Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5202–5218
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On best proximity points for various α-proximal contractions on metric-like spaces

Hassen Aydi[a,b,*], Abdelbasset Felhi[c]

[a]University of Dammam, Department of Mathematics, College of Education of Jubail, P. O. 12020, Industrial Jubail 31961, Saudi Arabia.

[b]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

[c]King Faisal University, Department of Mathematics, College of Sciences, Al-Hassa, Saudi Arabia.

Communicated by R. Saadati

Abstract

We establish some best proximity points for various α-proximal contractive non-self-mappings in the class of metric-like spaces. We provide concrete examples. We also present some best proximity point theorems in metric (metric-like) spaces endowed with a graph and in partially ordered metric spaces.

Keywords: Metric-like, best proximity point, fixed point, controlled proximal contraction.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

The notion of a metric-like (or a dislocated metric) was rediscovered by Harandi [1]. In the last years, many (common) fixed point results by using the concept of metric-like have been proved, see for example [2–6].

Definition 1.1. Let X be a nonempty set. A function $\sigma : X \times X \to \mathbb{R}^+$ is said to be a b-metric-like (or a dislocated b-metric) on X if for any $x, y, z \in X$, the following conditions hold:

(σ_1) $\sigma(x, y) = 0 \Longrightarrow x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

*Corresponding author
Email addresses: hmaydi@uod.edu.sa (Hassen Aydi), afelhi@kfu.edu.sa (Abdelbasset Felhi)

Received 2016-03-20

(σ_3) $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$.

Then the pair (X,σ) is called a metric-like space.

Example 1.2. Let $X = [0,\infty)$. Consider the mapping $\sigma : X \times X \to [0,\infty)$ defined by $\sigma(x,y) = (x+y)$ for all $x,y \in X$. Then (X,σ) is a metric-like space.

Mention that each metric-like on X generates a T_0 topology τ_σ on X which has a base the family of open σ-balls $\{B_\sigma(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Definition 1.3. Let (X,σ) be a metric-like space, $\{x_n\}$ be a sequence in X and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x).$$

In a metric-like space, the limit for a convergent sequence is not unique in general.

Definition 1.4. Let (X,σ) be a metric-like space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ exists and is finite.

Definition 1.5. Let (X,σ) be a metric-like space. We say that (X,σ) is complete if and only if each Cauchy sequence in X is convergent.

In what follows, we recall some notations and definitions which will be needed in the sequel. For A and B two nonempty subsets of a metric-like space (X,σ), define

$$\begin{aligned}\sigma(A,B) &= \inf\{\sigma(a,b) :\ a \in A,\ b \in B\},\\ A_0 &= \inf\{a \in A : \sigma(a,b) = \sigma(A,B),\ \text{for some}\, b \in B\},\\ B_0 &= \inf\{b \in B : \sigma(a,b) = \sigma(A,B),\ \text{for some}\, a \in A\}.\end{aligned}$$

The concept of (P)-property was introduced by Raj and Veeramani [14]. This concept was weakened later by Zhang et al. [18] where the concept of weak P-property was introduced. In the class of metric-like spaces, we have the following.

Definition 1.6. Let A and B be nonempty subsets of a metric-like space (X,σ) with $A_0 \neq \emptyset$. The pair (A,B) is said to have the weak (P)-property if and only if

$$\begin{cases}\sigma(x_1,y_1) = \sigma(A,B),\\ \sigma(x_2,y_2) = \sigma(A,B),\end{cases} \Rightarrow \sigma(x_1,x_2) \leq \sigma(y_1,y_2),$$

where $x_1,x_2 \in A$ and $y_1,y_2 \in B$.

Example 1.7. Let $X = \{(1,2),(0,1),(1,3),(3,1)\}$ endowed with the metric-like $\sigma((x_1,x_2),(y_1,y_2)) = x_1 + x_2 + y_1 + y_2$ for all $(x_1,x_2),(y_1,y_2) \in X$. Let $A = \{(1,2),(0,1)\}$ and $B = \{(1,3),(3,1)\}$. We have

$$\sigma((0,1),(1,3)) = 5 = \sigma(A,B) \quad \text{and} \quad \sigma((0,1),(3,1)) = \sigma(A,B).$$

Moreover,

$$\sigma((0,1),(0,1)) = 2 < 8 = \sigma((1,3),(3,1)).$$

Also, $A_0 \neq \emptyset$. Hence, the pair (A,B) satisfies the weak (P)-property.

As in [8], we introduce in the setting of metric-like spaces the following.

Definition 1.8. Let A and B be nonempty subsets of a metric-like space (X,σ) and $\alpha : X \times X \to [0,\infty)$. A mapping $T : A \to B$ is named α-proximal admissible if

$$\begin{cases} \alpha(x_1,x_2) \geq 1, \\ \sigma(u_1,Tx_1) = \sigma(A,B), \\ \sigma(u_2,Tx_2) = \sigma(A,B), \end{cases} \Rightarrow \alpha(u_1,u_2) \geq 1,$$

for all $x_1,x_2,u_1,u_2 \in A$.

If $\sigma(A,B) = 0$, mention that T is α-proximal admissible implies that T is α-admissible [17].

We introduce the following notions.

Definition 1.9. Let A and B be nonempty subsets of a metric-like space (X,σ) and $\alpha : X \times X \to [0,\infty)$. A mapping $T : A \to B$ is named triangular α-proximal admissible if

(T_1) T is α-proximal admissible;

(T_2) $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1$, $x,y,z \in A$.

Now, let Ψ be the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:

(ψ_1) ψ is nondecreasing;

(ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t \in \mathbb{R}^+$, where ψ^n is the nth iterate of ψ.

Consider also Φ as the set of functions $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying:

(ϕ_1) ϕ is continuous;

(ϕ_2) $\phi(x,y) = 0$ if and only if $x = y = 0$.

In the following, we give some generalized α-proximal contractions.

Definition 1.10. Let A and B be two nonempty subsets of a metric-like space (X,σ). Consider a non-self-mapping $T : A \to B$ and a given function $\alpha : X \times X \to [0,\infty)$.

(i) T is called an α-proximal contraction if

$$\alpha(x,y)\sigma(Tx,Ty) \leq \psi(\sigma(x,y)) \tag{1.1}$$

for all $x,y \in A$, where $\psi \in \Psi$.

(ii) T is called an α-proximal C-contraction if

$$\alpha(x,y)\sigma(Tx,Ty) \leq \frac{\sigma(x,Ty) + \sigma(y,Tx) - 2\sigma(A,B)}{2} - \phi(\sigma(x,Ty) - \sigma(A,B), \sigma(y,Tx) - \sigma(A,B)) \tag{1.2}$$

for all $x,y \in A$, where $\phi \in \Phi$.

On the other hand, the definition of a best proximity point is as follows.

Definition 1.11. Let (X,σ) be a metric-like space. Consider A and B as the two nonempty subsets of X. An element $a \in X$ is said to be a best proximity point of $T : A \to B$ if

$$\sigma(a,Ta) = \sigma(A,B).$$

Note that a fixed point coincides with a best proximity point in the case of $\sigma(A,B) = 0$. For some results on above concept, see for example [7, 9–13, 15, 16].

In this paper, we establish some existence results on best proximity points for various α-proximal contractions in the setting of metric-like spaces. We will support the obtained theorems by some concrete examples. Some nice consequences are also provided.

2. Main results

The first main result is:

Theorem 2.1. *Let A and B be nonempty closed subsets of a complete metric-like space (X,σ) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-mapping. Suppose that*

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the weak (P)-property;*
(ii) *T is α-proximal admissible;*
(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1, Tx_0) = \sigma(A,B), \quad \text{and} \quad \alpha(x_0,x_1) \geq 1;$$

(iv) *T is a continuous α-proximal contraction.*

Then, there exists $u \in A$ such that $\sigma(u,u) = 0$. Assume in addition that

(v) *$\alpha(z,z) \geq 1$ for each $z \in A_0$ such that $\sigma(z,z) = 0$.*

Then, such u is a best proximity point of T, that is,

$$\sigma(u,Tu) = \sigma(A,B).$$

Proof. By assumption (iii), there exist x_0 and $x_1 \in A_0$ such that

$$\sigma(x_1, Tx_0) = \sigma(A,B), \quad \text{and} \quad \alpha(x_0,x_1) \geq 1. \tag{2.1}$$

From condition (i), we have $T(A_0) \subseteq B_0$, so there exists $x_2 \in A_0$ such that

$$\sigma(x_2, Tx_1) = \sigma(A,B). \tag{2.2}$$

By (2.1), (2.2) and the fact that T is α-proximal admissible, we have

$$\alpha(x_1,x_2) \geq 1.$$

By repeating the above strategy, by the induction, we arrive to construct a sequence $\{x_n\}$ in A_0 such that

$$\sigma(x_{n+1}, Tx_n) = \sigma(A,B), \quad \text{and} \quad \alpha(x_n,x_{n+1}) \geq 1, \quad \text{for all } n \geq 0. \tag{2.3}$$

From condition (i), the pair (A,B) satisfies the weak (P)-property, so

$$\sigma(x_n, x_{n+1}) \leq \sigma(Tx_{n-1}, Tx_n), \quad \text{for all } n \geq 1. \tag{2.4}$$

The non-self-mapping T is an α-proximal contraction, so for all $n \geq 1$, by using (2.3) and (2.4)

$$\begin{aligned}\sigma(x_n,x_{n+1}) &\leq \sigma(Tx_{n-1},Tx_n)\\ &\leq \alpha(x_{n-1},x_n)\sigma(Tx_{n-1},Tx_n)\\ &\leq \psi(\sigma(x_{n-1},x_n))\\ &\leq \psi^n(\sigma(x_0,x_1)).\end{aligned}$$

Since $\psi \in \Psi$, so the right-hand side of above inequality tends to 0 as $n \to \infty$, that is,

$$\lim_{n\to\infty} \sigma(x_n,x_{n+1}) = 0.$$

For all $k \in \mathbb{N}$, we have

$$\sigma(x_n,x_{n+k}) \leq \sum_{m=n}^{n+k-1} \sigma(x_m,x_{m+1}) \leq \sum_{m=n}^{n+k-1} \psi^m(\sigma(x_0,x_1))$$

$$\leq \sum_{m=n}^{\infty} \psi^m(\sigma(x_0, x_1)) \to 0, \quad \text{as } n \to \infty.$$

It follows that $\lim_{n\to\infty} \sigma(x_n, x_{n+k}) = 0$ for all $k \in \mathbb{N}$, that is, $\{x_n\}$ is a Cauchy sequence in A. Since A is a closed subset of the complete metric-like space (X, σ), then there exists $u \in A$ such that $x_n \to u$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \sigma(x_n, u) = \sigma(u, u) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.5}$$

We have obtained $\sigma(u, u) = 0$. Thus, by condition (v), $\alpha(u, u) \geq 1$. Consequently, from condition (iv),

$$\sigma(Tu, Tu) \leq \alpha(u, u)\sigma(Tu, Tu) \leq \psi(\sigma(u, u)) = \psi(0) = 0,$$

which implies that $\sigma(Tu, Tu) = 0$. The mapping T is continuous at u, so

$$\lim_{n\to\infty} \sigma(Tx_n, Tu) = \sigma(Tu, Tu) = 0. \tag{2.6}$$

On the other hand, by triangular inequality and by using (2.3),

$$\begin{aligned}\sigma(A, B) \leq& \sigma(u, Tu)\\ \leq& \sigma(u, x_{n+1}) + \sigma(x_{n+1}, Tx_n) + \sigma(Tx_n, Tu)\\ =& \sigma(u, x_{n+1}) + \sigma(A, B) + \sigma(Tx_n, Tu).\end{aligned}$$

By letting $n \to \infty$ in above inequalities, by (2.5) and (2.6),

$$\sigma(A, B) \leq \sigma(u, Tu) \leq \sigma(A, B),$$

that is, $\sigma(A, B) = \sigma(u, Tu)$, i.e., u is a best proximity point of T. □

In the next result, we replace the continuity hypothesis by the following condition in A:

(H) if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$, for all k.

Theorem 2.2. *Let A and B be nonempty closed subsets of a complete metric-like space (X, σ) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-mapping. Suppose that*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak (P)-property;*
(ii) *T is α-proximal admissible;*
(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1, Tx_0) = \sigma(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iv) *T is an α-proximal contraction;*
(iv) *(H) holds.*

Then, there exists $u \in A$ such that

$$\sigma(u, Tu) = \sigma(A, B) \quad \textit{and} \quad \sigma(u, u) = 0.$$

Proof. By following the proof of Theorem 2.1, there exists a sequence $\{x_n\}$ in A_0 such that (2.3) holds. Also, $\{x_n\}$ is Cauchy in the subset A, which is closed in the complete metric-like space (X, σ), then there exists $u \in A$ such that $x_n \to u$ as $n \to \infty$. By hypothesis (H), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Now, from condition (iv), we have

$$\sigma(Tx_{n(k)}, Tu) \leq \alpha(x_{n(k)}, u)\sigma(Tx_{n(k)}, Tu) \leq \psi(\sigma(x_{n(k)}, u)).$$

On the other hand, we have

$$\begin{aligned}\sigma(A,B) \leq& \sigma(u,Tu)\\ \leq& \sigma(u,x_{n(k)+1})+\sigma(x_{n(k)+1},Tx_{n(k)})+\sigma(Tx_{n(k)},Tu)\\ =& \sigma(u,x_{n(k)+1})+\sigma(A,B)+\sigma(Tx_{n(k)},Tu).\end{aligned}$$

Therefore,

$$\sigma(A,B)\leq \sigma(u,Tu)\leq \sigma(u,x_{n(k)+1})+\sigma(A,B)+\psi(\sigma(x_{n(k)},u)).$$

By (2.5) and a property of ψ, as $n\to\infty$, we get

$$\sigma(A,B)\leq \sigma(u,Tu)\leq \sigma(A,B),$$

that is, $\sigma(A,B)=\sigma(u,Tu)$, i.e., u is a best proximity point of T. □

Now, we prove the uniqueness of such best proximity point. For this, we need the following additional condition.

(U) For all $x,y\in B(T)$, we have $\alpha(x,y)\geq 1$, where $B(T)$, denotes the set of best proximity points of T.

Theorem 2.3. *By adding condition* (U) *to the hypotheses of Theorem* 2.1 *(resp. Theorem* 2.2*), we obtain that u is the unique best proximity point of T.*

Proof. We argue by contradiction, that is, there exist $u,v\in A$ such that $\sigma(A,B)=\sigma(u,Tu)=\sigma(v,Tv)$ with $u\neq v$. By assumption (U), we have $\alpha(u,v)\geq 1$. So, as the pair (A,B) satisfies the weak (P)-property, then by (1.1), we have

$$0<\sigma(u,v)\leq \sigma(Tu,Tv)\leq \alpha(u,v)\sigma(Tu,Tv)\leq \psi(\sigma(u,v))<\sigma(u,v),$$

which is a contradiction. Hence, $u=v$.

□

We provide the following example.

Example 2.4. Let $X=\{0,1,2,3\}$ endowed with the metric-like σ given as

$$\sigma(0,0)=\frac{3}{2},\quad \sigma(1,1)=\sigma(3,3)=0,\quad \sigma(2,2)=2,$$

$$\sigma(0,1)=\sigma(1,0)=2,\quad \sigma(0,2)=\sigma(2,0)=\sigma(3,1)=\sigma(1,3)=\frac{3}{2},$$

$$\sigma(0,3)=\sigma(2,3)=\frac{5}{2},\quad \text{and}\quad \sigma(1,2)=\sigma(2,1)=3,$$

Take $A=\{1,2\}$ and $B=\{2,3\}$. Consider the mapping $T:A\to B$ defined by

$$T2=2,\quad \text{and}\quad T1=3.$$

Remark that $\sigma(A,B)=\sigma(1,3)=\frac{3}{2}$. Also, $A_0=\{1\}$ and $B_0=\{3\}$. Note that $T(A_0)\subseteq B_0$. Now, let $x_1,x_2\in A$ and $y_1,y_2\in B$ such that

$$\begin{cases}\sigma(x_1,y_1)=\sigma(A,B)=\frac{3}{2},\\ \sigma(x_2,y_2)=\sigma(A,B)=\frac{3}{2}.\end{cases}$$

Then, we have $(x_1=1,y_1=3)$ and $(x_2=1,y_2=3)$. In this case,

$$\sigma(x_1,x_2)=0=\sigma(y_1,y_2),$$

that is, the pair (A, B) has the weak (P)-property.

Take $\psi(t) = \frac{11}{12}t$, for all $t \geq 0$. Define $\alpha : X \times X \to [0, \infty)$ as follows

$$\begin{cases} \alpha(x, y) = 1, & \text{if} \quad (x, y) \in \{(1, 2), (2, 1), (1, 1)\}, \\ \alpha(x, y) = 0, & \text{if not.} \end{cases}$$

Let x_1, x_2, u_1 and u_2 in $A = \{1, 2\}$ such that

$$\begin{cases} \alpha(x_1, x_2) \geq 1, \\ \sigma(u_1, Tx_1) = \sigma(A, B) = \frac{3}{2}, \\ \sigma(u_2, Tx_2) = \sigma(A, B) = \frac{3}{2}. \end{cases}$$

Then, necessarily, we have $(x_1 = x_2 = u_1 = u_2 = 1)$. So

$$\alpha(u_1, u_2) \geq 1,$$

that is, T is α-proximal admissible. By the symmetry of α and σ, it suffices to study the cases $(x = 1, y = 2)$ and $(x = y = 1)$.

If $(x = 1, y = 2)$, we have

$$\alpha(x, y)\sigma(Tx, Ty) = \sigma(3, 2) = \frac{5}{2} \leq \psi(3) = \psi(\sigma(1, 2) = \psi(\sigma(x, y)).$$

If $(x = y = 1)$, we have

$$\alpha(x, y)\sigma(Tx, Ty) = \sigma(3, 3) = 0 = \psi(0) = \psi(\sigma(1, 1) = \psi(\sigma(x, y)).$$

Thus, (1.1) is satisfied for all $x, y \in A$. Moreover, the conditions (H) and (iii) with $x_0 = x_1 = 1$ in Theorem 2.2 are verified. So T has a best proximity point which is $u = 1$. It is also unique and verifies $\sigma(u, u) = 0$.

Theorem 2.5. *Let A and B be nonempty closed subsets of a complete metric-like space (X, σ) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-mapping. Suppose that*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak (P)-property;*
(ii) *T is triangular α-proximal admissible;*
(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1, Tx_0) = \sigma(A, B), \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iv) *T is a continuous α-proximal C-contraction.*

Then, there exists $u \in A$ such that $\sigma(u, u) = 0$. Assume in addition that

(v) *$\alpha(z, z) \geq 1$ for each $z \in A$ such that $\sigma(z, z) = 0$.*

Then, such u is a best proximity point of T, that is,

$$\sigma(u, Tu) = \sigma(A, B).$$

Proof. By following the proof of Theorem 2.1, we construct a sequence $\{x_n\}$ in A_0 such that (2.3) holds, that is,

$$\sigma(x_{n+1}, Tx_n) = \sigma(A, B), \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n \geq 0.$$

Since T is triangular α-proximal admissible, then

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ and } \alpha(x_{n+1}, x_{n+2}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}) \geq 1.$$

Thus by the induction, we get

$$\alpha(x_n, x_m) \geq 1, \quad \text{for all } m > n \geq 0.$$

Again, (2.4) is satisfied, that is,

$$\sigma(x_n, x_{n+1}) \leq \sigma(Tx_{n-1}, Tx_n), \quad \text{for all } n \geq 1.$$

By condition (iv), T is an α-proximal C-contraction, so

$$\begin{aligned}
\sigma(x_n, x_{n+1}) &\leq \sigma(Tx_{n-1}, Tx_n) \\
&\leq \alpha(x_{n-1}, x_n)\sigma(Tx_{n-1}, Tx_n) \\
&\leq \frac{\sigma(x_{n-1}, Tx_n) + \sigma(x_n, Tx_{n-1}) - 2\sigma(A, B)}{2} \\
&\quad - \phi(\sigma(x_{n-1}, Tx_n) - \sigma(A, B), \sigma(x_n, Tx_{n-1}) - \sigma(A, B)) \\
&= \frac{\sigma(x_{n-1}, Tx_n) - \sigma(A, B)}{2} - \phi(\sigma(x_{n-1}, Tx_n) - \sigma(A, B), 0) \\
&\leq \frac{\sigma(x_{n-1}, Tx_n) - \sigma(A, B)}{2} \\
&\leq \frac{\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, Tx_n) - \sigma(A, B)}{2} \\
&= \frac{\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1})}{2}.
\end{aligned}$$

One can write

$$\sigma(x_n, x_{n+1}) \leq \sigma(x_{n-1}, x_n), \quad \text{for all } n \geq 1,$$

which allows to say that $\{\sigma(x_n, x_{n+1})\}$ is an nonincreasing sequence in $[0, \infty)$. Then, there exists $t \geq 0$ such that

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = t. \tag{2.7}$$

We obtained

$$\sigma(x_n, x_{n+1}) \leq \frac{\sigma(x_{n-1}, Tx_n) - \sigma(A, B)}{2} \leq \frac{\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1})}{2}.$$

By (2.7), we have

$$\lim_{n\to\infty} \sigma(x_{n-1}, Tx_n) - \sigma(A, B) = 2t.$$

Moreover, we have

$$\sigma(x_n, x_{n+1}) \leq \frac{\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1})}{2} - \phi(\sigma(x_{n-1}, Tx_n) - \sigma(A, B), 0).$$

By letting $n \to \infty$, we get

$$t \leq t - \phi(2t, 0),$$

which holds unless $\phi(2t, 0) = 0$, so by a property of ϕ, $t = 0$, i.e.,

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{2.8}$$

Now, we shall prove that

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.9}$$

Suppose to the contrary that there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k

$$\sigma(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{2.10}$$

Moreover, corresponding to $n(k)$ we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k)$ and satisfying (2.10). Then

$$\sigma(x_{n(k)}, x_{m(k)-1}) < \varepsilon. \tag{2.11}$$

By using (2.10), (2.11) and the triangular inequality, we get

$$\begin{aligned}\varepsilon \leq \sigma(x_{n(k)}, x_{m(k)}) &\leq \sigma(x_{n(k)}, x_{m(k)-1}) + \sigma(x_{m(k)-1}, x_{m(k)})\\ &< \sigma(x_{m(k)-1}, x_{m(k)}) + \varepsilon.\end{aligned}$$

By letting $k \to \infty$ in the above inequality and using (2.8), we obtain

$$\lim_{k\to\infty} \sigma(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} \sigma(x_{n(k)}, x_{m(k)-1}) = \varepsilon. \tag{2.12}$$

Also, by the triangular inequality, we have

$$\sigma(x_{n(k)}, x_{m(k)-1}) - \sigma(x_{n(k)}, x_{n(k)-1}) - \sigma(x_{m(k)}, x_{m(k)-1}) \leq \sigma(x_{n(k)-1}, x_{m(k)}).$$

$$\sigma(x_{n(k)-1}, x_{m(k)}) \leq \sigma(x_{n(k)-1}, x_{n(k)}) + \sigma(x_{n(k)}, x_{m(k)}).$$

By letting $k \to \infty$ in the above inequalities and by using (2.8) and (2.12), we obtain

$$\lim_{k\to\infty} \sigma(x_{n(k)-1}, x_{m(k)}) = \varepsilon. \tag{2.13}$$

On the other hand, we have

$$\sigma(x_{n(k)}, Tx_{n(k)-1}) = \sigma(A, B), \quad \text{and } \sigma(x_{m(k)}, Tx_{m(k)-1}) = \sigma(A, B), \quad \text{for all } k \geq 1.$$

Since the pair (A, B) satisfies the (P)-property, it follows that

$$\sigma(x_{n(k)}, x_{m(k)}) \leq \sigma(Tx_{n(k)-1}, Tx_{m(k)-1}), \quad \text{for all } k \geq 1.$$

Consider

$$a_k := \sigma(x_{n(k)-1}, Tx_{m(k)-1}) - \sigma(A, B), \quad \text{and} \quad b_k := \sigma(x_{m(k)-1}, Tx_{n(k)-1}) - \sigma(A, B).$$

By (1.2) and as $\alpha(x_{n(k)-1}, x_{m(k)} - 1) \geq 1$ for all $k \geq 1$, we get

$$\begin{aligned}\sigma(x_{n(k)}, x_{m(k)}) &\leq \sigma(Tx_{n(k)-1}, Tx_{m(k)-1}) \leq \alpha(x_{n(k)-1}, x_{m(k)} - 1)\sigma(Tx_{n(k)-1}, Tx_{m(k)-1})\\ &\leq 2^{-1}[\sigma(x_{n(k)-1}, Tx_{m(k)-1}) + \sigma(x_{m(k)-1}, Tx_{n(k)-1}) - 2\sigma(A, B)]\\ &\quad - \phi(\sigma(x_{n(k)-1}, Tx_{m(k)-1}) - \sigma(A, B), \sigma(x_{m(k)-1}, Tx_{n(k)-1}) - \sigma(A, B))\\ &\leq 2^{-1}[\sigma(x_{n(k)-1}, x_{m(k)}) + \sigma(x_{m(k)}, Tx_{m(k)-1})\\ &\quad + \sigma(x_{m(k)-1}, x_{n(k)}) + \sigma(x_{n(k)}, Tx_{n(k)-1}) - 2\sigma(A, B)] - \phi(a_k, b_k)\\ &= 2^{-1}[\sigma(x_{n(k)-1}, x_{m(k)}) + \sigma(x_{m(k)-1}, x_{n(k)})] - \phi(a_k, b_k)\\ &\leq 2^{-1}[\sigma(x_{n(k)-1}, x_{m(k)}) + \sigma(x_{m(k)-1}, x_{n(k)})].\end{aligned}$$

By letting $k \to \infty$ and taking into account (2.12) and (2.13), we get

$$\varepsilon \leq 2^{-1}[\varepsilon + \varepsilon] - \lim_{k\to\infty} \phi(a_k, b_k) \leq \varepsilon.$$

Thus,

$$\lim_{k\to\infty} \phi(a_k, b_k) = 0.$$

Also

$$\lim_{k\to\infty} (a_k + b_k) = 2\varepsilon. \tag{2.14}$$

By (2.14), $\{a_k\}$ and $\{b_k\}$ are bounded in $[0,\infty)$. Then, $\{(a_k, b_k)\}$ is bounded in $[0,\infty) \times [0,\infty)$. Consequently, there exists a subsequence of $\{(a_k, b_k)\}$ denoted by $\{(a_{n_k}, b_{n_k})\}$ and to be convergent. It follows that $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are convergent. Again, by (2.14), we have

$$\lim_{k\to\infty} \phi(a_{n_k}, b_{n_k}) = \lim_{k\to\infty} \phi(a_k, b_k) = 0.$$

Since ϕ is continuous, then

$$\phi(\lim_{k\to\infty} a_{n_k}, \lim_{k\to\infty} b_{n_k}) = 0.$$

From the fact that $\phi(x, y) = 0$ if and only if $x = y = 0$, we obtain

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} b_{n_k} = 0. \tag{2.15}$$

By (2.14) and (2.15),

$$2\varepsilon = \lim_{k\to\infty} (a_k + b_k) = \lim_{k\to\infty} (a_{n_k} + b_{n_k}) = 0.$$

This yields $\varepsilon = 0$, which is a contradiction. This completes the proof of (2.9). It follows that $\{x_n\}$ is a Cauchy sequence in A. Since A is a closed subset of the complete metric-like space (X, σ), then there exists $u \in A$ such that $x_n \to u$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \sigma(x_n, u) = \sigma(u, u) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.16}$$

We have obtained $\sigma(u, u) = 0$. Thus, by condition (v), $\alpha(u, u) \geq 1$. Consequently, from condition (iv),

$$\sigma(Tu, Tu) \leq \alpha(u, u)\sigma(Tu, Tu) \leq \sigma(u, Tu) - \sigma(A, B) - \phi\Big(\sigma(u, Tu) - \sigma(A, B), \sigma(u, Tu) - \sigma(A, B)\Big).$$

The mapping T is continuous at u, so

$$\lim_{n\to\infty} \sigma(Tx_n, Tu) = \sigma(Tu, Tu). \tag{2.17}$$

On the other hand, by triangular inequality,

$$\begin{aligned}\sigma(u, Tu) \leq& \sigma(u, x_{n+1}) + \sigma(x_{n+1}, Tx_n) + \sigma(Tx_n, Tu)\\ =& \sigma(u, x_{n+1}) + \sigma(A, B) + \sigma(Tx_n, Tu).\end{aligned}$$

By letting $n \to \infty$ in above inequalities, by (2.16) and (2.17),

$$\sigma(u, Tu) \leq \sigma(A, B) + \sigma(Tu, Tu).$$

Then

$$\begin{aligned}\sigma(u, Tu) - \sigma(A, B) \leq \sigma(Tu, Tu) &\leq \sigma(u, Tu) - \sigma(A, B)\\ &\quad - \phi\Big(\sigma(u, Tu) - \sigma(A, B), \sigma(u, Tu) - \sigma(A, B)\Big)\\ &\leq \sigma(u, Tu) - \sigma(A, B).\end{aligned}$$

We deduce that $\phi\Big(\sigma(u,Tu)-\sigma(A,B),\sigma(u,Tu)-\sigma(A,B)\Big)=0$. Again from the fact that $\phi(x,y)=0$ if and only if $x=y=0$, we obtain $\sigma(u,Tu)-\sigma(A,B)=0$, that is, $\sigma(A,B)=\sigma(u,Tu)$, i.e., u is a best proximity point of T.

□

Theorem 2.6. *Let A and B be nonempty closed subsets of a complete metric-like space (X,σ) such that $A_0\neq\emptyset$. Let $T:A\to B$ be a given non-self-mapping. Suppose that*

(i) *$T(A_0)\subseteq B_0$ and (A,B) satisfies the weak (P)-property;*
(ii) *T is triangular α-proximal admissible;*
(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1,Tx_0)=\sigma(A,B),\quad \textit{and}\quad \alpha(x_0,x_1)\geq 1;$$

(iv) *T is an α-proximal C-contraction;*
(v) *(H) holds.*

Then, there exists $u\in A$ such that u is a best proximity point of T and $\sigma(u,u)=0$.

Proof. By following the proof of Theorem 2.5, there exists a sequence $\{x_n\}$ in A_0 such that

$$\sigma(x_{n+1},Tx_n)=\sigma(A,B),\quad \text{and}\quad \alpha(x_n,x_m)\geq 1,\quad \text{for all } m>n\geq 0.$$

Also, $\{x_n\}$ is Cauchy in the subset A, which is closed in the complete metric-like space (X,σ), then there exists $u\in A$ such that $x_n\to u$ as $n\to\infty$. By hypothesis (H), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},u)\geq 1$ for all k. Now, from condition (iv), we have

$$\begin{aligned}\sigma(Tx_{n(k)},Tu)&\leq \alpha(x_{n(k)},u)\sigma(Tx_{n(k)},Tu)\\&\leq 2^{-1}[\sigma(x_{n(k)},Tu)+\sigma(u,Tx_{n(k)})-2\sigma(A,B)]\\&\quad-\phi(\sigma(x_{n(k)},Tu)-\sigma(A,B),\sigma(u,Tx_{n(k)})-\sigma(A,B))\\&\leq 2^{-1}[\sigma(x_{n(k)},u)+\sigma(u,Tu)+\sigma(u,x_{n(k)+1})-\sigma(A,B)].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sigma(A,B)\leq\sigma(u,Tu)&\leq\sigma(u,x_{n(k)+1})+\sigma(x_{n(k)+1},Tx_{n(k)})+\sigma(Tx_{n(k)},Tu)\\&=\sigma(u,x_{n(k)+1})+\sigma(A,B)+\sigma(Tx_{n(k)},Tu).\end{aligned}$$

Therefore,

$$\begin{aligned}\sigma(A,B)\leq\sigma(u,Tu)\leq{}&\sigma(u,x_{n(k)+1})+\sigma(A,B)\\&+2^{-1}[\sigma(x_{n(k)},u)+\sigma(u,Tu)+\sigma(u,x_{n(k)+1})-\sigma(A,B)].\end{aligned}$$

By (2.5), as $n\to\infty$, we get

$$\sigma(A,B)\leq\sigma(u,Tu)\leq\sigma(A,B)+2^{-1}[\sigma(u,Tu)-\sigma(A,B)]=2^{-1}[\sigma(u,Tu)+\sigma(A,B)].$$

Hence

$$\sigma(A,B)\leq\sigma(u,Tu)\leq\sigma(A,B),$$

that is, $\sigma(A,B)=\sigma(u,Tu)$, i.e., u is a best proximity point of T. □

Theorem 2.7. *By adding condition* (U) *to the hypotheses of Theorem* 2.5 *(resp. Theorem* 2.6*), we obtain that u is the unique best proximity point of T.*

Proof. Suppose that there exist $u, v \in A$ such that $\sigma(A,B) = \sigma(u,Tu) = \sigma(v,Tv)$. By assumption (U), we have $\alpha(u,v) \geq 1$. So, as the pair (A,B) satisfies the weak (P)-property, then by (1.2), we have

$$\begin{aligned}\sigma(u,v) \leq \sigma(Tu,Tv) \leq \alpha(u,v)\sigma(Tu,Tv) &\leq 2^{-1}[\sigma(u,Tv) + \sigma(Tu,v) - 2\sigma(A,B)] \\ &\quad - \phi(\sigma(u,Tv) - \sigma(A,B), \sigma(v,Tu) - \sigma(A,B)) \\ &\leq 2^{-1}[\sigma(u,v) + \sigma(v,Tv) + \sigma(v,u) + \sigma(u,Tu) - 2\sigma(A,B)] \\ &\quad - \phi(\sigma(u,Tv) - \sigma(A,B), \sigma(v,Tu) - \sigma(A,B)) \\ &= \sigma(u,v) - \phi(\sigma(u,Tv) - \sigma(A,B), \sigma(v,Tu) - \sigma(A,B)) \\ &\leq \sigma(u,v).\end{aligned}$$

Therefore,

$$\phi(\sigma(u,Tv) - \sigma(A,B), \sigma(v,Tu) - \sigma(A,B)) = 0.$$

From the fact that $\phi(x,y) = 0$ iff $x = y = 0$, we obtain $\sigma(u,Tv) - \sigma(A,B) = 0$ and $\sigma(v,Tu) - \sigma(A,B) = 0$. Then, we have $\sigma(u,Tu) = \sigma(v,Tu) = \sigma(A,B)$, and since the pair (A,B) satisfies the weak (P)-property, then

$$\sigma(u,v) \leq \sigma(Tu,Tu).$$

Also

$$\begin{cases} \alpha(u,v) \geq 1, \\ \sigma(u,Tu) = \sigma(A,B), \\ \sigma(u,Tv) = \sigma(A,B). \end{cases}$$

The mapping T is α-proximal admissible, then $\alpha(u,u) \geq 1$. It follows from (1.2) that

$$\begin{aligned}\sigma(u,v) \leq \sigma(Tu,Tu) \leq \alpha(u,u)\sigma(Tu,Tu) &\leq 2^{-1}[2\sigma(u,Tu) - 2\sigma(A,B)] \\ &\quad - \phi(\sigma(u,Tu) - \sigma(A,B), \sigma(u,Tu) - \sigma(A,B)) \\ &= -\phi(0,0) = 0.\end{aligned}$$

This yields that $\sigma(u,v) = 0$ and so, $u = v$. □

The following example illustrates Theorem 2.6.

Example 2.8. Let $X = [0,\infty) \times [0,\infty)$ endowed with the metric-like $\sigma : X \times X \to [0,\infty)$ given as

$$\sigma((x_1,x_2),(y_1,y_2)) = \begin{cases} |x_1 - y_1| + |x_2 - y_2|, & \text{if } \quad (x_1,x_2),(y_1,y_2) \in [0,1]^2, \\ x_1 + x_2 + y_1 + y_2, & \text{if not.} \end{cases}$$

It is easy to prove that (X,σ) a complete metric-like space. Take $A = \{0\} \times [0,\infty)$ and $B = \{1\} \times [0,\infty)$. Remark that $\sigma(A,B) = \sigma((0,0),(1,0)) = 1$. Also, $A_0 = \{0\} \times [0,1]$ and $B_0 = \{1\} \times [0,1]$. Consider the mapping $T : A \to B$ defined by

$$T(0,x) = (1, \frac{x}{4}), \quad \forall\, x \geq 0.$$

We have $T(A_0) \subseteq B_0$. Now, let $(0,x_1),(0,x_2) \in A$ and $(1,u_1),(1,u_2) \in B$ such that

$$\begin{cases} \sigma((0,x_1),(1,u_1)) = \sigma(A,B) = 1, \\ \sigma((0,x_2),(1,u_2)) = \sigma(A,B) = 1. \end{cases}$$

Necessarily, $(x_1 = u_1 \in [0,1])$ and $(x_2 = u_2 \in [0,1])$. In this case,

$$\sigma((0,x_1),(0,x_2)) = \sigma((1,u_1),(1,u_2)),$$

that is, the pair (A, B) has the weak (P)-property.

Take $\phi(u,v)=\frac{1}{20}(u+v)$ for all $u,v\geq 0$. Define $\alpha : X\times X\to[0,\infty)$ as follows

$$\begin{cases}\alpha((x,y),(s,t))=1, & \text{if} \quad (x,y),(s,t)\in[0,1]\times[0,1],\\ \alpha((x,y),(s,t))=0, & \text{if not.}\end{cases}$$

Let $(0,x_1),(0,x_2),(0,u_1)$ and $(0,u_2)$ in A such that

$$\begin{cases}\alpha((0,x_1),(0,x_2))\geq 1,\\ \sigma((0,u_1),T(0,x_1))=\sigma(A,B)=1,\\ \sigma((0,u_2),T(0,x_2))=\sigma(A,B)=1.\end{cases}$$

Then, necessarily, $(x_1,x_2)\in[0,1]\times[0,1]$. Also, we have ($u_1=\frac{x_1}{4}$ and $u_2=\frac{x_2}{4}$). So

$$\alpha(u_1,u_2)\geq 1,$$

that is, T is α-proximal admissible. Moreover, the condition (T_2) in Definition 1.9 is satisfied, so the mapping $T : A\to B$ is triangular α-proximal admissible.

Let $(0,x)$ and $(0,y)\in A$ such that $\alpha(x,y)=1$. Then, $x,y\in[0,1]$. In this case, we have

$$\begin{aligned}\alpha((0,x),(0,y))\sigma(T(0,x),T(0,y)) &= \sigma(T(0,x),T(0,y))\\ &= \sigma((1,\frac{x}{4}),(1,\frac{y}{4}))\\ &= |\frac{x}{4}-\frac{y}{4}|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\tfrac{\sigma((0,x),T(0,y))+\sigma((0,y),T(0,x))-2\sigma(A,B)}{2}-\phi\Big(\sigma((0,x),T(0,y))-\sigma(A,B),\sigma((0,y),T(0,x))-\sigma(A,B)\Big)\\ &=\tfrac{\sigma((0,x),(1,\frac{y}{4}))+\sigma((0,y),(1,\frac{x}{4}))-2}{2}-\phi\Big(\sigma((0,x),(1,\tfrac{y}{4}))-1,\sigma((0,y),(1,\tfrac{x}{4}))-1\Big)\\ &=\tfrac{1}{2}\Big(1+|x-\tfrac{y}{4}|+1+|y-\tfrac{x}{4}|-2\Big)-\phi\Big(1+|x-\tfrac{y}{4}|-1,1+|y-\tfrac{x}{4}|-1\Big)\\ &=\tfrac{1}{2}\Big(|x-\tfrac{y}{4}|+|y-\tfrac{x}{4}|\Big)-\phi(|x-\tfrac{y}{4}|,|y-\tfrac{x}{4}|).\end{aligned}$$

Without loss of generality, take $x\leq y$. We have the following cases:

Case 1: If $x\leq\frac{y}{4}$, we have

$$\begin{aligned}&\tfrac{\sigma((0,x),T(0,y))+\sigma((0,y),T(0,x))-2\sigma(A,B)}{2}-\phi\Big(\sigma((0,x),T(0,y))-\sigma(A,B),\sigma((0,y),T(0,x))-\sigma(A,B)\Big)\\ &=\tfrac{1}{2}\Big(\tfrac{y}{4}-x+y-\tfrac{x}{4}\Big)-\phi\Big(\tfrac{y}{4}-x,y-\tfrac{x}{4}\Big)\\ &=\tfrac{5}{8}y-\tfrac{5}{8}x-\tfrac{1}{20}(\tfrac{5}{4}y-\tfrac{5}{4}x)\\ &=\tfrac{9}{16}(y-x).\end{aligned}$$

We deduce from above that (1.2) holds.

Case 2: If $x\geq\frac{y}{4}$, we have

$$\begin{aligned}&\tfrac{\sigma((0,x),T(0,y))+\sigma((0,y),T(0,x))-2\sigma(A,B)}{2}-\phi\Big(\sigma((0,x),T(0,y))-\sigma(A,B),\sigma((0,y),T(0,x))-\sigma(A,B)\Big)\\ &=\tfrac{1}{2}\Big(x-\tfrac{y}{4}+y-\tfrac{x}{4}\Big)-\phi\Big(x-\tfrac{y}{4},y-\tfrac{x}{4}\Big)\\ &=\tfrac{3}{8}x-\tfrac{3}{8}y-\tfrac{1}{20}(\tfrac{3}{4}x-\tfrac{3}{4}y)\\ &=\tfrac{27}{80}(y-x).\end{aligned}$$

Again, (1.2) holds.

We conclude that (1.2) is satisfied for all $x,y\in A$. Moreover, the conditions (H) and (iii) in Theorem 2.6 are verified. So, T has a best proximity point which is $u=(0,0)$. It is also unique and verifies $\sigma(u,u)=0$.

3. Consequences

In this paragraph, we present some consequences on our obtained results.

3.1. Some classical best proximity point results

We have the following results.

Corollary 3.1. *Let A and B be nonempty closed subsets of a complete metric-like space (X,σ) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-mapping such that*

$$\sigma(Tx,Ty) \leq \psi(\sigma(x,y))$$

for all $x,y \in A$, where $\psi \in \Psi$. Suppose that

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the weak (P)-property;*

(ii) *there exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1,Tx_0) = \sigma(A,B);$$

(iv) *T is continuous.*

Then, there exists a unique $u \in A$ such that

$$\sigma(u,Tu) = \sigma(A,B), \quad \text{and} \quad \sigma(u,u) = 0.$$

Proof. It suffices to take $\alpha(x,y) = 1$ in Theorem 2.1. The uniqueness of u holds since (U) is satisfied. □

Corollary 3.2. *Let A and B be nonempty closed subsets of a complete metric-like space (X,σ) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-mapping such that*

$$\sigma(Tx,Ty) \leq \frac{\sigma(x,Ty)+\sigma(y,Tx)-2\sigma(A,B)}{2} - \phi(\sigma(x,Ty)-\sigma(A,B),\sigma(y,Tx)-\sigma(A,B))$$

for all $x,y \in A$, where $\phi \in \Phi$. Suppose that

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the weak (P)-property;*

(ii) *there exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1,Tx_0) = \sigma(A,B).$$

Then, there exists $u \in A$ such that

$$\sigma(u,Tu) = \sigma(A,B), \quad \text{and} \quad \sigma(u,u) = 0.$$

Proof. It suffices to take $\alpha(x,y) = 1$ in Theorem 2.6. □

3.2. Some best proximity results on a metric-like endowed with a partial order

Let (X,σ) be a metric-like space endowed with a partial order $\leq$. We introduce the following definition.

Definition 3.3. Let A and B be nonempty subsets of a metric-like space (X,σ) and $\leq$ a partial order on X, $T : A \to B$ is named a proximal nondecreasing mapping if

$$\begin{cases} x_1 \leq x_2, \\ \sigma(u_1,Tx_1) = \sigma(A,B), \quad & \Rightarrow u_1 \leq u_2, \\ \sigma(u_2,Tx_2) = \sigma(A,B), \end{cases}$$

for all $x_1,x_2,u_1,u_2 \in A$.

Wa also need the following hypothesis.

(H$_1$) if $\{x_n\}$ is a sequence in A such that $x_n \leq x_{n+1}$ for all n and $x_n \to x \in A$, as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \leq x$ for all k.

We state the following.

Corollary 3.4. *Let A and B be nonempty closed subsets of a complete metric-like space (X, σ) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-mapping such that*

$$\sigma(Tx, Ty) \leq \psi(\sigma(x, y))$$

for all $x, y \in A$ such that $x \leq y$, where $\psi \in \Psi$. Suppose that

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak (P)-property;*
(ii) *T is a proximal nondecreasing mapping;*
(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1, Tx_0) = \sigma(A, B), \quad \text{and} \quad x_0 \leq x_1;$$

(iv) *T is continuous or (H_1) holds.*

Then, there exists $u \in A$ such that

$$\sigma(u, Tu) = \sigma(A, B), \quad \text{and} \quad \sigma(u, u) = 0.$$

Proof. It suffices to consider $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x \leq y, \\ 0 & \text{if not.} \end{cases}$$

All hypotheses of Theorem 2.1 (resp. Theorem 2.2) are satisfied. This completes the proof. □

Similar to Corollary 3.4, we may state:

Corollary 3.5. *Let A and B be nonempty closed subsets of a complete metric-like space (X, σ) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-mapping such that*

$$\sigma(Tx, Ty) \leq \frac{\sigma(x, Ty) + \sigma(y, Tx) - 2\sigma(A, B)}{2} - \phi(\sigma(x, Ty) - \sigma(A, B), \sigma(y, Tx) - \sigma(A, B))$$

for all $x, y \in A$ such that $x \leq y$, where $\phi \in \Phi$. Suppose that

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak (P)-property;*
(ii) *T is a proximal nondecreasing mapping;*
(ii) *There exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1, Tx_0) = \sigma(A, B), \quad \text{and} \quad x_0 \leq x_1;$$

(iv) *T is continuous or* (H$_1$) *holds.*

Then, there exists $u \in A$

$$\sigma(u, Tu) = \sigma(A, B), \quad \text{and} \quad \sigma(u, u) = 0.$$

3.3. Some best proximity results on a metric-like with a graph

Let (X,σ) be a metric-like space and let $G=(V(G),E(G))$ be a directed graph such that $V(G)=X$ and $E(G)$ contains all loops, i.e., $\Delta:=\{(x,x):x\in X\}\subset E(G)$. We need in the sequel the following hypothesis:

($\mathrm{H_G}$) if $\{x_n\}$ is a sequence in A, such that $(x_n,x_{n+1})\in E(G)$ for all n and $x_n\to x\in A$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $(x_{n(k)},x)\in E(G)$ for all k.

Again, we introduce the following definition.

Definition 3.6. Let A and B be nonempty subsets of a metric-like space (X,σ) endowed with a graph G. $T:A\to B$ is named a G-proximal mapping if

$$\begin{cases}(x_1,x_2)\in E(G),\\ \sigma(u_1,Tx_1)=\sigma(A,B), \qquad \Rightarrow (u_1,u_2)\in E(G),\\ \sigma(u_2,Tx_2)=\sigma(A,B),\end{cases}$$

for all $x_1,x_2,u_1,u_2\in A$.

We also introduce the following.

Definition 3.7. Let A and B be nonempty subsets of a metric-like space (X,σ) and $\alpha:X\times X\to[0,\infty)$. A mapping $T:A\to B$ is named triangular G-proximal admissible if

($\mathrm{T_1}$) T is G-proximal admissible;

($\mathrm{T_2}$) $(x,y)\in E(G)$ and $(y,z)\in E(G)\Rightarrow(x,z)\in E(G)$, $x,y,z\in A$.

We have the two following best proximity point results on a metric-like endowed with a graph.

Corollary 3.8. *Let A and B be nonempty closed subsets of a complete metric-like space (X,σ) such that $A_0\neq\emptyset$. Let $T:A\to B$ be a given non-self-mapping such that*

$$\sigma(Tx,Ty)\le\psi(\sigma(x,y))$$

for all $x,y\in A$ such that $(x,y)\in E(G)$, where $\psi\in\Psi$. Suppose that

(i) *$T(A_0)\subseteq B_0$ and (A,B) satisfies the weak (P)-property;*
(ii) *T is a G-proximal mapping;*
(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1,Tx_0)=\sigma(A,B), \quad \text{and} \quad (x_0,x_1)\in E(G);$$

(iv) *T is continuous or (H_G) holds.*

Then, there exists $u\in A$ such that

$$\sigma(u,Tu)=\sigma(A,B) \quad \text{and} \quad \sigma(u,u)=0.$$

Proof. It suffices to consider $\alpha:X\times X\to[0,\infty)$ such that

$$\alpha(x,y)=\begin{cases}1 & \text{if} \quad (x,y)\in E(G),\\ 0 & \text{if not.}\end{cases}$$

All hypotheses of Theorem 2.1 (resp. Theorem 2.2) are satisfied. This completes the proof. □

Similar to Corollary 3.8, we may state the following.

Corollary 3.9. *Let A and B be nonempty closed subsets of a complete metric-like space (X,σ) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a given non-self-mapping such that*

$$\sigma(Tx,Ty) \leq \frac{\sigma(x,Ty)+\sigma(y,Tx)-2\sigma(A,B)}{2} - \phi(\sigma(x,Ty)-\sigma(A,B), \sigma(y,Tx)-\sigma(A,B))$$

for all $x, y \in A$ such that $(x,y) \in E(G)$, where $\phi \in \Phi$. Suppose that

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the weak (P)-property;*

(ii) *T is a triangular G-proximal mapping;*

(ii) *there exist elements x_0 and x_1 in A_0 such that*

$$\sigma(x_1,Tx_0) = \sigma(A,B), \quad \text{and} \quad (x_0,x_1) \in E(G);$$

(iv) *T is continuous or (H_G) holds.*

Then, there exists $u \in A$

$$\sigma(u,Tu) = \sigma(A,B), \quad \text{and} \quad \sigma(u,u) = 0.$$

References

[1] A. Amini-Harandi, *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory Appl., **2012** (2012), 10 pages. 1

[2] H. Aydi, A. Felhi, E. Karapinar, S. Sahmim, Hausdorff metric-like, generalized Nadler's fixed point theorem on metric-like spaces and application, Micolc Math. Notes, (In press). 1

[3] H. Aydi, A. Felhi, S. Sahmim, *Fixed points of multivalued nonself almost contractions in metric-like spaces*, Math. Sci. (Springer), **9** (2015), 103–108.

[4] H. Aydi, E. Karapinar, *Fixed point results for generalized α-ψ-contractions in metric-like spaces and applications*, Electron. J. Differential Equations, **2015** (2015), 15 pages.

[5] C. Chen, J. Dong, C. Zhu, *Some fixed point theorems in b-metric-like spaces*, Fixed Point Theory Appl., **2015** (2015), 10 pages.

[6] M. Cvetković, E. Karapınar, V. Rakocević, *Some fixed point results on quasi-b-metric-like spaces*, J. Inequal. Appl., **2015** (2015), 17 pages. 1

[7] A. Felhi, H. Aydi, *Best proximity points and stability results for controlled proximal contractive set valued mappings*, Fixed Point Theory Appl., **2016** (2016), 23 pages. 1

[8] M. Jleli, E. Karapınar, B. Samet, *Best proximity points for generalized α-ψ-proximal contractive type mappings*, J. Appl. Math., **2013** (2013), 10 pages. 1

[9] S. Karpagam, S. Agrawal, *Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps*, Nonlinear Anal., **74** (2011), 1040–1046. 1

[10] W. K. Kim, S. Kum, K. H. Lee, *On general best proximity pairs and equilibrium pairs in free abstract economies*, Nonlinear Anal., **68** (2008), 2216–2227.

[11] W. A. Kirk, S. Reich, P. Veeramani, *Proximinal retracts and best proximity pair theorems*, Numer. Funct. Anal. Optim., **24** (2003), 851–862.

[12] C. Mongkolkeha, P. Kumam, *Best proximity point theorems for generalized cyclic contractions in ordered metric spaces*, J. Optim. Theory Appl., **155** (2012), 215–226.

[13] H. K. Nashine, P. Kumam, C. Vetro, *Best proximity point theorems for rational proximal contractions*, Fixed Point Theory Appl., **2013** (2013), 11 pages. 1

[14] V. S. Raj, P. Veeramani, *best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal., **74** (2011), 4804–4808. 1

[15] S. Sadiq Basha, P. Veeramani, *Best proximity pairs and best approximations*, Acta Sci. Math. (Szeged), **63** (1997), 289–300. 1

[16] S. Sadiq Basha, P. Veeramani, *Best proximity pair theorems for multifunctions with open fibres*, J. Approx. Theory, **103** (2000), 119–129. 1

[17] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha\psi$-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 1

[18] J. Zhang, Y. Su, Q. Cheng, *A note on 'A best proximity point theorem for Geraghty-contractions'*, Fixed Point Theory Appl., **2013** (2013), 4 pages. 1