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Essential norm of weighted composition operators from H^∞ to the Zygmund space

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Abstract

Let φ be an analytic self-map of the unit disk $\mathbb{D}$ and $u \in H(\mathbb{D})$, the space of analytic functions on $\mathbb{D}$. The weighted composition operator, denoted by uC_φ, is defined by $(uC_\varphi f)(z) = u(z)f(\varphi(z))$, $f \in H(\mathbb{D})$, $z \in \mathbb{D}$. In this paper, we give three different estimates for the essential norm of the operator uC_φ from H^∞ into the Zygmund space, denoted by $\mathcal{Z}$. In particular, we show that $\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \approx \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}}$.

Keywords: Zygmund space, essential norm, weighted composition operator.
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1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. Let H^∞ denote the bounded analytic function space, i.e.,

$$H^\infty = \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)| < \infty\}.$$

The Bloch space, denoted by $\mathcal{B}$, is the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

For more details of the Bloch space we refer the reader to [21].

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Let $\mathcal{Z}$ denote the set of all functions $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\| = \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $h > 0$. By Theorem 5.3 of [3], we see that $f \in \mathcal{Z}$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty$. $\mathcal{Z}$, called the Zygmund space, a Banach space with the norm defined by

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|.$$

See [1, 3, 6] for more details on the space $\mathcal{Z}$.

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

Let $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Let X, Y be Banach spaces and $\|\cdot\|_{X\to Y}$ denotes the operator norm. Recall that the essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators K mapping X into Y, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\|_{X\to Y} : K \ \text{ is a compact operator}\}.$$

It is well-known that $\|T\|_{e,X\to Y} = 0$ if and only if $T : X \to Y$ is compact.

The composition operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded for any φ by the Schwarz-Pick Lemma. Madigan and Matheson studied the compactness of the operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ in [11]. Montes-Rodrieguez [12] studied the essential norm of the operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ and got the exact value for it, i.e.,

$$\|C_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} = \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}.$$

Tjani [16] proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{|a|\to 1} \|C_\varphi \sigma_a\|_{\mathcal{B}} = 0$, where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. Wulan et al. [17] proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{j\to\infty} \|\varphi^j\|_{\mathcal{B}} = 0$. In [20], Zhao obtained that

$$\|C_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} = \frac{e}{2} \limsup_{n\to\infty} \|\varphi^n\|_{\mathcal{B}}.$$

The boundedness and compactness of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ were studied in [13]. The essential norm of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ was studied in [5, 10].

The composition operators, weighted composition operators and related operators on the Zygmund space were studied in [1, 2, 4, 6–9, 14, 15, 18, 19]. In [2], the authors studied the operator $uC_\varphi : H^\infty \to \mathcal{Z}$. Among others, they showed that $uC_\varphi : H^\infty \to \mathcal{Z}$ is compact if and only if $\lim_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}} = 0$. In fact, from the proof of Theorem 2 in [2], or [14, 19], we find that they obtained the following result.

Theorem 1.1 ([2, 14, 19])**.** *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ such that the operator $uC_\varphi : H^\infty \to \mathcal{Z}$ is bounded. Then the following statements are equivalent:*

(a) *The operator $uC_\varphi : H^\infty \to \mathcal{Z}$ is compact.*

(b) $\lim_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}} = 0$.

(c)
$$\limsup_{|\varphi(w)|\to 1} \|uC_\varphi f_{\varphi(w)}\|_{\mathcal{Z}} = \limsup_{|\varphi(w)|\to 1} \|uC_\varphi g_{\varphi(w)}\|_{\mathcal{Z}} = \limsup_{|\varphi(w)|\to 1} \|uC_\varphi h_{\varphi(w)}\|_{\mathcal{Z}} = 0,$$

where

$$f_a(z) = \frac{1-|a|^2}{1-\bar{a}z}, \; g_a(z) = \frac{(1-|a|^2)^2}{(1-\bar{a}z)^2}, \; h_a(z) = \frac{(1-|a|^2)^3}{(1-\bar{a}z)^3}, a \in \mathbb{D}.$$

(d)

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u''(z)| = \lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}$$
$$= \lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} = 0.$$

Motivated by the above result, in this paper, we completely characterize the essential norm of the operator uC_φ from H^∞ to the Zygmund space.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \le CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main results and proofs

In this section, we give some estimates of the essential norm for the operator $uC_\varphi : H^\infty \to \mathcal{Z}$. For this purpose, we need to state a lemma.

Lemma 2.1 ([16]). *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that*

(1) *the point evaluation functionals on Y are continuous,*
(2) *the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets,*
(3) *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Theorem 2.2. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ such that $uC_\varphi : H^\infty \to \mathcal{Z}$ is bounded. Then*

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \approx \max\{A, B, C\} \approx \max\{E, F, G\},$$

where

$$A := \limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{Z}}, \quad B := \limsup_{|a|\to 1}\|uC_\varphi g_a\|_{\mathcal{Z}}, \quad C := \limsup_{|a|\to 1}\|uC_\varphi h_a\|_{\mathcal{Z}},$$

$$E := \limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}, \quad F := \limsup_{|\varphi(z)|\to 1}(1-|z|^2)|u''(z)|,$$

and

$$G := \limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}.$$

Proof. First, we prove that $\max\{A, B, C\} \lesssim \|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}}$. Let $a \in \mathbb{D}$. It is easy to see that $f_a, g_a, h_a \in H^\infty$ and f_a, g_a, h_a converge to 0 uniformly on compact subsets of $\mathbb{D}$. Thus, for any compact operator $K : H^\infty \to \mathcal{Z}$, by Lemma 2.1 we have

$$\lim_{|a|\to 1}\|Kf_a\|_{\mathcal{Z}} = 0, \quad \lim_{|a|\to 1}\|Kg_a\|_{\mathcal{Z}} = 0, \quad \lim_{|a|\to 1}\|Kh_a\|_{\mathcal{Z}} = 0.$$

Hence

$$\|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} \gtrsim \limsup_{|a|\to 1}\|(uC_\varphi - K)f_a\|_{\mathcal{Z}}$$

$$
\begin{aligned}
&\geq \limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{Z}} - \limsup_{|a|\to 1}\|Kf_a\|_{\mathcal{Z}} = A,\\
\|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} &\gtrsim \limsup_{|a|\to 1}\|(uC_\varphi - K)g_a\|_{\mathcal{Z}}\\
&\geq \limsup_{|a|\to 1}\|uC_\varphi g_a\|_{\mathcal{Z}} - \limsup_{|a|\to 1}\|Kg_a\|_{\mathcal{Z}} = B,
\end{aligned}
$$

and

$$
\begin{aligned}
\|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} &\gtrsim \limsup_{|a|\to 1}\|(uC_\varphi - K)h_a\|_{\mathcal{Z}}\\
&\geq \limsup_{|a|\to 1}\|uC_\varphi h_a\|_{\mathcal{Z}} - \limsup_{|a|\to 1}\|Kh_a\|_{\mathcal{Z}} = C.
\end{aligned}
$$

Therefore, we obtain

$$
\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} = \inf_K \|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} \gtrsim \max\big\{A, B, C\big\}.
$$

Next, we will prove that $\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \gtrsim \max\big\{E, F, G\big\}$. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Define

$$
\begin{aligned}
k_j(z) &= \frac{1-|\varphi(z_j)|^2}{(1-\overline{\varphi(z_j)}z)} - \frac{5}{3}\frac{(1-|\varphi(z_j)|^2)^2}{(1-\overline{\varphi(z_j)}z)^2} + \frac{2}{3}\frac{(1-|\varphi(z_j)|^2)^3}{(1-\overline{\varphi(z_j)}z)^3},\\
p_j(z) &= \frac{1-|\varphi(z_j)|^2}{(1-\overline{\varphi(z_j)}z)} - \frac{(1-|\varphi(z_j)|^2)^2}{(1-\overline{\varphi(z_j)}z)^2} + \frac{1}{3}\frac{(1-|\varphi(z_j)|^2)^3}{(1-\overline{\varphi(z_j)}z)^3},
\end{aligned}
$$

and

$$
q_j(z) = \frac{1-|\varphi(z_j)|^2}{(1-\overline{\varphi(z_j)}z)} - 2\frac{(1-|\varphi(z_j)|^2)^2}{(1-\overline{\varphi(z_j)}z)^2} + \frac{(1-|\varphi(z_j)|^2)^3}{(1-\overline{\varphi(z_j)}z)^3}.
$$

It is easy to see that all k_j, p_j and q_j belong to H^∞ and converge to 0 uniformly on compact subsets of $\mathbb{D}$. Moreover,

$$
\begin{aligned}
&k_j(\varphi(z_j)) = 0, \quad k_j''(\varphi(z_j)) = 0, \quad |k_j'(\varphi(z_j))| = \frac{1}{3}\frac{|\varphi(z_j)|}{(1-|\varphi(z_j)|^2)},\\
&p_j'(\varphi(z_j)) = 0, \quad p_j''(\varphi(z_j)) = 0, \quad |p_j(\varphi(z_j))| = \frac{1}{3},\\
&q_j(\varphi(z_j)) = 0, \quad q_j'(\varphi(z_j)) = 0, \quad |q_j''(\varphi(z_j))| = \frac{2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2}.
\end{aligned}
$$

Then for any compact operator $K : H^\infty \to \mathcal{Z}$, by Lemma 2.1 we obtain

$$
\begin{aligned}
\|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi(k_j)\|_{\mathcal{Z}} - \limsup_{j\to\infty}\|K(k_j)\|_{\mathcal{Z}}\\
&\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2},\\
\|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi(p_j)\|_{\mathcal{Z}} - \limsup_{j\to\infty}\|K(p_j)\|_{\mathcal{Z}}\\
&\gtrsim \limsup_{j\to\infty}(1-|z_j|^2)|u''(z_j)|,
\end{aligned}
$$

and

$$
\|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} \gtrsim \limsup_{j\to\infty}\|uC_\varphi(q_j)\|_{\mathcal{Z}} - \limsup_{j\to\infty}\|K(q_j)\|_{\mathcal{Z}}
$$

$$\gtrsim \limsup_{j\to\infty} \frac{(1-|z_j|^2)|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2}.$$

From the definition of the essential norm, we obtain

$$\begin{aligned}
\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} &= \inf_K \|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} \\
&\gtrsim \limsup_{j\to\infty} \frac{(1-|z_j|^2)|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2} \\
&= \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} = E, \\
\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} &= \inf_K \|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} \gtrsim \limsup_{j\to\infty} (1-|z_j|^2)|u''(z_j)| \\
&= \limsup_{|\varphi(z)|\to 1} (1-|z|^2)|u''(z)| = F,
\end{aligned}$$

and

$$\begin{aligned}
\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} &= \inf_K \|uC_\varphi - K\|_{H^\infty\to\mathcal{Z}} \\
&\gtrsim \limsup_{j\to\infty} \frac{(1-|z_j|^2)|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2} \\
&= \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} = G.
\end{aligned}$$

Hence,

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \gtrsim \max\{E, F, G\}.$$

Finally, we prove that

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \lesssim \max\{A, B, C\} \quad \text{and} \quad \|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \lesssim \max\{E, F, G\}.$$

For $r \in [0,1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(\mathbb{D}).$$

It is obvious that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, the operator K_r is compact on H^∞ and $\|K_r\|_{H^\infty\to H^\infty} \le 1$. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for all positive integers j, the operator $uC_\varphi K_{r_j} : H^\infty \to \mathcal{Z}$ is compact. By the definition of the essential norm, we get

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \le \limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{H^\infty\to\mathcal{Z}}. \tag{2.1}$$

Therefore, we only need to prove that

$$\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{H^\infty\to\mathcal{Z}} \lesssim \max\{A, B, C\}$$

and

$$\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{H^\infty\to\mathcal{Z}} \lesssim \max\{E, F, G\}.$$

For any $f \in H^\infty$ such that $\|f\|_\infty \le 1$, consider

$$\begin{aligned}
\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{Z}} &= |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| + \|u(f-f_{r_j})\circ\varphi\|_{**} \\
&\quad + |u'(0)(f-f_{r_j})(\varphi(0)) + u(0)(f-f_{r_j})'(\varphi(0))\varphi'(0)|.
\end{aligned} \tag{2.2}$$

Here $\|g\|_{**} = \sup_{z\in\mathbb{D}}(1-|z|^2)|g''(z)|$. It is obvious that

$$\lim_{j\to\infty} |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0 \tag{2.3}$$

and

$$\lim_{j\to\infty} |u'(0)(f-f_{r_j})(\varphi(0)) + u(0)(f-f_{r_j})'(\varphi(0))\varphi'(0)| = 0. \tag{2.4}$$

Now, we consider

$$\limsup_{j\to\infty} \|u\cdot(f-f_{r_j})\circ\varphi\|_{**} \le Q_1+Q_2+Q_3+Q_4+Q_5+Q_6, \tag{2.5}$$

where

$$\begin{aligned}
Q_1 &:= \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)|(f-f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)|,\\
Q_2 &:= \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|(f-f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)|,\\
Q_3 &:= \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)|(f-f_{r_j})(\varphi(z))||u''(z)|,\\
Q_4 &:= \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|(f-f_{r_j})(\varphi(z))||u''(z)|,\\
Q_5 &:= \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)|(f-f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)|,\\
Q_6 &:= \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|(f-f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)|,
\end{aligned}$$

and $N\in\mathbb{N}$ is large enough such that $r_j \ge \frac{1}{2}$ for all $j\ge N$. Since $uC_\varphi : H^\infty \to \mathcal{Z}$ is bounded, from the proof of Theorem 1 in [2], we see that $u\in\mathcal{Z}$,

$$\widetilde{J}_1 := \sup_{z\in\mathbb{D}}(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)| < \infty$$

and

$$\widetilde{J}_2 := \sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)|^2|u(z)| < \infty.$$

Since $r_jf'_{r_j}\to f'$, as well as $r_j^2f''_{r_j}\to f''$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, we have

$$Q_1 \le \widetilde{J}_1 \limsup_{j\to\infty} \sup_{|w|\le r_N} |f'(w)-r_jf'(r_jw)| = 0, \tag{2.6}$$

$$Q_5 \le \widetilde{J}_2 \limsup_{j\to\infty} \sup_{|w|\le r_N} |f''(w)-r_j^2f''(r_jw)| = 0, \tag{2.7}$$

and

$$Q_3 \le \|u\|_{\mathcal{Z}} \limsup_{j\to\infty} \sup_{|w|\le r_N} |f(w)-f(r_jw)| = 0. \tag{2.8}$$

Next, we consider Q_2. We have $Q_2 \le \limsup_{j\to\infty}(S_1+S_2)$, where

$$S_1 := \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)|$$

and

$$S_2 := \sup_{|\varphi(z)|>r_N} (1-|z|^2)r_j|f'(r_j\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)|.$$

First we estimate S_1. Using the fact that $\|f\|_\infty \le 1$, we have

$$\begin{aligned}
S_1 &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)|\frac{3(1-|\varphi(z)|^2)}{|\varphi(z)|}\frac{|\varphi(z)|}{3(1-|\varphi(z)|^2)} \\
&\lesssim \frac{\|f\|_\infty}{r_N} \sup_{|\varphi(z)|>r_N} \frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)||\varphi(z)|}{3(1-|\varphi(z)|^2)} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \sup_{|a|>r_N} \frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)||\varphi(z)|}{3(1-|\varphi(z)|^2)} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi(f_a - \frac{5}{3}g_a + \frac{2}{3}h_a)\|_{\mathcal{Z}} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi(f_a)\|_{\mathcal{Z}} + \frac{5}{3}\sup_{|a|>r_N} \|uC_\varphi(g_a)\|_{\mathcal{Z}} + \frac{2}{3}\sup_{|a|>r_N} \|uC_\varphi(h_a)\|_{\mathcal{Z}}.
\end{aligned} \tag{2.9}$$

Here we used the fact that $\sup_{w\in\mathbb{D}}(1-|w|^2)|f'(w)| \lesssim \|f\|_\infty$ for any $f \in H^\infty$, since $H^\infty \subset \mathcal{B}$ and $\|f\|_{\mathcal{B}} \le \|f\|_\infty$. Taking limit as $N \to \infty$ we obtain

$$\begin{aligned}
\limsup_{j\to\infty} S_1 &\lesssim \limsup_{|a|\to 1} \|uC_\varphi(f_a)\|_{\mathcal{Z}} + \limsup_{|a|\to 1} \|uC_\varphi(g_a)\|_{\mathcal{Z}} + \limsup_{|a|\to 1} \|uC_\varphi(h_a)\|_{\mathcal{Z}} \\
&= A + B + C.
\end{aligned}$$

Similarly, we have $\limsup_{j\to\infty} S_2 \lesssim A+B+C$, i.e., we get that

$$Q_2 \lesssim A+B+C \lesssim \max\{A,B,C\}. \tag{2.10}$$

From (2.9), we see that

$$\limsup_{j\to\infty} S_1 \lesssim \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} = E.$$

Similarly, we have $\limsup_{j\to\infty} S_2 \lesssim E$. Therefore,

$$Q_2 \lesssim E. \tag{2.11}$$

Also for Q_4, we have $Q_4 \le \limsup_{j\to\infty}(S_3+S_4)$, where

$$S_3 := \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f(\varphi(z))||u''(z)|, \quad S_4 := \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f(r_j\varphi(z))||u''(z)|.$$

After a calculation, we have

$$\begin{aligned}
S_3 &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f(\varphi(z))||u''(z)| \\
&\lesssim \|f\|_\infty \sup_{|\varphi(z)|>r_N} \frac{1}{3}(1-|z|^2)|u''(z)| \\
&\lesssim \sup_{|\varphi(z)|>r_N} \sup_{|a|>r_N} \frac{1}{3}(1-|z|^2)|u''(z)| \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi(f_a)\|_{\mathcal{Z}} + \sup_{|a|>r_N} \|uC_\varphi(g_a)\|_{\mathcal{Z}} + \frac{1}{3}\sup_{|a|>r_N} \|uC_\varphi(h_a)\|_{\mathcal{Z}} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi(f_a)\|_{\mathcal{Z}} + \sup_{|a|>r_N} \|uC_\varphi(g_a)\|_{\mathcal{Z}} + \sup_{|a|>r_N} \|uC_\varphi(h_a)\|_{\mathcal{Z}}.
\end{aligned} \tag{2.12}$$

Taking limit as $N \to \infty$ we obtain

$$\begin{aligned}\limsup_{j\to\infty} S_3 &\lesssim \limsup_{|a|\to 1} \|uC_\varphi(f_a)\|_{\mathcal{Z}} + \limsup_{|a|\to 1} \|uC_\varphi(g_a)\|_{\mathcal{Z}} + \limsup_{|a|\to 1} \|uC_\varphi(h_a)\|_{\mathcal{Z}} \\ &= A + B + C.\end{aligned}$$

Similarly, we have $\limsup_{j\to\infty} S_4 \lesssim A + B + C$, i.e., we get that

$$Q_4 \lesssim A + B + C \lesssim \max\big\{A, B, C\big\}. \tag{2.13}$$

From (2.12), we see that

$$\limsup_{j\to\infty} S_3 \lesssim \limsup_{|\varphi(z)|\to 1} (1 - |z|^2)|u''(z)| = F.$$

Similarly, we have that $\limsup_{j\to\infty} S_4 \lesssim F$. Therefore,

$$Q_4 \lesssim F. \tag{2.14}$$

Also, for Q_6, we have $Q_6 \le \limsup_{j\to\infty}(S_5 + S_6)$, where

$$S_5 := \sup_{|\varphi(z)|>r_N} (1 - |z|^2)|f^{''}(\varphi(z))||\varphi'(z)|^2|u(z)|, \quad S_6 := \sup_{|\varphi(z)|>r_N} (1 - |z|^2) r_j^2 |f^{''}(r_j\varphi(z))||\varphi'(z)|^2|u(z)|.$$

After a calculation, we have

$$\begin{aligned}S_5 &\lesssim \|f\|_\infty \sup_{|\varphi(z)|>r_N} (1 - |z|^2)|\varphi'(z)|^2|u(z)| \frac{2|\varphi(z)|^2}{(1 - |\varphi(z)|^2)^2} \\ &\lesssim \sup_{|\varphi(z)|>r_N} (1 - |z|^2)|\varphi'(z)|^2|u(z)| \frac{2|\varphi(z)|^2}{(1 - |\varphi(z)|^2)^2} \\ &\lesssim \sup_{|a|>r_N} \|uC_\varphi(f_a - 2g_a + h_a)\|_{\mathcal{Z}} \\ &\lesssim \sup_{|a|>r_N} \Big(\|uC_\varphi(f_a)\|_{\mathcal{Z}} + \|uC_\varphi(g_a)\|_{\mathcal{Z}} + \|uC_\varphi(h_a)\|_{\mathcal{Z}} \Big).\end{aligned} \tag{2.15}$$

Taking limit as $N \to \infty$ we obtain

$$\begin{aligned}\limsup_{j\to\infty} S_5 &\lesssim \limsup_{|a|\to 1} \|uC_\varphi(f_a)\|_{\mathcal{Z}} + \limsup_{|a|\to 1} \|uC_\varphi(g_a)\|_{\mathcal{Z}} + \limsup_{|a|\to 1} \|uC_\varphi(h_a)\|_{\mathcal{Z}} \\ &= A + B + C.\end{aligned}$$

Similarly, we get $\limsup_{j\to\infty} S_6 \lesssim A + B + C$, i.e., we have

$$Q_6 \lesssim A + B + C \lesssim \max\big\{A, B, C\big\}. \tag{2.16}$$

From (2.15), we obtain

$$\limsup_{j\to\infty} S_5 \lesssim \limsup_{|\varphi(z)|\to 1} \frac{(1 - |z|^2)|\varphi'(z)|^2|u(z)|}{(1 - |\varphi(z)|^2)^2} = G.$$

Similarly, we obtain $\limsup_{j\to\infty} S_6 \lesssim G$. Therefore,

$$Q_6 \lesssim G. \tag{2.17}$$

Hence, by (2.2)-(2.8), (2.10), (2.13) and (2.16) we get

$$\begin{aligned}\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{H^\infty\to\mathcal{Z}} &= \limsup_{j\to\infty} \sup_{\|f\|_\infty\le 1} \|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{Z}} \\ &= \limsup_{j\to\infty} \sup_{\|f\|_\infty\le 1} \|u\cdot(f - f_{r_j})\circ\varphi\|_{**} \lesssim \max\big\{A, B, C\big\}.\end{aligned} \tag{2.18}$$

Similarly, by (2.2)-(2.8), (2.11), (2.14) and (2.17) we get

$$\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{H^\infty\to\mathcal{Z}} \lesssim \max\{E, F, G\}. \tag{2.19}$$

Therefore, by (2.1), (2.18) and (2.19), we obtain

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \lesssim \max\{A, B, C\} \quad \text{and} \quad \|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \lesssim \max\{E, F, G\}.$$

This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ such that $uC_\varphi : H^\infty \to \mathcal{Z}$ is bounded. Then*

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \approx \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}}.$$

Proof. First, we prove that

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \geq \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}}.$$

Let n be any positive integer and $f_n(z) = z^n$. Then $\|f_n\|_\infty = 1$ and f_n uniformly converges to zero on compact subsets of $\mathbb{D}$. By Lemma 2.1, we have $\lim_{n\to\infty} \|Kf_n\|_{\mathcal{Z}} = 0$. Hence,

$$\|uC_\varphi - K\| \geq \limsup_{n\to\infty} \|(uC_\varphi - K)f_n\|_{\mathcal{Z}} \geq \limsup_{n\to\infty} \|uC_\varphi f_n\|_{\mathcal{Z}}.$$

Therefore, by the definition of essential norm we get

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \geq \limsup_{n\to\infty} \|uC_\varphi f_n\|_{\mathcal{Z}} = \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}}. \tag{2.20}$$

Next, we prove that

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}}.$$

Since $uC_\varphi : H^\infty \to \mathcal{Z}$ is bounded, by Theorem 1 of [2] we see that

$$P := \sup_{k\geq 0} \|u\varphi^k\|_{\mathcal{Z}} < \infty.$$

Consider the Maclaurin expansion of f_a, where

$$f_a(z) = (1 - |a|^2) \sum_{k=0}^{\infty} \bar{a}^k z^k.$$

For any fix positive integer $n \geq 2$, it follows from the linearity of uC_φ and the triangle inequality that

$$\begin{aligned}
\|uC_\varphi f_a\|_{\mathcal{Z}} &\leq (1 - |a|^2) \sum_{k=0}^{\infty} |a|^k \|u\varphi^k\|_{\mathcal{Z}} \\
&= (1 - |a|^2) \sum_{k=0}^{n-1} |a|^k \|u\varphi^k\|_{\mathcal{Z}} + (1 - |a|^2) \sum_{k=n}^{\infty} |a|^k \|u\varphi^k\|_{\mathcal{Z}} \\
&\leq Pn(1 - |a|^2) + (1 - |a|^2) \sum_{k=n}^{\infty} |a|^k \|u\varphi^k\|_{\mathcal{Z}} \\
&\leq Pn(1 - |a|^2) + 2 \sup_{k\geq n} \|u\varphi^k\|_{\mathcal{Z}}.
\end{aligned}$$

Letting $|a| \to 1$ in the above inequality leads to

$$\limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}} \le 2 \sup_{k\ge n} \|u\varphi^k\|_{\mathcal{Z}}$$

for any positive integer $n \ge 2$. Thus,

$$\limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}} \lesssim \limsup_{k\to\infty} \|u\varphi^k\|_{\mathcal{Z}}.$$

Similarly, we can prove that

$$\limsup_{|a|\to 1} \|uC_\varphi g_a\|_{\mathcal{Z}} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}}, \quad \limsup_{|a|\to 1} \|uC_\varphi h_a\|_{\mathcal{Z}} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}}.$$

Hence,

$$\max\{A, B, C\} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}}.$$

Therefore, by Theorem 2.2 we obtain

$$\|uC_\varphi\|_{e,H^\infty\to\mathcal{Z}} \lesssim \max\{A, B, C\} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}}. \tag{2.21}$$

By (2.20) and (2.21), we get the desired result. The proof is completed. □

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