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Variational principle for a three-point boundary value problem

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Abstract

A variational principle is established for a three-point boundary value problem. The stationary condition includes not only the governing equation but also the natural boundary conditions. The paper reveals that not every boundary condition adopts a variational formulation, and the existence and uniqueness of the solutions of a three-point boundary value problem can be revealed by its variational formulation.

Keywords: Variational theory, boundary value problem, semi-inverse method, natural boundary condition.
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1. Introduction

Two-point boundary value problems are widely studied and applied in various fields, while multiple-point problems are relatively difficult to be solved either numerically or analytically. For example, a differential equation describing a truss bridge requires multiple boundary conditions. An unsuitable boundary condition might make the problem ill-posed, though the solution does exist, and this is the reason that existence of solution was widely studied for three-point boundary problems [1, 10]. It becomes an important issue in multiple point problems to incorporate a suitable boundary condition into the governing equations. In this

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paper, we study the following nonlinear three-point boundary value problems for a second-order ordinary differential equation

$$y'' + p(x)y' + q(x)y + r(x)y^2 + f(x) = 0, \quad x \in [x_1, x_3], \tag{1.1}$$

with the following general boundary conditions:

$$\sum_{i=1}^{3} k_i y(x_i) + \sum_{i=1}^{3} h_i y'(x_i) = \alpha, \tag{1.2}$$

$$\sum_{i=1}^{3} m_i y(x_i) + \sum_{i=1}^{3} n_i y'(x_i) = \beta, \quad x_2 \in [x_1, x_3], \tag{1.3}$$

where $p(x)$, $q(x)$, $f(x)$ and $r(x)$ are known functions, k_i, h_i, m_i, n_i, α and β are constants. When $r(x) = 0$, Eq. (1.1) is a linear one.

There are some effective approaches to three-point boundary value problems for a second-order ordinary differential equation [2, 9, 11]. In this paper, a variational formulation is to be established via the semi-inverse method [3, 4], and how to suitably incorporate boundary conditions will be discussed.

2. Variational formulation

In case $p(x) = 0$, without considering the boundary conditions, we have the following variational principle:

$$J(y) = \int_a^b \{\frac{1}{2}y'^2 - \frac{1}{2}q(x)y^2 - f(x)y - \frac{1}{3}r(x)y^3\}dx. \tag{2.1}$$

In order to establish a variational formulation for Eq. (1.1) , according to the semi-inverse method [3], we begin with the following trial-functional:

$$J(y) = \int_a^b \{\frac{1}{2}\sigma(x)y'^2 + F(x, y)\}dx,$$

where $\sigma(x)$ is an unknown function of x to be further determined, and F is an unknown function of x and y and/or derivatives of y. There are alternative approaches to construct the trial-functionals, illustrating examples are available in the review article [5]. The semi-inverse method becomes an effective method for establishment of variational formulation directly from governing equations [6–8].

The Euler-Lagrange equation of Eq. (2.1) reads

$$\frac{d}{dx}[\sigma(x)y'] - \frac{\partial F}{\partial y} = 0,$$

or

$$\sigma(x)y'' + \sigma'(x)y' - \frac{\partial F}{\partial y} = 0. \tag{2.2}$$

Eq. (1.1) can be equivalently written as

$$\sigma(x)y'' + \sigma(x)p(x)y' + \sigma(x)q(x)y + \sigma(x)r(x)y^2 + \sigma(x)f(x) = 0. \tag{2.3}$$

Comparison of Eq. (2.2) with Eq. (2.3) results in

$$\sigma'(x) = \sigma(x)p(x),$$

$$\frac{\partial F}{\partial y} = -\sigma(x)q(x)y - \sigma(x)r(x)y^2 - \sigma(x)f(x).$$

We can identify $\sigma(x)$ and $F(y)$ as follows

$$\sigma(x) = exp\{\int p(x)dx\}, \tag{2.4}$$

$$F(x,y) = -\frac{1}{2}\sigma(x)q(x)y^2 - \frac{1}{3}\sigma(x)r(x)y^3 - \sigma(x)f(x)y.$$

We, therefore, obtain the following variational formulation for Eq. (1.1):

$$J(y) = \int_a^b \{\frac{1}{2}\sigma(x)y'^2 - \frac{1}{2}\sigma(x)q(x)y^2 - \frac{1}{3}\sigma(x)r(x)y^3 - \sigma(x)f(x)y\}dx, \tag{2.5}$$

where $\sigma(x)$ is called integral factor, and it is defined in Eq. (2.4).

3. Boundary conditions

In order to incorporate the boundary conditions into the variational formulation, we consider the case $r(x) = 0$.

We write Eq. (2.5) in the form

$$\begin{aligned} J(y) = &\int_{x_1}^{x_2} \{\frac{1}{2}\sigma(x)y'^2 - \frac{1}{2}\sigma(x)q(x)y^2 - \sigma(x)f(x)y\}dx \\ &+ \int_{x_2}^{x_3} \{\frac{1}{2}\sigma(x)y'^2 - \frac{1}{2}\sigma(x)q(x)y^2 - \sigma(x)f(x)y\}dx \\ &+ \sum_{i=1}^{3} F_i(y(x_1), y(x_2), y(x_3), y'(x_1), y'(x_2), y'(x_3)), \end{aligned} \tag{3.1}$$

where $F_i (i = 1, 2, 3)$ are introduced to match the boundary conditions of Eq. (1.2) and Eq. (1.3). The variation of Eq. (3.1) reads

$$\begin{aligned} \delta J(y) = &\int_{x_1}^{x_2} \{\sigma(x)y'\delta y' - \sigma(x)q(x)y\delta y - \sigma(x)f(x)\delta y\}dx \\ &+ \int_{x_2}^{x_3} \{\sigma(x)y'\delta y' - \sigma(x)q(x)y\delta y - \sigma(x)f(x)\delta y\}dx \\ &+ \sum_{i=1}^{3} \delta F_i(y(x_1), y(x_2), y(x_3), y'(x_1), y'(x_2), y'(x_3)), \end{aligned}$$

or

$$\begin{aligned} \delta J(y) = &\int_{x_1}^{x_2} \{\sigma(x)y'\frac{d}{dx}(\delta y) - \sigma(x)q(x)y\delta y - \sigma(x)f(x)\delta y\}dx \\ &+ \int_{x_2}^{x_3} \{\sigma(x)y'\frac{d}{dx}(\delta y) - \sigma(x)q(x)y\delta y - \sigma(x)f(x)\delta y\}dx \\ &+ \sum_{i=1}^{3} \delta F_i(y(x_1), y(x_2), y(x_3), y'(x_1), y'(x_2), y'(x_3)). \end{aligned}$$

Integration by parts results in

$$\begin{aligned} \delta J(y) = &\int_{x_1}^{x_2} \{-\delta y\frac{d}{dx}(\sigma(x)y') - \sigma(x)q(x)y\delta y - \sigma(x)f(x)\delta y\}dx \\ &+ \int_{x_2}^{x_3} \{-\delta y\frac{d}{dx}(\sigma(x)y') - \sigma(x)q(x)y\delta y - \sigma(x)f(x)\delta y\}dx \end{aligned}$$

$$+\{\sigma(x)y'\delta y\}|_{x=x_1}^{x=x_2}+\{\sigma(x)y'\delta y\}|_{x=x_2}^{x=x_3}$$
$$+\sum_{i=1}^{3}\delta F_i(y(x_1),y(x_2),y(x_3),y'(x_1),y'(x_2),y'(x_3)).$$

By setting $\delta J(y)=0$, we can obtain the following natural boundary conditions:
On $x=x_1$:

$$\delta y(x_1):-\sigma(x_1)y'(x_1)+\frac{\partial F_1}{\partial y(x_1)}+\frac{\partial F_2}{\partial y(x_1)}+\frac{\partial F_3}{\partial y(x_1)}=0,$$

$$\delta y'(x_1):\frac{\partial F_1}{\partial y'(x_1)}+\frac{\partial F_2}{\partial y'(x_1)}+\frac{\partial F_3}{\partial y'(x_1)}=0.$$

On $x=x_2$:

$$\delta y(x_2):\frac{\partial F_1}{\partial y(x_2)}+\frac{\partial F_2}{\partial y(x_2)}+\frac{\partial F_3}{\partial y(x_2)}=0, \tag{3.2}$$

$$\delta y'(x_2):\frac{\partial F_1}{\partial y'(x_2)}+\frac{\partial F_2}{\partial y'(x_2)}+\frac{\partial F_3}{\partial y'(x_2)}=0. \tag{3.3}$$

On $x=x_3$:

$$\delta y(x_3):\sigma(x_3)y'(x_3)+\frac{\partial F_1}{\partial y(x_3)}+\frac{\partial F_2}{\partial y(x_3)}+\frac{\partial F_3}{\partial y(x_3)}=0,$$

$$\delta y'(x_3):\frac{\partial F_1}{\partial y'(x_3)}+\frac{\partial F_2}{\partial y'(x_3)}+\frac{\partial F_3}{\partial y'(x_3)}=0. \tag{3.4}$$

Incorporating the boundary conditions, Eqs. (1.2) and (1.3), we have

$$\frac{\partial F_1}{\partial y(x_1)}+\frac{\partial F_2}{\partial y(x_1)}+\frac{\partial F_3}{\partial y(x_1)}=\sigma(x_1)y'(x_1)=\frac{\sigma(x_1)}{h_1}[\alpha-\sum_{i=1}^{3}k_iy(x_i)-\sum_{i=2}^{3}h_iy'(x_i)], \tag{3.5}$$

$$\frac{\partial F_1}{\partial y(x_3)}+\frac{\partial F_2}{\partial y(x_3)}+\frac{\partial F_3}{\partial y(x_3)}=-\sigma(x_3)y'(x_3)=-\frac{\sigma(x_3)}{n_3}[\beta-\sum_{i=1}^{3}m_iy(x_i)+\sum_{i=1}^{2}n_iy'(x_i)]. \tag{3.6}$$

The integrability condition for identification of $F_i(i=1,2,3)$ requires

$$\frac{\partial}{\partial y(x_i)}\frac{\partial F_k}{\partial y(x_j)}=\frac{\partial}{\partial y(x_j)}\frac{\partial F_k}{\partial y(x_i)}, \tag{3.7}$$

$$\frac{\partial}{\partial y'(x_i)}\frac{\partial F_k}{\partial y'(x_j)}=\frac{\partial}{\partial y'(x_j)}\frac{\partial F_k}{\partial y'(x_i)}, \tag{3.8}$$

$$\frac{\partial}{\partial y(x_i)}\frac{\partial F_k}{\partial y'(x_j)}=\frac{\partial}{\partial y'(x_j)}\frac{\partial F_k}{\partial y(x_i)}, \tag{3.9}$$

where $i,j,k=1,2,3$. For example, for $j=1,i=3$ and $k=1$, from Eq. (3.7), it requires that

$$-\frac{\sigma(x_1)k_3}{h_1}=\frac{\sigma(x_3)m_1}{n_3}.$$

To show the identification of $F_i(i=1,2,3)$, we consider a simple case of the boundary conditions:

$$y'(x_1)=\alpha_1,$$
$$y'(x_3)+\lambda y'(x_2)=\beta_1. \tag{3.10}$$

Setting

$$\frac{\partial F_2}{\partial y(x_1)}=\frac{\partial F_3}{\partial y(x_1)}=0,$$

in Eq. (3.5), F_1 can be identified, which reads

$$F_1 = \sigma(x_1)\alpha_1 y(x_1).$$

By setting

$$\frac{\partial F_1}{\partial y(x_3)} = \frac{\partial F_2}{\partial y(x_3)} = 0,$$
$$\frac{\partial F_1}{\partial y'(x_3)} = \frac{\partial F_2}{\partial y'(x_3)} = 0,$$

in Eq. (3.6) and Eq. (3.4), respectively, and by using the boundary condition, Eq. (3.10), we have

$$\frac{\partial F_3}{\partial y(x_3)} = -\sigma(x_3)y'(x_3) = -\sigma(x_3)[\beta_1 - \lambda y'(x_2)].$$

We, therefore, can identify F_3 as follows

$$F_3 = -\sigma(x_3)[\beta_1 - \lambda y'(x_2)]y(x_3).$$

From Eqs.(3.2) and (3.3), and using the above identified results, we have

$$\frac{\partial F_2}{\partial y(x_2)} = 0,$$

$$\frac{\partial F_2}{\partial y'(x_2)} = -\frac{\partial F_3}{\partial y'(x_2)} = -\sigma(x_3)\lambda y(x_3).$$

We, therefore, identify F_2 as follows

$$F_2 = -\sigma(x_3)\lambda y(x_3)y'(x_2).$$

The variational formulation reads

$$J(y) = \int_{x_1}^{x_2} \{\frac{1}{2}\sigma(x)y'^2 - \frac{1}{2}\sigma(x)q(x)y^2 - \sigma(x)f(x)y\}dx$$
$$+ \sigma(x_1)\alpha_1 y(x_1) - \sigma(x_3)\lambda y(x_3)y'(x_2) - \sigma(x_3)[\beta_1 - \lambda y'(x_2)]y(x_3).$$

4. Discussion and conclusions

The integral factor, $\sigma(x)$, defined in Eq. (2.4), is of great importance, it affects the boundary conditions of the three-point boundary value problem. The identification of integral boundary in the variational formulation requires the integrability conditions, that is given in Eqs. (3.7)-(3.9). A variational formulation reveals the existence and uniqueness of the solutions of a three-point boundary value problem, and it also suggests some suitable boundary conditions for a practical problem. The present theory can be easily extended to multiple point boundary value problems.

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