + k\phi(d(y_{n+1}, y_{n+2})) + \cdots + k\phi(d(y_{m-1}, y_m))\\ &\preceq k(d_n + d_{n+1} + d_{n+2} + d_{n+3} + \cdots)\\ &\preceq k(\lambda^{\frac{n-1}{2}}\mu^{\frac{n-1}{2}+1}(e+\mu)d_0 + \lambda^{\frac{n+1}{2}}\mu^{\frac{n+1}{2}+1}(e+\lambda)d_0 + \cdots).\end{aligned}$$

So

$$\phi(d(y_n, y_m)) \preceq k(\lambda\mu)^{\frac{n-1}{2}}(e+\lambda)(e-\lambda\mu)^{-1}d_0.$$

Since $0 < r(\lambda) < 1$, $0 < r(\mu) < 1$, $0 < r(\lambda\mu) < 1$, in both cases $\phi(d(y_n, y_m)) \to \theta$ as $n \to \infty$, and we have $d(y_n, y_m) \to \theta$ as $n \to \infty$. Then by Lemmas 2.4 and 2.5, $\{y_n\} = \{hx_{n-1}\}$ is a Cauchy sequence. Since $h(X)$ is complete, there exist $v \in h(X)$ and $u \in X$ such that $\lim\limits_{n\to\infty} y_n = v$ and $v = hu$.

Next we show that u is a coincidence point of pairs (f, h) and (g, h), i.e., $fu = gu = hu$.

If $fu \neq hu$ then $\theta < d(fu, hu)$. Using (5.1) we obtain

$$\begin{aligned}\phi(d(fu, y_{2n+1})) &= \phi(d(fu, gx_{2n+1}))\\ &\preceq a\phi(d(hu, hx_{2n+1})) + b\phi(d(hu, fu)) + c\phi(d(hx_{2n+1}, gx_{2n+1}))\\ &= a\phi(d(hu, y_{2n})) + b\phi(d(hu, fu)) + c\phi(d(y_{2n}, y_{2n+1}))\\ &= a\phi(d(hu, y_{2n})) + b\phi(d(hu, fu) + cd_{2n}.\end{aligned}$$

Since $y_{2n} \to hu, d_{2n} \to \theta, d(fu, y_{2n+1}) \to d(fu, hu)$ as $n \to \infty$, letting $n \to \infty$ in above inequality we get

$$\begin{aligned}\phi(d(fu, hu)) &\preceq b\phi(d(hu, fu))\\ &< \phi(d(hu, fu)),\end{aligned}$$

a contradiction. Therefore $fu = hu$. Similarly it can be shown that $gu = hu$. Therefore

$$fu = gu = hu = v.$$

Thus v is a point of coincidence of pairs (f, h) and (g, h).

In the following, we show that the point of coincidence of pairs (f, h) and (g, h) is unique.

Suppose w is another point of coincidence of (f, h) and (g, h), i.e., $fz = gz = hz = w$ for some $z \in X$. Then from (5.1) it follows that

$$\begin{aligned}\phi(d(w, v)) &= \phi(d(fz, gu))\\ &\preceq a\phi(d(hz, hu)) + b\phi(d(hz, fz)) + c\phi(d(hu, gu))\\ &= a\phi(d(w, v)) + b\phi(d(w, w)) + c\phi(d(v, v))\\ &= a\phi(d(w, v)).\end{aligned}$$

So $\phi(d(w, v)) = \theta$, i.e., $w = v$. Namely, the point of coincidence of pairs (f, h) and (g, h) is unique. □

6. Applications

In the section, we give a simple application of one of the main results. Also the presented example shows that the given conditions of the main results are realizable and valid.

Let $C^2_R([0, 1])$ be the space of all real functions on $[0, 1]$ whose second derivative is continuous. We recall that for $a, b > 0$, the space $C^2_R([0, 1])$ with the norm

$$\|f\| = \|f\|_\infty + a\|f'\|_\infty + b\|f''\|_\infty$$

is a Banach space, where $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$. This space is a Banach algebra if and only if $2b \le a^2$ (see [13, page 272]), and henceforth, we assume that $0 < a, 0 < 2b \le a^2$.

If we take $X = C^2_R([0, 1])$ with the above norm and $P = \{u \in X : u \ge 0\}$, then (X, d) becomes a cone metric space where $d(x, y) = (\sup\limits_{t\in[0,1]} |x(t) - y(t)|)f(t)$ and $f : [0, 1] \to R, f(t) = e^t$.

We now study the existence of solution for the nonlinear Volterra integral equation

$$x(t) = z_0(t) + \int_0^t K(t, s, x(s))ds, \quad z_0(t), x(t) \in C^2_R([0, 1]), \quad t \in [0, 1]. \tag{6.1}$$

If the following conditions are satisfied, then the equation (6.1) has a unique solution in $C^2_R([0, 1])$,

(i) $K : [0, 1] \times [0, 1] \times X \to X$ has a continuous derivative;

(ii) for any $x(t), y(t) \in C^2_R([0, 1])$, $t, s \in [0, 1]$,

$$|K(t, s, x(s)) - K(t, s, y(s))| \preceq \frac{1}{3}\left(1 + t^3\right)|x(s) - y(s)|.$$

In fact, let $A(x(t)) = z_0(t) + \int_0^t K(t, s, x(s))ds,\ z_0(t), x(t) \in C^2_R([0, 1]),\ t \in [0, 1]$ and take $k(t) = \frac{1}{3}(1 + t^3)$, $t \in [0, 1]$, then $k \in P$ and $r(k) < 1$. Moreover, we can check that $d(A(x), A(y)) \preceq kd(x, y)$ for all $x(t), y(t) \in C^2_R([0, 1])$. Thus by Corollary 4.4, the equation (6.1) has a unique solution in $C^2_R([0, 1])$.

Remark 6.1. Under the current conditions, it is not easy to claim that the equation (6.1) has a unique solution in $C^2_R([0, 1])$ by other known results rather than ours. But by our results, it becomes a no-brainer problem.

Acknowledgement

The authors thank the editor and the referees for their valuable comments and suggestions. This work was supported by the NSF of Education Department of Jiangxi Province of China (No. GJJ150028) and the Special Innovation Foundation of Graduate Student of Nanchang University (No. cx2016149).

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