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Semicontinuity of solution mappings for a class of parametric generalized vector equilibrium problems

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Abstract

In this paper, we discuss the upper and lower semicontinuity of the strong efficient solution mapping, the weakly efficient solution mapping and the efficient solution mapping to a class of parametric generalized vector equilibrium problems by using scalarization methods and a new density result.

Keywords: Parametric generalized vector equilibrium problem, solution mapping, lower semicontinuity, upper semicontinuity.
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1. Introduction

Vector equilibrium problem, as a generalization of the equilibrium problem [7] and the vector variational inequality [16], plays a very important role in many fields such as mathematical physics, economics theory, operations research, management science, engineering design and others. The existence theory concerned with solutions for the vector variational inequalities and the vector equilibrium problems has been extensively studied by many authors under quite different conditions (see, for example, [4, 5, 8, 12, 14, 15, 17, 18, 26, 28, 30, 32, 35] and the references therein).

On the other hand, the stability analysis in connection with the solution mappings to vector equilibrium problems is an important topic in vector optimization theory. Recently, the lower semicontinuity and the upper semicontinuity of the solution mappings to parametric vector equilibrium problems have been intensively

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studied in the literature, for instance, we refer the reader to [1–3, 9–11, 13, 19, 20, 22, 23, 27, 29, 31, 33, 34]. We note that, in order to get the semicontinuity of the solution mappings for the parametric vector equilibrium problems, the authors of [3, 9–11, 19, 20, 29, 31, 34] employed the monotonicity of mappings or the information about the solution mappings. It is worth mentioning that the monotonicity of mappings may yield that the set of solutions is a singleton and the assumptions involving information of solution mappings are not reasonable from the view of real problems. Therefore, it is important and interesting to discuss the semicontinuity of the solution mappings for a parametric generalized vector equilibrium problem (for short, PGVEP) under some new conditions.

The rest of the paper is organized as follows. Section 2 presents some necessary notations and lemmas. In Section 3, we obtain a new scalarization result and a new density result for a generalized vector equilibrium problem. Then we establish the lower semicontinuity of strong efficient solution mapping, weakly efficient solution mapping and efficient solution mapping to (PGVEP) by using the scalarization methods and the density result. In Section 4, we discuss the upper semicontinuity of strong efficient solution mapping and weakly efficient solution mapping to (PGVEP). Moreover, we establish the Hausdorff upper semicontinuity of efficient solution mapping to (PGVEP), which is a generalization of Theorem 5.4 of [24] from the finite dimensional space to the infinite dimensional space.

2. Preliminaries

Throughout this paper, unless otherwise specified, let Λ, W, Δ, X and Y be five normed vector spaces. Assume that $C \subseteq Y$ is a closed, convex, pointed cone with nonempty interior, $P \subseteq \Delta$ is a convex, pointed cone, and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Let Y^* be the topological dual space of Y and C^* be defined by

$$C^* = \{f \in Y^* : f(c) \geq 0,\ \forall c \in C\}.$$

Denote the quasi-interior of C^* by $C^\#$, i.e.,

$$C^\# = \{f \in Y^* : f(c) > 0,\ \forall c \in C\setminus\{0\}\}.$$

Let D be a nonempty subset of Y. The cone hull of D is defined as

$$\mathrm{cone}(D) = \{td : t \geq 0, d \in D\}.$$

Denote the closure of D by $\mathrm{cl}(D)$ and the interior of D by $\mathrm{int}D$. A nonempty convex subset B of the convex cone C is called a base of C if $C = \mathrm{cone}(B)$ and $0 \notin \mathrm{cl}(B)$. It is easy to see that $C^\# \neq \emptyset$ if and only if C has a base. Let e be a fixed point in $\mathrm{int}C$,

$$B^* = \{f \in C^* : f(e) = 1\},$$

and

$$B^\# = \left\{f \in C^\# : f(e) = 1\right\}.$$

Then it is easy to see that B^* is a weak* compact base of C^*, $B^\#$ is a base of $C^\#$ and $B^* = \mathrm{cl}\left(B^\#\right)$ with respect to the weak* topology.

Let K be a nonempty subset of X and $S : X \rightrightarrows \Delta$ and $F : X \times \Delta \times X \rightrightarrows Y$ be two set-valued mappings. We consider the following generalized vector equilibrium problem consisting of finding $x_0 \in K$ such that

$$\text{(GVEP)} \qquad F(x_0, u, y) \cap (-\Omega) = \emptyset, \quad \forall u \in S(x_0), \forall y \in K,$$

where $\Omega \cup \{0\}$ is a cone in Y.

Let $W(F, S, K)$ denote the set of all weakly efficient solutions of (GVEP), i.e.,

$$W(F,S,K) = \{x \in K : F(x,u,y) \cap (-\mathrm{int}C) = \emptyset,\ \forall u \in S(x), \forall y \in K\}.$$

and $E(F,S,K)$ denote the set of all efficient solutions of (GVEP), i.e.,

$$E(F,S,K) = \{x \in K : F(x,u,y) \cap (-C \setminus \{0\}) = \emptyset,\ \forall u \in S(x), \forall y \in K\}.$$

For any $f \in C^*$, let $Q(f)$ denote the set of all f-solutions of (GVEP), i.e.,

$$Q(f) = \{x \in K : f(F(x,u,y)) \subseteq \mathbb{R}_+, \forall u \in S(x), \forall y \in K\}.$$

Let K be a nonempty subset of X and $S : X \rightrightarrows \Delta$ and $F : X \times \Delta \times X \rightrightarrows Y$ be two set-valued mappings. Let $F : X \times \Delta \times X \times W \rightrightarrows Y$ and $K : \Lambda \rightrightarrows X$ be two set-valued mappings. For any $(\alpha,\lambda) \in W \times \Lambda$, we consider the following parametric generalized vector equilibrium problem consisting of finding $x_0 \in K(\lambda)$ such that

$$\text{(PGVEP)} \qquad F(x_0,u,y,\alpha) \cap (-\Omega) = \emptyset, \quad \forall u \in S(x_0), \forall y \in K(\lambda),$$

where $\Omega \cup \{0\}$ is a cone in Y.

For any $(\alpha,\lambda) \in W \times \Lambda$, let $M(\alpha,\lambda)$ denote the set of all strong efficient solutions of (PGVEP), i.e.,

$$M(\alpha,\lambda) = \{x \in K(\lambda) : F(x,u,y,\alpha) \subseteq C, \forall u \in S(x), \forall y \in K(\lambda)\},$$

and $W(\alpha,\lambda)$ denote the set of all weakly efficient solutions of (PGVEP), i.e.,

$$W(\alpha,\lambda) = \{x \in K(\lambda) : F(x,u,y,\alpha) \cap (-\mathrm{int}C) = \emptyset,\ \forall u \in S(x), \forall y \in K(\lambda)\}.$$

For any $f \in C^*$ and $(\alpha,\lambda) \in W \times \Lambda$, let $S_f(\alpha,\lambda)$ denote the set of all f-solutions of (PGVEP), i.e.,

$$S_f(\alpha,\lambda) = \{x \in K(\lambda) : f(F(x,u,y,\alpha)) \subseteq \mathbb{R}_+, \forall u \in S(x), \forall y \in K(\lambda)\}.$$

Definition 2.1. A set-valued mapping $\Phi : \Delta \rightrightarrows Y$ is said to be P-C-increasing, if for any $u_1, u_2 \in \Delta$ with $u_1 - u_2 \in P$, one has

$$\Phi(u_1) \subseteq \Phi(u_2) + C.$$

Remark 2.2. The special case is as follows: a function $f : \mathbb{R} \to \mathbb{R}$ is said to be $\mathbb{R}_+$-$\mathbb{R}_+$-increasing, if for any $u_1, u_2 \in \mathbb{R}$ with $u_1 \geq u_2$, one has $f(u_1) \geq f(u_2)$.

Definition 2.3. Let D be a nonempty convex subset of X. A set-valued mapping $\Phi : D \rightrightarrows Y$ is said to be

(i) C-concave, if for any $x_1, x_2 \in D$ and $t \in [0,1]$, one has

$$\Phi(tx_1 + (1-t)x_2) \subseteq t\Phi(x_1) + (1-t)\Phi(x_2) + C;$$

(ii) strictly C-concave, if for any $x_1, x_2 \in D$ with $x_1 \neq x_2$ and, for any $t \in]0,1[$, one has

$$\Phi(tx_1 + (1-t)x_2) \subseteq t\Phi(x_1) + (1-t)\Phi(x_2) + \mathrm{int}C;$$

(iii) C-convexlike, if for any $x_1, x_2 \in D$ and, for any $t \in [0,1]$, there exists $x_3 \in D$ such that

$$t\Phi(x_1) + (1-t)\Phi(x_2) \subseteq \Phi(x_3) + C.$$

Now, we give the following example to illustrate that strictly C-concavity is easy to be verified.

Example 2.4. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+ = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\right\}$, $X = \mathbb{R}$ and $D = [-1,1]$. We denote by B_Y the closed unit ball in Y. Let a set-valued mapping $\Phi : D \rightrightarrows Y$ be defined as follows

$$\Phi(x) = \left(-x^2, 2\cos x\right) + B_Y.$$

Then it is easy to check that Φ is strictly C-concave.

Definition 2.5. A set-valued mapping $G : T \rightrightarrows T_1$ is said to be

(i) Hausdorff upper semicontinuous (H-u.s.c.) at $u_0 \in T$, if for any neighborhood V of $0 \in T_1$, there exists a neighborhood $U(u_0)$ of u_0 such that for every $u \in U(u_0)$, $G(u) \subseteq G(u_0) + V$;

(ii) upper semicontinuous (u.s.c.) at $u_0 \in T$, if for any neighborhood V of $G(u_0)$, there exists a neighborhood $U(u_0)$ of u_0 such that for every $u \in U(u_0)$, $G(u) \subseteq V$;

(iii) lower semicontinuous (l.s.c.) at $u_0 \in T$, if for any $x \in G(u_0)$ and any neighborhood V of x, there exists a neighborhood $U(u_0)$ of u_0 such that for every $u \in U(u_0)$, $G(u) \cap V \neq \emptyset$.

We say that G is H-u.s.c., u.s.c. and l.s.c. on T if it is H-u.s.c., u.s.c. and l.s.c. at each point $u \in T$, respectively. We say that G is continuous on T if it is both u.s.c. and l.s.c. on T.

Lemma 2.6 ([6])**.** *A set-valued mapping $\Phi : T \rightrightarrows T_1$ is l.s.c. at $u_0 \in T$ if and only if for any sequence $\{u_n\} \subseteq T$ with $u_n \to u_0$ and for any $x_0 \in \Phi(u_0)$, there exists $x_n \in \Phi(u_n)$ such that $x_n \to x_0$.*

Lemma 2.7 ([21])**.** *Let $\Phi : T \rightrightarrows T_1$ be a set-valued mapping. For any given $u_0 \in T$, if $\Phi(u_0)$ is compact, then Φ is u.s.c. at $u_0 \in T$ if and only if for any sequence $\{u_n\} \subseteq T$ with $u_n \to u_0$ and for any $x_n \in \Phi(u_n)$, there exist $x_0 \in \Phi(u_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$.*

Lemma 2.8 ([25])**.** *A set-valued mapping $G : T \rightrightarrows T_1$ is l.s.c. on T if and only if, for any $A \subseteq T$, one has*

$$\bigcup_{u \in \mathrm{cl}(A)} G(u) \subseteq \mathrm{cl}\left(\bigcup_{u \in A} G(u)\right).$$

3. Lower semicontinuity

In this section, we establish the lower semicontinuity of strong efficient solution mapping, weakly efficient solution mapping and efficient solution mapping to (PGVEP).

Lemma 3.1. *Let K be a nonempty compact convex subset of X. Assume that*

(i) *$S(\cdot)$ is l.s.c. and P-concave on K with nonempty compact values;*

(ii) *for any $(x, y) \in K \times K$, $F(x, \cdot, y)$ is P-C-increasing;*

(iii) *for any $y \in K$, $F(\cdot, \cdot, y)$ is strictly C-concave on $K \times \Delta$;*

(iv) *$F(\cdot, \cdot, \cdot)$ is continuous on $K \times \Delta \times K$ with nonempty compact values.*

Then $Q(\cdot)$ is l.s.c. on $C^* \setminus \{0_{Y^*}\}$, where the topology on $C^* \setminus \{0_{Y^*}\}$ is the weak topology.*

Proof. Suppose to the contrary that $Q(\cdot)$ is not l.s.c. at $f_0 \in C^* \setminus \{0_{Y^*}\}$. Then there exist $x_0 \in Q(f_0)$, a neighborhood W_0 of $0 \in X$ and a sequence $\{f_n\}$ with

$$f_n \xrightarrow{w^*} f_0,$$

such that

$$(x_0 + W_0) \cap Q(f_n) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

There are two cases to be considered.

Case 1. $Q(f_0)$ is singleton. Let

$$x_n \in Q(f_n), \quad \forall n \in \mathbb{N}. \tag{3.2}$$

It clear that $x_n \in K$. Since K is compact, without loss of generality, we can assume that $x_n \to \bar{x} \in K$. We claim that $\bar{x} \in Q(f_0)$. In fact, if not, then there exist $u_0 \in S(x_0)$ and $y_0 \in K$ such that

$$f_0(F(\bar{x}, u_0, y_0)) \not\subseteq \mathbb{R}_+.$$

Then there exists $z_0 \in F(\bar{x}, u_0, y_0)$ such that

$$f_0(z_0) < 0. \tag{3.3}$$

Since $S(\cdot)$ is l.s.c. at x_0, it follows from Lemma 2.6 that there exists $u_n \in S(x_n)$ such that $u_n \to u_0$. Noting that $F(\cdot,\cdot,y_0)$ is l.s.c. at (x_0, u_0), by Lemma 2.6, there exists $z_n \in F(x_n, u_n, y_0)$ such that $z_n \to z_0$. It follows from

$$f_n \xrightarrow{w^*} f_0,$$

that $f_n(z_n) \to f_0(z_0)$. By this together with (3.3), we have $f_n(z_n) < 0$ for n large enough, which contradicts (3.2). Therefore, $\bar{x} \in Q(f_0)$. It follows from $Q(f_0)$ is singleton that $\bar{x} = x_0$ and so $x_n \to x_0$. By this together with (3.2), we have

$$x_n \in (x_0 + W_0) \cap Q(f_n),$$

for n large enough, which contradicts (3.1).

Case 2. $Q(f_0)$ is not singleton. Then there exists $x' \in Q(f_0)$ such that $x' \neq x_0$. Since $x', x_0 \in Q(f_0)$, we have

$$f_0\left(F\left(x', u, y\right)\right) \subseteq \mathbb{R}_+, \ \ \forall u \in S\left(x'\right), \ \forall y \in K, \tag{3.4}$$

and

$$f_0(F(x_0, u, y)) \subseteq \mathbb{R}_+, \ \ \forall u \in S(x_0), \ \forall y \in K. \tag{3.5}$$

Since $S(\cdot)$ is P-concave on K, for any $t \in]0,1[$, we have

$$S\left(tx' + (1-t)x_0\right) \subseteq tS\left(x'\right) + (1-t)S(x_0) + P.$$

For any $u_t \in S(tx' + (1-t)x_0)$, there exist $u' \in S(x')$, $u_0 \in S(x_0)$ and $p_0 \in P$ such that

$$u_t = tu' + (1-t)u_0 + p_0.$$

By noting that $F(tx' + (1-t)x_0, \cdot, y)$ is P-C-increasing, we have

$$F\left(tx' + (1-t)x_0, u_t, y\right) \subseteq F\left(tx' + (1-t)x_0, tu' + (1-t)u_0, y\right) + C. \tag{3.6}$$

Since $F(\cdot,\cdot,y)$ is strictly C-concave on $K \times \Delta$, we have

$$F\left(tx' + (1-t)x_0, tu' + (1-t)u_0, y\right) \subseteq tF\left(x', u', y\right) + (1-t)F(x_0, u_0, y) + \mathrm{int}C. \tag{3.7}$$

Let $x(t) := tx' + (1-t)x_0$. Then it is clear that $x(t) \in K$. It is easy to see that there exists $t_0 \in]0,1[$ such that $x(t_0) \in x_0 + W_0$. It follows from (3.1) that $x(t_0) \notin Q(f_n)$. Then there exist $u_n \in S(x(t_0))$ and $y_n \in K$ such that

$$f_n(F(x(t_0), u_n, y_n)) \not\subseteq \mathbb{R}_+.$$

Thus, there exists $z_n \in F(x(t_0), u_n, y_n)$ such that

$$f_n(z_n) < 0. \tag{3.8}$$

Since $S(x(t_0))$ and K are compact, without loss of generality, we can assume that $u_n \to \bar{u} \in S(x(t_0))$ and $y_n \to y_0 \in K$. By Lemma 2.7, there exist $z_0 \in F(x(t_0), \bar{u}, y_0)$ and a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to z_0$. Without loss of generality, we can assume that $z_n \to z_0$. It follows that $f_n(z_n) \to f_0(z_0)$. By (3.8), we have

$$f_0(z_0) \leq 0. \tag{3.9}$$

On the other hand, from (3.4), (3.5), (3.6) and (3.7), we know that $f_0(z_0) > 0$, which contradicts (3.9). This completes the proof. □

Lemma 3.2. *Assume that, for each $x \in K$, $F(x,\cdot,\cdot)$ is C-convexlike on $S(x) \times K$. Then*

$$W(F,S,K) = \bigcup_{f \in B^*} Q(f).$$

Proof. For any $x \in \bigcup_{f \in B^*} Q(f)$, there exists $f_0 \in B^*$ such that $x \in Q(f_0)$. Thus,

$$f_0(F(x,u,y)) \subseteq \mathbb{R}_+, \quad \forall u \in S(x), \ \forall y \in K. \tag{3.10}$$

Suppose that $x \notin W(F,S,K)$. Then there exist $u_0 \in S(x)$ and $y_0 \in K$ such that

$$F(x,u_0,y_0) \cap (-\mathrm{int}C) \neq \emptyset,$$

and so there exists $z_0 \in F(x,u_0,y_0)$ such that $f_0(z_0) < 0$, which contradicts (3.10). Therefore, we know that $x \in W(F,S,K)$. Next, we show that

$$W(F,S,K) \subseteq \bigcup_{f \in B^*} Q(f).$$

Let $x \in W(F,S,K)$. Then

$$F(x,u,y) \cap (-\mathrm{int}C) = \emptyset, \quad \forall u \in S(x), \ \forall y \in K.$$

It is easy to see that

$$(F(x,S(x),K) + C) \cap (-\mathrm{int}C) = \emptyset.$$

For each $x \in K$, since $F(x,\cdot,\cdot)$ is C-convexlike on $S(x) \times K$, we can see that $F(x,S(x),K) + C$ is a convex set. By the separation theorem of convex sets, there exists $g \in Y^* \setminus \{0\}$ such that

$$\inf\{g(z+c) : u \in S(x), y \in K, z \in F(x,u,y), c \in C\} \geq \sup\{g(c') : c' \in -C\}.$$

It follows that $g \in C^*$ and

$$g(F(x,u,y)) \subseteq \mathbb{R}_+, \quad \forall u \in S(x), \ \forall y \in K.$$

Since $e \in \mathrm{int}C$ and $g \in C^* \setminus \{0\}$, it follows that $g(e) > 0$. Let $\psi = \frac{g}{g(e)}$. We can see that $\psi \in B^*$ and

$$\psi(F(x,u,y)) \subseteq \mathbb{R}_+, \quad \forall u \in S(x), \ \forall y \in K.$$

Thus, $x \in Q(\psi)$ and so $x \in \bigcup_{f \in B^*} Q(f)$. This completes the proof. □

Lemma 3.3. *Let K be a nonempty compact convex subset of X. Assume that*

(i) *$S(\cdot)$ is l.s.c. and P-concave on K with nonempty compact values;*

(ii) *for any $(x,y) \in K \times K$, $F(x,\cdot,y)$ is P-C-increasing;*

(iii) *for any $y \in K$, $F(\cdot,\cdot,y)$ is strictly C-concave on $K \times \Delta$;*

(iv) *$F(\cdot,\cdot,\cdot)$ is continuous on $K \times \Delta \times K$ with nonempty compact values;*

(v) *for each $x \in K$, $F(x,\cdot,\cdot)$ is C-convexlike on $S(x) \times K$.*

Then

$$\bigcup_{f \in B^{\#}} Q(f) \subseteq E(F,S,K) \subseteq W(F,S,K) = \bigcup_{f \in B^*} Q(f) \subseteq \mathrm{cl}\left(\bigcup_{f \in B^{\#}} Q(f)\right).$$

Proof. It follows from Lemma 3.2 and the definitions that

$$\bigcup_{f\in B^{\#}} Q(f) \subseteq E(F,S,K) \subseteq W(F,S,K) = \bigcup_{f\in B^{*}} Q(f).$$

By Lemma 3.1, we know that $Q(\cdot)$ is l.s.c. on $B^* = \mathrm{cl}\left(B^{\#}\right)$, by Lemma 2.8, one has

$$\bigcup_{f\in B^{*}} Q(f) \subseteq \mathrm{cl}\left(\bigcup_{f\in B^{\#}} Q(f)\right),$$

and so

$$\bigcup_{f\in B^{\#}} Q(f) \subseteq E(\Omega,\Gamma) \subseteq W(\Omega,\Gamma) = \bigcup_{f\in B^{*}} Q(f) \subseteq \mathrm{cl}\left(\bigcup_{f\in B^{\#}} Q(f)\right).$$

This completes the proof. □

Theorem 3.4. *Let $(\alpha_0,\lambda_0) \in W\times\Lambda$. Assume that*

(i) *$K(\lambda_0)$ is nonempty convex compact and $K(\cdot)$ is continuous at λ_0;*

(ii) *$S(\cdot)$ is continuous and P-concave on $K(\lambda_0)$ with nonempty compact values;*

(iii) *for any $(x,y)\in K(\lambda_0)\times K(\lambda_0)$, $F(x,\cdot,y,\alpha_0)$ is P-C-increasing;*

(iv) *for any $y\in K(\lambda_0)$, $F(\cdot,\cdot,y,\alpha_0)$ is strictly C-concave on $K(\lambda_0)\times\Delta$;*

(v) *$F(\cdot,\cdot,\cdot,\cdot)$ is continuous on $K(\lambda_0)\times\Delta\times K(\lambda_0)\times\{\alpha_0\}$ with nonempty compact values.*

Then $M(\cdot,\cdot)$ is l.s.c. at (α_0,λ_0).

Proof. Suppose to the contrary that $M(\cdot,\cdot)$ is not l.s.c. at (α_0,λ_0). Then there exist $x_0\in M(\alpha_0,\lambda_0)$ and a neighborhood W_0 of $0\in X$ such that, for any neighborhood $U'\times V'$ of (α_0,λ_0), there exists $(\alpha',\lambda')\in U'\times V'$ satisfying

$$(x_0+W_0)\cap M\left(\alpha',\lambda'\right) = \emptyset.$$

Hence, there exists a sequence $\{(\alpha_n,\lambda_n)\}$ with $(\alpha_n,\lambda_n)\to(\alpha_0,\lambda_0)$ such that

$$(x_0+W_0)\cap M(\alpha_n,\lambda_n) = \emptyset,\quad \forall n\in\mathbb{N}. \tag{3.11}$$

There are two cases to be considered.
Case 1. $M(\alpha_0,\lambda_0)$ is singleton. Let

$$x_n\in M(\alpha_n,\lambda_n),\quad \forall n\in\mathbb{N}. \tag{3.12}$$

It is clear that $x_n\in K(\lambda_n)$ for all $n\in\mathbb{N}$. By Lemma 2.7, there exist $\bar{x}\in K(\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to\bar{x}$. Without loss of generality, we can assume that $x_n\to\bar{x}$. We claim that $\bar{x}\in M(\alpha_0,\lambda_0)$. In fact, suppose to the contrary that $\bar{x}\notin M(\alpha_0,\lambda_0)$. Then there exist $u_0\in S(\bar{x})$ and $y_0\in K(\lambda_0)$ such that

$$F(\bar{x},u_0,y_0,\alpha_0)\not\subset C.$$

It follows that there exists $z_0\in F(\bar{x},u_0,y_0,\alpha_0)$ such that

$$z_0\notin C. \tag{3.13}$$

Since $S(\cdot)$ is l.s.c. at $\bar{x}$ and $K(\cdot)$ is l.s.c. at λ_0, it follows from Lemma 2.6 that there exists $u_n\in S(x_n)$ such that $u_n\to u_0$ and there exists $y_n\in K(\lambda_n)$ such that $y_n\to y_0$. By noting that $F(\cdot,\cdot,\cdot,\cdot)$ is l.s.c.

at $(\bar{x}, u_0, y_0, \alpha_0)$, by Lemma 2.6, there exists $z_n \in F(x_n, u_n, y_n, \alpha_n)$ such that $z_n \to z_0$. It follows from (3.13) that $z_n \notin C$ for n large enough, which contradicts (3.12). Therefore, $\bar{x} \in M(\alpha_0, \lambda_0)$. It follows from $M(\alpha_0, \lambda_0)$ is singleton that $\bar{x} = x_0$ and so $x_n \to x_0$. By this together with (3.12), we have

$$x_n \in (x_0 + W_0) \cap M(\alpha_n, \lambda_n),$$

for n large enough, which contradicts (3.11).

Case 2. $M(\alpha_0, \lambda_0)$ is not singleton. Then there exists $x' \in M(\alpha_0, \lambda_0)$ such that $x' \neq x_0$. Since $x', x_0 \in M(\alpha_0, \lambda_0)$, one has

$$F\left(x', u, y, \alpha_0\right) \subseteq C, \quad \forall u \in S\left(x'\right), \quad \forall y \in K(\lambda_0), \tag{3.14}$$

and

$$F(x_0, u, y, \alpha_0) \subseteq C, \quad \forall u \in S(x_0), \quad \forall y \in K(\lambda_0). \tag{3.15}$$

Since $S(\cdot)$ is P-concave on $K(\lambda_0)$, for any $t \in]0,1[$, we have

$$S\left(tx' + (1-t)x_0\right) \subseteq tS\left(x'\right) + (1-t)S(x_0) + P.$$

For any $u_t \in S(tx' + (1-t)x_0)$, there exist $u' \in S(x')$, $u_0 \in S(x_0)$ and $p_0 \in P$ such that

$$u_t = tu' + (1-t)u_0 + p_0.$$

By noting that $F(tx' + (1-t)x_0, \cdot, y, \alpha_0)$ is P-C-increasing, we have

$$F\left(tx' + (1-t)x_0, u_t, y, \alpha_0\right) \subseteq F\left(tx' + (1-t)x_0, tu' + (1-t)u_0, y, \alpha_0\right) + C. \tag{3.16}$$

Since $F(\cdot, \cdot, y, \alpha_0)$ is strictly C-concave on $K(\lambda_0) \times \Delta$, we have

$$F\left(tx' + (1-t)x_0, tu' + (1-t)u_0, y, \alpha_0\right) \subseteq tF\left(x', u', y, \alpha_0\right) + (1-t)F(x_0, u_0, y, \alpha_0) + \mathrm{int}C. \tag{3.17}$$

Let $x(t) := tx' + (1-t)x_0$. Then it is clear that $x(t) \in K(\lambda_0)$. For the above W_0, there exists a neighborhood W_1 of $0 \in X$ such that

$$W_1 + W_1 \subseteq W_0.$$

Obviously, there exists $t_0 \in]0,1[$ such that $x(t_0) \in x_0 + W_1$. Thus,

$$x(t_0) + W_1 \subseteq x_0 + W_1 + W_1 \subseteq x_0 + W_0. \tag{3.18}$$

Since $x(t_0) \in K(\lambda_0)$, by Lemma 2.6, there exists $x'_n \in K(\lambda_n)$ such that $x'_n \to x(t_0)$ and so $x'_n \in x(t_0) + W_1$ for n large enough. By noting (3.11) and (3.18), we have $x'_n \notin M(u_n, \lambda_n)$ and so there exist $y'_n \in K(\lambda_n)$ and $u'_n \in S(x'_n)$ such that

$$F\left(x'_n, u'_n, y'_n, \alpha_n\right) \not\subseteq C.$$

Thus, there exists $z'_n \in F(x'_n, u'_n, y'_n, \alpha_n)$ satisfying

$$z'_n \notin C. \tag{3.19}$$

Since $y'_n \in K(\lambda_n)$, it follows from Lemma 2.7 that there exist $y' \in K(\lambda_0)$ and a subsequence $\left\{y'_{n_k}\right\}$ of $\{y'_n\}$ such that $y'_{n_k} \to y'$. Without loss of generality, we can assume that $y'_n \to y'$. Since $u'_n \in S(x'_n)$, it follows from Lemma 2.7 that there exist $u' \in S(x(t_0))$ and a subsequence $\left\{u'_{n_k}\right\}$ of $\{u'_n\}$ such that $u'_{n_k} \to u'$. Without loss of generality, we can assume that $u'_n \to u'$. By noting the fact that $F(\cdot, \cdot, \cdot, \cdot)$ is u.s.c. at $(x(t_0), u', y', \alpha_0)$, there exist $z' \in F(x(t_0), u', y', \alpha_0)$ and a subsequence $\left\{z'_{n_k}\right\}$ of $\{z'_n\}$ such that $z'_{n_k} \to z'$. Without loss of generality, we can assume that $z'_n \to z'$. It follows from (3.19) that

$$z' \notin \mathrm{int}C. \tag{3.20}$$

On the other hand, from (3.14), (3.15), (3.16) and (3.17), we know that $z' \in \mathrm{int}C$, which contradicts (3.20). This completes the proof. □

Similar to the proof of Theorem 3.4, we can get the following lemma.

Lemma 3.5. *Let $f \in C^* \setminus \{0\}$ and $(\alpha_0, \lambda_0) \in W \times \Lambda$. Assume that*

(i) *$K(\lambda_0)$ is nonempty convex compact and $K(\cdot)$ is continuous at λ_0;*

(ii) *$S(\cdot)$ is continuous and P-concave on $K(\lambda_0)$ with nonempty compact values;*

(iii) *for any $(x, y) \in K(\lambda_0) \times K(\lambda_0)$, $F(x, \cdot, y, \alpha_0)$ is P-C-increasing;*

(iv) *for any $y \in K(\lambda_0)$, $F(\cdot, \cdot, y, \alpha_0)$ is strictly C-concave on $K(\lambda_0) \times \Delta$;*

(v) *$F(\cdot, \cdot, \cdot, \cdot)$ is continuous on $K(\lambda_0) \times \Delta \times K(\lambda_0) \times \{\alpha_0\}$ with nonempty compact values.*

Then $S_f(\cdot, \cdot)$ is l.s.c. at (α_0, λ_0).

Theorem 3.6. *Let $(\alpha_0, \lambda_0) \in W \times \Lambda$. Assume that*

(i) *$K(\lambda_0)$ is nonempty convex compact and $K(\cdot)$ is continuous at λ_0;*

(ii) *$S(\cdot)$ is continuous and P-concave on $K(\lambda_0)$ with nonempty compact values;*

(iii) *for any $(x, y) \in K(\lambda_0) \times K(\lambda_0)$, $F(x, \cdot, y, \alpha_0)$ is P-C-increasing;*

(iv) *for any $y \in K(\lambda_0)$, $F(\cdot, \cdot, y, \alpha_0)$ is strictly C-concave on $K(\lambda_0) \times \Delta$;*

(v) *$F(\cdot, \cdot, \cdot, \cdot)$ is continuous on $K(\lambda_0) \times \Delta \times K(\lambda_0) \times \{\alpha_0\}$ with nonempty compact values;*

(vi) *for any $x \in K(\lambda_0)$, $F(x, \cdot, \cdot, \alpha_0)$ is C-convexlike on $S(x) \times K(\lambda_0)$.*

Then $W(\cdot, \cdot)$ is l.s.c. at (α_0, λ_0). Moreover, $E(\cdot, \cdot)$ is l.s.c. at (α_0, λ_0).

Proof. It follows from Lemma 3.2 that

$$W(\alpha_0, \lambda_0) = \bigcup_{f \in B^*} S_f(\alpha_0, \lambda_0).$$

For any $x_0 \in W(\alpha_0, \lambda_0)$ and any neighborhood U of x_0, there exists $f_0 \in C^*$ such that $x_0 \in S_{f_0}(\alpha_0, \lambda_0)$. It follows from Lemma 3.5 that $S_{f_0}(\cdot, \cdot)$ is l.s.c. at (α_0, λ_0) and so there exists a neighborhood $U(\alpha_0) \times U(\lambda_0)$ of (α_0, λ_0) such that

$$U \cap S_{f_0}(\alpha, \lambda) \neq \emptyset, \ \ \forall (\alpha, \lambda) \in U(\alpha_0) \times U(\lambda_0).$$

It is easy to see that

$$S_{f_0}(\alpha, \lambda) \subseteq W(\alpha, \lambda),$$

and so

$$U \cap W(\alpha, \lambda) \neq \emptyset, \ \ \forall (\alpha, \lambda) \in U(\alpha_0) \times U(\lambda_0).$$

Therefore, $W(\cdot, \cdot)$ is l.s.c. at (α_0, λ_0). It follows from Lemma 3.3 that

$$\bigcup_{f \in B^\#} S_f(\alpha_0, \lambda_0) \subseteq E(\alpha_0, \lambda_0) \subseteq W(\alpha_0, \lambda_0) = \bigcup_{f \in B^*} S_f(\alpha_0, \lambda_0) \subseteq \mathrm{cl}\left(\bigcup_{f \in B^\#} S_f(\alpha_0, \lambda_0) \right).$$

For any $x \in E(\alpha_0, \lambda_0)$ and any open neighborhood V of x, since

$$x \in E(\alpha_0, \lambda_0) \subseteq \mathrm{cl}\left(\bigcup_{f \in B^\#} S_f(\alpha_0, \lambda_0) \right),$$

we have

$$V \cap \left(\bigcup_{f \in B^{\#}} S_f (\alpha_0, \lambda_0) \right) \neq \emptyset.$$

Then there exists $f \in B^{\#}$ such that

$$V \cap S_f (\alpha_0, \lambda_0) \neq \emptyset.$$

By Lemma 3.5, $S_f (\cdot, \cdot)$ is l.s.c. at (α_0, λ_0). Thus, there exists a neighborhood $U (\alpha_0) \times U (\lambda_0)$ of (α_0, λ_0) such that

$$V \cap S_f (\alpha, \lambda) \neq \emptyset, \ \forall (\alpha, \lambda) \in U (\alpha_0) \times U (\lambda_0).$$

Since $f \in B^{\#}$, it is clear that

$$S_f (\alpha, \lambda) \subseteq E (\alpha, \lambda).$$

Then,

$$V \cap E (\alpha, \lambda) \neq \emptyset, \ \forall (\alpha, \lambda) \in U (\alpha_0) \times U (\lambda_0).$$

Therefore, $E (\cdot, \cdot)$ is l.s.c. at (α_0, λ_0). This completes the proof. □

4. Upper semicontinuity

In this section, we establish the upper semicontinuity of strong efficient solution mapping and weakly efficient solution mapping to (PGVEP) and the Hausdorff upper semicontinuity of efficient solution mapping to (PGVEP).

Theorem 4.1. *Let $(\alpha_0, \lambda_0) \in W \times \Lambda$. Assume that $K (\lambda_0)$ is nonempty compact, $K (\cdot)$ is continuous at λ_0, $S (\cdot)$ is l.s.c. on $K (\lambda_0)$ and $F (\cdot, \cdot, \cdot, \cdot)$ is l.s.c. on $K (\lambda_0) \times \Delta \times K (\lambda_0) \times \{\alpha_0\}$. Then $M (\cdot, \cdot)$ is u.s.c. at (α_0, λ_0). Moreover, $W (\cdot, \cdot)$ is u.s.c. at (α_0, λ_0).*

Proof. Suppose to the contrary that $M (\cdot, \cdot)$ is u.s.c. at (α_0, λ_0). Then there exist a neighborhood W_0 of $M (\alpha_0, \lambda_0)$ and a sequence $\{(\alpha_n, \lambda_n)\}$ with $(\alpha_n, \lambda_n) \to (\alpha_0, \lambda_0)$ such that

$$M (\alpha_n, \lambda_n) \not\subset W_0.$$

Then there exists

$$x_n \in M (\alpha_n, \lambda_n), \tag{4.1}$$

such that

$$x_n \notin W_0, \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Since $x_n \in K (\lambda_n)$, by Lemma 2.7, there exist $x_0 \in K (\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. Without loss of generality, we can assume that $x_n \to x_0$.

We claim that $x_0 \in M (\alpha_0, \lambda_0)$. In fact, suppose to the contrary that $x_0 \notin M (\alpha_0, \lambda_0)$. Then there exist $u_0 \in S (x_0)$ and $y_0 \in K (\lambda_0)$ such that

$$F (x_0, u_0, y_0, \alpha_0) \not\subset C.$$

Then, there exists $z_0 \in F (x_0, u_0, y_0, \alpha_0)$ such that

$$z_0 \notin C. \tag{4.3}$$

Since $S (\cdot)$ is l.s.c. at x_0 and $K (\cdot)$ is l.s.c. at λ_0, it follows from Lemma 2.6 that there exists $u_n \in S (x_n)$ such that $u_n \to u_0$ and there exists $y_n \in K (\lambda_n)$ such that $y_n \to y_0$. By noting that $F (\cdot, \cdot, \cdot, \cdot)$ is l.s.c. at $(x_0, u_0, y_0, \alpha_0)$, by Lemma 2.6, there exists $z_n \in F (x_n, u_n, y_n, \alpha_n)$ such that $z_n \to z_0$. It follows from (4.3) that $z_n \notin C$ for n large enough, which contradicts (4.1). Therefore, $x_0 \in M (\alpha_0, \lambda_0)$. We can see that $x_n \to x_0 \in W_0$, which contradicts (4.2).

By the similar arguments, we can prove that $W (\cdot, \cdot)$ is u.s.c. at (α_0, λ_0). This completes the proof. □

Lemma 4.2. *Assume that K is a nonempty closed subset of X, $S(\cdot)$ is l.s.c. on K and for any $y \in K$, $F(\cdot,\cdot,y)$ is l.s.c. on $K \times \Delta$. Then $Q(f)$ is closed.*

Proof. Let $\{x_n\} \subseteq Q(f)$ with $x_n \to x_0$. Then

$$f(F(x_n,u,y)) \subseteq \mathbb{R}_+, \ \ \forall u \in S(x_n), \ \forall \in K. \tag{4.4}$$

It follows from the closedness of K that $x_0 \in K$. For any $\bar{u} \in S(x_0)$, since $S(\cdot)$ is l.s.c. at x_0, by Lemma 2.6, there exists $u_n \in S(x_n)$ such that $u_n \to \bar{u}$. For any $z \in F(x_0,\bar{u},y)$, by noting that $F(\cdot,\cdot,y)$ is l.s.c. at $(x_0,\bar{u})$, by Lemma 2.6, there exists $z_n \in F(x_n,u_n,y)$ such that $z_n \to z$. By (4.4), we have $f(z_n) \geq 0$. It follows from $f(z_n) \to f(z)$ that $f(z) \geq 0$. Then

$$f(F(x_0,\bar{u},y)) \subseteq \mathbb{R}_+, \ \ \forall \bar{u} \in S(x_0), \ \forall y \in K,$$

which means that $x_0 \in Q(f)$. Therefore, $Q(f)$ is closed. This completes the proof. □

Lemma 4.3. *Let $(f_0,\alpha_0,\lambda_0) \in B^* \times W \times \Lambda$. Assume that $K(\lambda_0)$ is nonempty compact, $K(\cdot)$ is continuous at λ_0, $S(\cdot)$ is l.s.c. on $K(\lambda_0)$ and $F(\cdot,\cdot,\cdot,\cdot)$ is l.s.c. on $K(\lambda_0) \times \Delta \times K(\lambda_0) \times \{\alpha_0\}$. Then $S_{.}(\cdot,\cdot)$ is u.s.c. at (f_0,α_0,λ_0), where the topology on B^* is the weak* topology.*

Proof. Suppose to the contrary that $S_{.}(\cdot,\cdot)$ is u.s.c. at (f_0,α_0,λ_0). Then there exist a neighborhood W_0 of $S_{f_0}(\alpha_0,\lambda_0)$ and a sequence $\{(f_n,\alpha_n,\lambda_n)\}$ with $(f_n,\alpha_n,\lambda_n) \to (f_0,\alpha_0,\lambda_0)$ such that

$$S_{f_n}(\alpha_n,\lambda_n) \not\subset W_0.$$

Then there exists

$$x_n \in S_{f_n}(\alpha_n,\lambda_n), \tag{4.5}$$

such that

$$x_n \notin W_0, \quad \forall n \in \mathbb{N}. \tag{4.6}$$

Since $x_n \in K(\lambda_n)$, by Lemma 2.7, there exist $x_0 \in K(\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. Without loss of generality, we can assume that $x_n \to x_0$.

We claim that $x_0 \in S_{f_0}(\alpha_0,\lambda_0)$. In fact, suppose to the contrary that $x_0 \notin S_{f_0}(\alpha_0,\lambda_0)$. Then there exist $u_0 \in S(x_0)$ and $y_0 \in K(\lambda_0)$ such that

$$f_0(F(x_0,u_0,y_0,\alpha_0)) \not\subset \mathbb{R}_+.$$

Then, there exists $z_0 \in F(x_0,u_0,y_0,\alpha_0)$ such that

$$f_0(z_0) < 0. \tag{4.7}$$

Since $S(\cdot)$ is l.s.c. at x_0 and $K(\cdot)$ is l.s.c. at λ_0, it follows from Lemma 2.6 that there exists $u_n \in S(x_n)$ such that $u_n \to u_0$ and there exists $y_n \in K(\lambda_n)$ such that $y_n \to y_0$. By noting that $F(\cdot,\cdot,\cdot,\cdot)$ is l.s.c. at (x_0,u_0,y_0,α_0), by Lemma 2.6, there exists $z_n \in F(x_n,u_n,y_n,\alpha_n)$ such that $z_n \to z_0$. By noting the fact that

$$f_n \xrightarrow{w^*} f_0,$$

it is easy to see that $f_n(z_n) \to f_0(z_0)$. By this together with (4.7), we have $f_n(z_n) < 0$ for n large enough, which contradicts (4.5). Therefore, $x_0 \in S_{f_0}(\alpha_0,\lambda_0)$. We can see that $x_n \to x_0 \in W_0$, which contradicts (4.6). This completes the proof. □

Theorem 4.4. *Let $(\alpha_0,\lambda_0) \in W \times \Lambda$. Assume that*

(i) *$K(\lambda_0)$ is nonempty convex compact and $K(\cdot)$ is continuous at λ_0;*

(ii) *$S(\cdot)$ is l.s.c. and P-concave on $K(\lambda_0)$ with nonempty compact values;*

(iii) *for any* $(x, y) \in K(\lambda_0) \times K(\lambda_0)$, $F(x, \cdot, y, \alpha_0)$ *is P-C-increasing;*

(iv) *for any* $y \in K(\lambda_0)$, $F(\cdot, \cdot, y, \alpha_0)$ *is strictly C-concave on* $K(\lambda_0) \times \Delta$*;*

(v) $F(\cdot, \cdot, \cdot, \cdot)$ *is continuous on* $K(\lambda_0) \times \Delta \times K(\lambda_0) \times \{\alpha_0\}$ *with nonempty compact values;*

(vi) *for any* $(\alpha, \lambda) \in W \times \Lambda$ *and for any* $x \in K(\lambda)$, $F(x, \cdot, \cdot, \alpha)$ *is C-convexlike on* $S(x) \times K(\lambda)$.

Then, $E(\cdot, \cdot)$ *is H-u.s.c. at* (α_0, λ_0).

Proof. Suppose to the contrary that $E(\cdot, \cdot)$ is not H-u.s.c. at (α_0, λ_0). Then there exist a neighborhood W_0 of $0 \in X$ and a sequence $\{(\alpha_n, \lambda_n)\}$ with $(\alpha_n, \lambda_n) \to (\alpha_0, \lambda_0)$ such that

$$E(\alpha_n, \lambda_n) \not\subset E(\alpha_0, \lambda_0) + W_0, \quad \forall n \in \mathbb{N}.$$

Thus, there exists

$$x_n \in E(\alpha_n, \lambda_n), \tag{4.8}$$

satisfying

$$x_n \notin E(\alpha_0, \lambda_0) + W_0, \quad \forall n \in \mathbb{N}. \tag{4.9}$$

From Lemma 3.2, one has

$$W(\alpha_n, \lambda_n) = \bigcup_{f \in B^*} S_f(\alpha_n, \lambda_n).$$

It is clear that

$$E(\alpha_n, \lambda_n) \subseteq W(\alpha_n, \lambda_n), \quad \forall n \in \mathbb{N}.$$

This together with (4.8) implies that

$$x_n \in \bigcup_{f \in B^*} S_f(\alpha_n, \lambda_n), \quad \forall n \in \mathbb{N},$$

and so there exists $f_n \in B^*$ such that

$$x_n \in S_{f_n}(\alpha_n, \lambda_n). \tag{4.10}$$

Since B^* is weak* compact, without loss of generality, we can assume that

$$f_n \xrightarrow{w^*} f_0 \in B^*.$$

It follows from Lemma 4.2 that $S_{f_0}(\alpha_0, \lambda_0)$ is closed. Since $S_{f_0}(\alpha_0, \lambda_0) \subseteq K(\lambda_0)$ and $K(\lambda_0)$ is compact, we can see that $S_{f_0}(\alpha_0, \lambda_0)$ is compact. By Lemma 4.3, we can see that $S_{\cdot}(\cdot, \cdot)$ is u.s.c. at $(f_0, \alpha_0, \lambda_0)$. By noting (4.10) and Lemma 2.7, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_0 \in S_{f_0}(\alpha_0, \lambda_0)$ such that $x_{n_k} \to x_0$. It follows from Lemma 3.3 that

$$\bigcup_{f \in B^\#} S_f(\alpha_0, \lambda_0) \subseteq E(\alpha_0, \lambda_0) \subseteq W(\alpha_0, \lambda_0) = \bigcup_{f \in B^*} S_f(\alpha_0, \lambda_0) \subseteq \mathrm{cl}\left(\bigcup_{f \in B^\#} S_f(\alpha_0, \lambda_0)\right).$$

Thus, one has

$$x_0 \in \bigcup_{f \in B^*} S_f(\alpha_0, \lambda_0) \subseteq \mathrm{cl}\left(\bigcup_{f \in B^\#} S_f(\alpha_0, \lambda_0)\right) = \mathrm{cl}\left(E(\alpha_0, \lambda_0)\right) \subseteq E(\alpha_0, \lambda_0) + W_0.$$

This together with $x_{n_k} \to x_0$ shows that

$$x_{n_k} \in E(\alpha_0, \lambda_0) + W_0,$$

for k large enough, which contradicts (4.9). This completes the proof. □

Remark 4.5. Theorem 4.4 is a generalization of Theorem 5.4 of [24] from the finite dimensional space to the infinite dimensional space.

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