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A new integrable symplectic map and the lie point symmetry associated with nonlinear lattice equations

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Abstract

A discrete matrix spectral problem is proposed, the hierarchy of discrete integrable system is inferred, which are Liouville integrable. And the Hamiltonian structures of the hierarchy are constructed. A family of finite-dimensional completely integrable systems and a new integrable symplectic map are provided in terms of the binary nonlinearity of spectral problem. In particular, two explicit formulations are acquired under the condition of the bargmann constraints. After that, the symmetry of the discrete integrable systems is given on the basis of the seed symmetry and its prolongation. Moreover, the solution of the discrete lattice equation can be gained by the way of the infinitesimal generator.

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1. Introduction

In the last several years, with the deepening of theoretical research on the discrete integrable systems such as the Toda lattice, Ablowitz-Ladik lattice the differential-difference KdV equation and so on [6, 9, 10, 19, 21, 23, 26], many people have made many outstanding research results which are widely used in photology and hydromechanics. After studying the integrable systems, we find that the discrete integrable systems can better explain the natural phenomenon than the continuous integrable system from the aspects of nature. There are two very important issues, one of which is to find a new Lax integrable nonlinear lattice systems and discuss their Hamiltonian structures [1, 4, 5, 11, 14, 17, 18, 22] and the other is to obtain integrable symplectic map, which has been proposed and developed in Refs. [3, 20, 24, 25, 27]. We can

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solve the discrete integrable system through various methods, such as Bäcklund transformation, Dârboux transformation, Hirota approach, the inverse scattering method, etc. Lie symmetry provides a systematic approach to the purpose of reducing the order of the differential equations. Usually, the standard method is used to solve the differential equation. However, using lie point symmetry can simplify solution, and it can also be used to solve the equation. For differential equations, it is important to study the group-invariant solution and symmetry reduction, because it provides a powerful tool for the study of differential equations. And for some equations, it can greatly reduce the calculation of the equation, which can be used to solve the equation. In this paper, we will use the symmetry theory to solve the discrete integrable systems [2, 7, 8, 12, 13, 15, 16].

General discrete integrable systems are as follows:

$$E_a(x_{n-1}, x_n, x_{n+1}, u_{n-1}, u_n, u_{n+1}) = 0, a = 1, 2, \cdots, N, \tag{1.1}$$

where

$$\det\left(\frac{\partial(E_a, E_b)}{\partial(x_{n+1}, u_{n+1})}\right) \neq 0, \quad \det\left(\frac{\partial(E_a, E_b)}{\partial(x_{n-1}, u_{n-1})}\right) \neq 0.$$

Set infinitesimal generator ν as follows:

$$\nu = \sum_{i=1}^{p} \xi_i(x, u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u_\alpha},$$

and

$$Pr^{(n)}\nu = \nu + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J \frac{\partial}{\partial u_\alpha^J}.$$

Here $Pr^{(n)}\nu$ is the infinitesimal generator of nth order prolonged space, where n indicates the highest order, in which

$$\phi_\alpha^J(x, u^{(n)}) = D_J(\phi_\alpha - \sum_{i=1}^{p} \xi_i u_{\alpha,i}) + \sum_{i=1}^{p} \xi_i u_{\alpha,i}^J,$$

where u_α^J,$u_{\alpha,i}$ satisfies $u_\alpha^J \in U^{(n)}(n \geq 1)$, $u_{\alpha,i} = \partial u_\alpha/\partial x_i$; and operator D is the total derivative, which is the differential operator of prolonged space,

$$D_i P = \frac{\partial P}{\partial x_i} + \sum_{\alpha=1}^{q} \frac{\partial P}{\partial u_\alpha} \cdot \frac{\partial u_\alpha}{\partial x_i} + \sum_{\alpha=1}^{q}\sum_{J} \frac{\partial P}{\partial u_\alpha^J} \cdot u_{\alpha,i}^J,$$

where $J = (J_1, \cdots, J_k)$ and $D_J = D_{J_1} D_{J_2} \cdots D_{J_k}$.

Substituting prolongation operator $Pr^{(n)}\nu$ into Eq. (1.1) and handling the coefficients of all $u_n, u_{n+1}\cdots$, we gain a linear independent expression and get the extension of the solution through setting the coefficient as zero.

In this paper, we would like to consider a new hierarchy of integrable nonlinear lattice equation which is inferred from a new discrete spectrum problem. Then, we will construct its Hamiltonian structure and test its properties of Liouville integrability. After that, a new integrable symplectic map and a family of finite-dimension completely integrable systems would be given according to the binary nonlinearization of the spectral problem. At the end of the paper, we use the symmetry theory and the gâteaux derivative to solve the symmetry and the infinitesimal generator of the discrete Lattice equation. We explain our results by some figures.

2. A new discrete integrable hierarchy and its Hamiltonian structure

Consider the following discrete spectrum problem,

$$E\varphi = U\varphi, \quad U = U(u,\lambda) = \begin{pmatrix} \lambda & \lambda q \\ \frac{1}{p} & 1+\frac{q}{p} \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{2.1}$$

The shift operator and the difference operator are defined as follows

$$\begin{aligned} &(Ef)(n) = f(n+1), \ (E^{-1}f)(n) = f(n-1), \ n \in Z, \\ &(Df)(n) = f(n+1) - f(n) = (E-1)f(n), \quad n \in Z, \end{aligned}$$

where $f^{(j)} = E^j f, \ j \in Z$, λ is the spectral parameter, $\lambda_t = 0$.

We give a static discrete zero curvature equation for obtaining the discrete integrable systems

$$(E\Gamma)U - U\Gamma = 0, \tag{2.2}$$

and choose

$$\Gamma = \begin{pmatrix} A & \lambda B \\ C & -A \end{pmatrix}.$$

We have four equations from Eq. (2.2) as follows:

$$\begin{cases} \lambda A^{(1)} + \lambda\frac{1}{p}B^{(1)} - \lambda A - \lambda q C = 0, \\ \lambda q A^{(1)} + \lambda(1+\frac{q}{p})B^{(1)} - \lambda^2 B + \lambda q A = 0, \\ \lambda C^{(1)} - \frac{1}{p}(A^{(1)} + A) - (1+\frac{q}{p})C = 0, \\ \lambda q C^{(1)} - \lambda\frac{1}{p}B - (1+\frac{q}{p})(A^{(1)} - A) = 0. \end{cases} \tag{2.3}$$

Furthermore, substituting $A = \sum\limits_{m=0}^{\infty} A_m\lambda^{-m}, B = \sum\limits_{m=0}^{\infty} B_m\lambda^{-m}, C = \sum\limits_{m=0}^{\infty} C_m\lambda^{-m}$ into Eq. (2.3), we have

$$C_0^{(1)} = 0, \ B_0 = 0, \ A_0^{(1)} - A_0 + \frac{1}{p}B_0^{(1)} - qC_0 = 0,$$

and

$$\begin{cases} A_m^{(1)} - A_m + \frac{1}{p}B_m^{(1)} - qC_m = 0, \\ q(A_m^{(1)} + A_m) + (1+\frac{q}{p})B_m^{(1)} - B_{m+1} = 0, \\ qC_{m+1}^{(1)} - \frac{1}{p}(A_m^{(1)} + A_m) - (1+\frac{q}{p})C_m = 0, \\ qC_{m+1}^{(1)} - \frac{1}{p}B_{m+1} - (1+\frac{q}{p})(A_m^{(1)} - A_m) = 0, \ m \geq 0. \end{cases} \tag{2.4}$$

By setting $A_0 = \dfrac{1}{2}, B_0 = 0$, the coefficients $A_m, B_m, C_m, (m \geq 1)$ can be obtained according to the Eq. (2.4). A set of coefficients are as follows

$$A_1 = -\frac{q}{p^{(-1)}}, \quad B_1 = q, \quad C_1 = \frac{1}{p^{(-1)}}, \cdots.$$

For any integer $m \geq 0$, we let $f = \sum\limits_{m\in Z} f_m\lambda^m$ and denote $f_+ = \sum\limits_{m\geq 0} f_m\lambda^m$, choose

$$\Gamma_+^{(n)} = \sum_{m=0}^{n} \begin{pmatrix} A_m & \lambda B_m \\ C_m & -A_m \end{pmatrix} \lambda^{n-m}, \Gamma_-^{(n)} = \lambda^n\Gamma - \Gamma_+^{(n)},$$

and rewrite Eq. (2.2) into

$$(\Gamma_+^{(n)})U - U\Gamma_+^{(n)} = -(\Gamma_-^{(n)})U + U\Gamma_-^{(n)}.$$

A direct calculation reads that

$$(\Gamma_+^{(n)})U - U\Gamma_+^{(n)} = \begin{pmatrix} 0 & \lambda B_{n+1} \\ -C_{n+1}^{(1)} & \frac{1}{p}B_{n+1} - qC_{n+1}^{(1)} \end{pmatrix}.$$

Let $\Gamma^{(n)} = \Gamma^{(n)}_{+}$, then the discrete zero curvature equation meets the following Lax integrable system

$$\begin{cases} p_t = C^{(1)}_{n+1} p^2, \\ q_t = B_{n+1}. \end{cases} \tag{2.5}$$

According to Eq. (2.4), we acquire the recurrence operator L as follows

$$L = \begin{pmatrix} E^{-1}(1+\frac{q}{p}) + \frac{q}{p}(E+1)(E-1)^{-1}E^{-1} & E^{-1}(E+1)(E-1)^{-1} \\ -\frac{q^2}{p^2}(E+1)(E-1)^{-1} & (1+\frac{q}{p}) + \frac{q}{p}(E+1)(E-1)^{-1} \end{pmatrix}.$$

We rewrite System (2.5) into

$$\begin{pmatrix} q \\ p \end{pmatrix}_{nt} = J \begin{pmatrix} C_{n+1} \\ -\frac{B^{(1)}_{n+1}}{p^2} \end{pmatrix} = JL^n \begin{pmatrix} C_1 \\ -\frac{B^{(1)}_1}{p^2} \end{pmatrix} = JL^n \begin{pmatrix} \frac{1}{p^{(-1)}} \\ -\frac{q^{(1)}}{p^2} \end{pmatrix}, \tag{2.6}$$

where

$$J = p^2 \begin{pmatrix} 0 & -E^{-1} \\ E & 0 \end{pmatrix}.$$

When we take $n = 1$, system (2.6) reduces to

$$\begin{cases} p_t = \frac{p^2}{p^{(-1)}} - q^{(1)}, \\ q_t = q^{(1)} - \frac{q^2}{p^{(-1)}}. \end{cases} \tag{2.7}$$

For purpose of constructing the Hamiltonian structure of system (2.6), we define

$$V = \Gamma U^{-1} = \begin{pmatrix} \lambda^{-1}(1+\frac{q}{p})A - \frac{B}{p} & \lambda B - qA \\ \lambda^{-1}(1+\frac{q}{p})C + \frac{1}{p\lambda}A & -qC - A \end{pmatrix}.$$

We have

$$\frac{\partial U}{\partial \lambda} = \begin{pmatrix} 1 & q \\ 0 & 0 \end{pmatrix}, \frac{\partial U}{\partial p} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{p^2} & -\frac{q}{p^2} \end{pmatrix}, \frac{\partial U}{\partial q} = \begin{pmatrix} 0 & \lambda \\ 0 & \frac{1}{p} \end{pmatrix}.$$

Therefore,

$$\begin{cases} \langle V, \frac{\partial U}{\partial \lambda} \rangle = \frac{1}{\lambda}(1+\frac{q}{p})(A+qC) - \frac{B}{p} + \frac{q}{p\lambda}A, \\ \left\langle V, \frac{\partial U}{\partial p} \right\rangle = \frac{1}{p^2}(2qA - \lambda B + q^2 C), \\ \left\langle V, \frac{\partial U}{\partial q} \right\rangle = C, \end{cases}$$

and $\langle A, B \rangle = Tr(AB)$, where A and B are the same order square matrix. By applying the discrete trace identity

$$\frac{\delta}{\delta u} \sum_{n \in Z} \left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = \left(\lambda^{-\varepsilon} \left(\frac{\partial}{\partial \lambda} \right) \lambda^{\varepsilon} \right) \left\langle V, \frac{\partial U}{\partial u^i} \right\rangle, \ i = 1, 2,$$

we obtain that

$$\frac{\delta}{\delta u} \frac{A}{\lambda} = \lambda^{-\varepsilon} \frac{\partial}{\partial \lambda} \lambda^{\varepsilon} \begin{pmatrix} C \\ -\frac{B^{(1)}}{p^2} \end{pmatrix}.$$

By comparing with the coefficient of λ^{-n-1}, we get

$$\frac{\delta}{\delta u} A_n = (\varepsilon - n) \begin{pmatrix} C_n \\ -\frac{B^{(1)}_n}{p^2} \end{pmatrix}.$$

When $n = 1$, let $\varepsilon = 0$, then we have

$$\frac{\delta}{\delta u} \left(-\frac{A_n}{n} \right) = \begin{pmatrix} C_n \\ -\frac{B^{(1)}_n}{p^2} \end{pmatrix},$$

and

$$\begin{pmatrix} C_{n+1} \\ -\frac{B^{(1)}_{n+1}}{p^2} \end{pmatrix} = \frac{\delta H_n}{\delta u}, H_n = -\frac{A_{n+1}}{n+1},$$

where J is a Hamiltonian operator. Hence, Eq. (2.6) can be written as the Hamiltonian structure.

$$\begin{pmatrix} q \\ p \end{pmatrix}_{nt} = J\frac{\delta H_n}{\delta u}. \tag{2.8}$$

Furthermore, we test the following result

$$(JL)^* = -JL,$$

where

$$\begin{aligned} K = JL &= \begin{pmatrix} q^2(E+1)(E-1)^{-1}E^{-1} & qp(E+1)(E-1)^{-1}E^{-1} - (p^2+qp)E^{-1} \\ qp(E+1)(E-1)^{-1} + (p^2+qp) & p^2(E+1)(E-1)^{-1} \end{pmatrix} \\ \left\{\tilde{H}_m, \tilde{H}_l\right\}_J &= 0,\ m,l \geq 1, \\ (\tilde{H}_m)_{t_l} &= \sum_{n\in Z}\left(\frac{\delta \tilde{H}_m}{\delta u}u_{t_l}\right)(n) = \sum_{n\in Z}(\frac{\delta \tilde{H}_m}{\delta u}, J\frac{\delta \tilde{H}_l}{\delta u}) = \left\{\tilde{H}_m, \tilde{H}_l\right\}_J = 0,\ m,l\geq 1. \end{aligned} \tag{2.9}$$

Hence, the conserved densities $\left\{\tilde{H}_m\right\}_{m=1}^{\infty}$ are the involution with respect to Poisson bracket (2.9) and we conclude that each nonlinear difference-differential equation of the discrete hierarchy is Liouville integrable.

3. A new integrable symplectic map and representation of solutions for Eq. (2.7)

In the subsection, we will discuss the symmetry constraint of Eq. (2.5). Consider the adjoint spectral problem of Eq. (2.1)

$$E^{-1}\psi = (E^{-1}U^T(a,\lambda))\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.1}$$

and the auxiliary problem

$$\psi_{t_m} = -(V_m(a,\lambda))^T\psi. \tag{3.2}$$

In terms of the compatibility condition of Eq. (2.6) and Eq. (2.7) $(E^{-1}\psi)_{t_m} = E^{-1}(\psi_{t_m})$, we gain

$$E^{-1}U^T_{t_m} = (E^{-1}U^T)(V_m)^T - (E^{-1}(V_m)^T)(E^{-1}U^T). \tag{3.3}$$

It is easy to test that Eq. (3.3) and $U_{t_m} = (EV_m)U - UV_m$ are equivalent. Hence, Eq. (3.3) is the another kind of zero curvature representation of the discrete soliton Eq. (2.8), where (3.1) and (3.2) are regarded as the adjoint Lax pairs of discrete soliton Eq. (2.8).

Assuming $\lambda_1, \lambda_2, \cdots, \lambda_N$ are n different eigenvalues of the spectral problem (2.1), we gain

$$\begin{aligned} \begin{pmatrix} E\varphi_{1j} \\ E\varphi_{2j} \end{pmatrix} &= U(a,\lambda_j)\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}, \\ \begin{pmatrix} E^{-1}\psi_{1j} \\ E^{-1}\psi_{2j} \end{pmatrix} &= (E^{-1}U^T(a,\lambda_j))\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, 1\leq j\leq N, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}_{t_m} &= V_m(a,\lambda_j)\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \end{pmatrix}, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_{t_m} &= -V_m^T(a,\lambda_j)\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \qquad 1\leq j\leq N, \end{aligned} \tag{3.5}$$

$$(E\varphi_{1j}, E\varphi_{2j}) = (\varphi_{1j}, \varphi_{2j})U(a,\lambda_j)^T, \quad 1\leq j\leq N,$$

$$(E\psi_{1j}, E\psi_{2j}) = (\psi_{1j}, \psi_{2j})U(a,\lambda_j)^{-1}, \quad 1\leq j\leq N.$$

According to Ref. [3], we infer

$$\begin{aligned}
&\frac{\delta\lambda_j}{\delta\, p} = \frac{q}{p^2}\varphi_{1j}\psi_{1j} - \frac{1}{p^2}\varphi_{1j}\psi_{2j} + \frac{q^2}{p^2}\varphi_{2j}\psi_{1j} - \frac{q}{p^2}\varphi_{1j}\psi_{2j}, \quad 1 \le j \le N,\\
&\frac{\delta\lambda_j}{\delta\, q} = \varphi_{2j}\psi_{1j}, \quad 1 \le j \le N,
\end{aligned}$$

where $\langle .\,,\,.\rangle$ denotes the inner product in R^N. By making use of the discrete Bargmann constraint

$$J\frac{\delta \tilde{H}_0}{\delta u} = J\sum_{j=0}^{N}\frac{\delta\lambda_j}{\delta u},$$

where $\alpha_j = 1,\ 1 \le j \le N$. That is,

$$\begin{aligned}
&\frac{\delta \tilde{H}_0}{\delta\, p} = \frac{q}{p^2}\left\langle \Phi_1, \Psi_1\right\rangle - \frac{1}{p^2}\left\langle \Phi_1, \Psi_2\right\rangle + \frac{q^2}{p^2}\left\langle \Phi_2, \Psi_1\right\rangle - \frac{q}{p^2}\left\langle \Phi_2, \Psi_2\right\rangle,\\
&\frac{\delta \tilde{H}_0}{\delta\, q} = \left\langle \Phi_2, \Psi_1\right\rangle,
\end{aligned}$$

where

$$\Phi_i = (\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{iN})^T, \quad \Psi_i = (\psi_{i1}, \psi_{i2}, \cdots, \psi_{iN})^T, \qquad i = 1, 2.$$

We obtain two explict constraints

$$\begin{cases}
q = \Lambda^{-1}\left\langle \Phi_1, \Psi_2\right\rangle,\\
p = \dfrac{\Lambda^2 + \Lambda\langle\Phi_2, \Psi_2\rangle - \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle - \Lambda\langle\Phi_1, \Psi_1\rangle}{\Lambda\left\langle \Phi_2, \Psi_1\right\rangle},
\end{cases} \tag{3.6}$$

or

$$\begin{cases}
q = \Lambda^{-1}\left\langle \Phi_1, \Psi_2\right\rangle,\\
p = -\Lambda^{-1}\left\langle \Phi_1, \Psi_2\right\rangle.
\end{cases} \tag{3.7}$$

Substituting Eqs. (3.6) and (3.7) into Eq. (3.4), we obtain a discrete Bargmann system

$$\begin{cases}
E\phi_{1j} = \lambda_j\phi_{1j} + \lambda_j q\phi_{2j} = \Lambda\phi_{1j} + \langle\Phi_1, \Psi_2\rangle\phi_{2j},\\
E\phi_{2j} = \dfrac{1}{p}\phi_{1j} + (1 + \dfrac{q}{p})\phi_{2j}\\
\qquad = \Lambda\langle\Phi_2, \Psi_1\rangle(\Lambda^2 - \Lambda\langle\Phi_1, \Psi_1\rangle - \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle + \Lambda\langle\Phi_2, \Psi_2\rangle)^{-1}\phi_{1j}\\
\qquad\quad + (1 + \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle(\Lambda^2 - \Lambda\langle\Phi_1, \Psi_1\rangle - \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle + \Lambda\langle\Phi_2, \Psi_2\rangle)^{-1})\phi_{2j},\\
E\psi_{1j} = \dfrac{1}{\lambda_j}(1 + \dfrac{q}{p})\psi_{1j} - \dfrac{1}{\lambda_j p}\psi_{2j}\\
\qquad = \Lambda^{-1}(1 + \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle(\Lambda^2 - \Lambda\langle\Phi_1, \Psi_1\rangle - \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle\\
\qquad\quad + \Lambda\langle\Phi_2, \Psi_2\rangle)^{-1})\psi_{1j} - \langle\Phi_2, \Psi_1\rangle(\Lambda^2 - \Lambda\langle\Phi_1, \Psi_1\rangle - \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle + \Lambda\langle\Phi_2, \Psi_2\rangle)^{-1}\psi_{2j},\\
E\psi_{2j} = -q\psi_{1j} + \psi_{2j} = -\Lambda^{-1}\langle\Phi_1, \Psi_2\rangle\psi_{1j} + \psi_{2j},
\end{cases} \tag{3.8}$$

or

$$\begin{cases}
E\Phi_{1j} = \lambda_j\phi_{1j} + \lambda_j q\phi_{2j} = \Lambda\phi_{1j} + \langle\Phi_1, \Psi_2\rangle\phi_{2j},\\
E\Phi_{2j} = \dfrac{1}{p}\phi_{1j} + (1 + \dfrac{q}{p})\phi_{2j} = -\Lambda\langle\Phi_1, \Psi_2\rangle^{-1}\phi_{1j},\\
E\Psi_{1j} = \dfrac{1}{\lambda_j}(1 + \dfrac{q}{p})\psi_{1j} - \dfrac{1}{\lambda_j p}\psi_{2j} = -\langle\Phi_1, \Psi_2\rangle^{-1}\psi_{2j},\\
E\Psi_{2j} = -q\psi_{1j} + \psi_{2j} = -\Lambda^{-1}\langle\Phi_1, \Psi_2\rangle\psi_{1j} + \psi_{2j}.
\end{cases} \tag{3.9}$$

Let $f_i = f_i(\Phi_1, \Phi_2, \Psi_1, \Psi_2),\ g_i = g_i(\Phi_1, \Phi_2, \Psi_1, \Psi_2),\ 1 \le i \le 2N$. We present

$$\begin{cases} f_j = \Lambda^{-1}(1 + \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle(\Lambda^2 - \Lambda\langle\Phi_1, \Psi_1\rangle - \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle + \Lambda\langle\Phi_2, \Psi_2\rangle)^{-1})\psi_{1j} \\ \qquad - \langle\Phi_2, \Psi_1\rangle(\Lambda^2 - \Lambda\langle\Phi_1, \Psi_1\rangle - \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle + \Lambda\langle\Phi_2, \Psi_2\rangle)^{-1}\psi_{2j}, \\ f_{N+j} = -\Lambda^{-1}\langle\Phi_1, \Psi_2\rangle\psi_{1j} + \psi_{2j}, \\ g_j = \Lambda\phi_{1j} + \langle\Phi_1, \Psi_2\rangle\phi_{2j}, \\ g_{N+j} = \Lambda\langle\Phi_2, \Psi_1\rangle(\Lambda^2 - \Lambda\langle\Phi_1, \Psi_1\rangle - \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle + \Lambda\langle\Phi_2, \Psi_2\rangle)^{-1}\phi_{1j} \\ \qquad + (1 + \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle(\Lambda^2 - \Lambda\langle\Phi_1, \Psi_1\rangle - \langle\Phi_1, \Psi_2\rangle\langle\Phi_2, \Psi_1\rangle + \Lambda\langle\Phi_2, \Psi_2\rangle)^{-1})\phi_{2j}, \end{cases}$$

or

$$\begin{cases} f_j = -\langle\Phi_1, \Psi_2\rangle^{-1}\psi_{2j}, \\ f_{N+j} = -\Lambda^{-1}\langle\Phi_1, \Psi_2\rangle\psi_{1j} + \psi_{2j}, \\ g_j = \Lambda\phi_{1j} + \langle\Phi_1, \Psi_2\rangle\phi_{2j}, \\ g_{N+j} = -\Lambda\langle\Phi_1, \Psi_2\rangle^{-1}\phi_{1j},\ 1 \le j \le N. \end{cases}$$

We define Poisson bracket as

$$\begin{aligned} \{f\,,\,g\} &= \sum_{i=1}^{2}\sum_{j=1}^{N}\left(\frac{\partial f}{\partial\psi_{ij}}\frac{\partial g}{\partial\varphi_{ij}} - \frac{\partial f}{\partial\varphi_{ij}}\frac{\partial g}{\partial\psi_{ij}}\right) \\ &= \sum_{i=1}^{2}\left(\left\langle\frac{\partial f}{\partial\Psi_i}, \frac{\partial g}{\partial\Phi_i}\right\rangle - \left\langle\frac{\partial f}{\partial\Phi_i}, \frac{\partial g}{\partial\Psi_i}\right\rangle\right) \end{aligned}$$

on any pair of functions $f = f(\Phi_1, \Phi_2, \Psi_1, \Psi_2)$ and $g = g(\Phi_1, \Phi_2, \Psi_1, \Psi_2)$. Let

$$H(\Psi_1, \Psi_2, \Phi_1, \Phi_2) = (E\Psi_1, E\Psi_2, E\Phi_1, E\Phi_2),$$

then, via a direct calculation, we infer

$$\{f_i, f_j\} = \{g_i, g_j\} = 0\,,\ \{f_i, g_j\} = \delta_{ij}, \qquad 1 \le i, j \le 2N,$$

we define Eqs. (3.8) and (3.9) as an integrable symplectic map. Then we take the binary nonlinearization of the Lax pairs and the adjoint Lax pairs into account. According to the Eq. (2.4), the following values could be selected as

$$\begin{cases} \tilde{A}_0 = \frac{1}{2}, \tilde{B}_0 = \tilde{C}_0 = 0, \tilde{A}_1 = 0, \\ \tilde{B_m} = \Lambda^{m-2}\langle\Phi_1, \Psi_2\rangle, \tilde{C_m} = \Lambda^{m-1}\langle\Phi_2, \Psi_1\rangle, \\ \tilde{A_m} = \dfrac{\Lambda^{m-1}\langle\Phi_1, \Psi_1\rangle - \Lambda^{m-1}\langle\Phi_2, \Psi_2\rangle}{2},\ m \ge 1. \end{cases} \tag{3.10}$$

Let

$$\tilde{\Gamma} = \begin{pmatrix} \tilde{A} & \lambda\tilde{B} \\ \tilde{C} & \tilde{-A} \end{pmatrix} = \sum_{m=0}^{\infty}\begin{pmatrix} \tilde{A}_m & \lambda\tilde{B}_m \\ \tilde{C}_m & -\tilde{A}_m \end{pmatrix}\lambda^{-m}$$

and

$$\tilde{F}_m = \det\tilde{\Gamma} = \frac{1}{2}\mathrm{tr}\tilde{\Gamma}^2 = \tilde{A}^2 + \Lambda\tilde{B}\tilde{C} = \sum_{0}^{m}(\tilde{A}_i\tilde{A}_{m-i} + \Lambda\tilde{B}_i\tilde{C}_{m-i}), \tag{3.11}$$

then we have

$$\tilde{F}_0=\frac{1}{4},\tilde{F}_1=0,\tilde{F}_m=\frac{\Lambda^{m-1}\langle\Phi_1,\Psi_1\rangle-\Lambda^{m-1}\langle\Phi_2,\Psi_2\rangle}{2}+\sum_{i=1}^{m-1}(\Lambda^{i-2}\langle\Phi_1,\Psi_2\rangle\Lambda^{m-i}\langle\Phi_2,\Psi_1\rangle$$
$$+\frac{\Lambda^{i-1}\langle\Phi_1,\Psi_1\rangle-\Lambda^{i-1}\langle\Phi_2,\Psi_2\rangle}{2}\frac{\Lambda^{m-i-1}\langle\Phi_1,\Psi_1\rangle-\Lambda^{m-i-1}\langle\Phi_2,\Psi_2\rangle}{2}).$$

We could obtain a family of finite-dimensional integrable systems and an integrable symplectic map via the binary nonlinearity of the isospectral problem. Bringing Eq. (3.10) into Eq. (3.5), we have

$$\begin{aligned}&\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\end{pmatrix}_{t_m}=V^{(m)}(u,\lambda_j)\big|_B\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\end{pmatrix},\\ &\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_{t_m}=-V^{(m)T}(u,\lambda_j)\big|_B\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix},\qquad j=1,2,\cdots,N,\end{aligned}\tag{3.12}$$

where subscript B stands for Eqs. (3.6) and (3.7).

Rewriting Eq. (3.12) as the Hamilton systems, we infer

$$D(\tilde{A}^2+\tilde{B}\tilde{C})=0,\quad \Psi_{it_m}=-\frac{\partial(\tilde{F}_{m+1})}{\partial\Phi_i},\quad \Phi_{it_m}=\frac{\partial(\tilde{F}_{m+1})}{\partial\Psi_i},i=1,\,2.$$

Setting

$$\overline{F_j}=\varphi_{1j}\psi_{1j}+\varphi_{2j}\psi_{2j},\quad 1\le j\le N,$$

we have

$$\{\ \tilde{F}_{m+1},\ \ \bar{F}_j\}=\frac{d\bar{F}_j}{dt_m}=0,\quad 1\le j\le N,m\ge 0;\ \ \{\ \bar{F}_i,\ \ \bar{F}_j\}=0,\quad 1\le i,j\le N.$$

Therefore, we get

$$\frac{\partial\tilde{F}_m}{\partial\Phi_1}=\frac{\Lambda^{m-1}\Psi_1}{2}+\sum_{i=1}^{m-1}\tilde{C}_{m+1-i}\Lambda^{i-2}\Psi_2+\sum_{i=1}^{m-1}\tilde{A}_i\Lambda^{m-i-1}\Psi_1,$$
$$\frac{\partial\tilde{F}_m}{\partial\Phi_2}=-\frac{\Lambda^{m-1}\Psi_2}{2}+\sum_{i=1}^{m-1}\tilde{B}_{m-i}\Lambda^{m-i}\Psi_1+\sum_{i=1}^{m-1}\tilde{A}_i\Lambda^{m-i-1}\Psi_2.$$

According to Eq. (3.11), the following conclusions can be drawn directly

$$\left.\frac{\partial\tilde{F}_{m+1}}{\partial\Phi_1}\right|_{\Phi_1=\Phi_2=0}=\frac{\Lambda^{m-1}\Psi_1}{2},$$
$$\left.\frac{\partial\tilde{F}_{m+1}}{\partial\Phi_2}\right|_{\Phi_1=\Phi_2=0}=-\frac{\Lambda^{m-1}\Psi_2}{2}.$$

Set $\sum_{j=1}^{N}\psi_{1j}^2\neq 0$, we suppose that $\Phi_1=(\varphi_{11,}\varphi_{12},\cdots\varphi_{1N})^T$ is satisfied with the following nonlinear equation

$$\begin{cases}\varphi_{11}\psi_{11}+\varphi_{12}\psi_{12}+\cdots\varphi_{1N}\psi_{1N}=0,\\ \lambda_1\varphi_{11}\psi_{11}+\lambda_2\varphi_{12}\psi_{12}+\cdots+\lambda_N\varphi_{1N}\psi_{1N}=0,\\ \quad\vdots\\ \lambda_1^{N-1}\varphi_{11}\psi_{11}+\lambda_2^{N-1}\varphi_{12}\psi_{12}+\cdots+\lambda_N^{N-1}\varphi_{1N}\psi_{1N}=0.\end{cases}\tag{3.13}$$

Through the observation of Vandermode determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ & \cdots & \cdots & \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1} \end{vmatrix} \neq 0,$$

we know $\Phi_1 = (\varphi_{11}, \varphi_{12,} \cdots \varphi_{1N})^T$ is only decided by Eq. (3.13). In addition, we obtain the following proposition by direct calculation

$$\frac{\partial \bar{F}_j}{\partial \varphi_{il}} = \psi_{ij}\delta_{jl}, \quad i = 1, 2, \quad j, l = 1, 2 \cdots N.$$

$$\det \begin{pmatrix} \frac{\partial \bar{F}_1}{\partial \Phi_1} & \cdots & \frac{\partial \bar{F}_N}{\partial \Phi_1} & \frac{\partial \tilde{F}_1}{\partial \Phi_1} & \cdots & \frac{\partial \tilde{F}_N}{\partial \Phi_1} \\ \frac{\partial \bar{F}_1}{\partial \Phi_2} & \cdots & \frac{\partial \bar{F}_N}{\partial \Phi_2} & \frac{\partial \tilde{F}_1}{\partial \Phi_2} & \cdots & \frac{\partial \tilde{F}_N}{\partial \Phi_2} \end{pmatrix} = \det \begin{pmatrix} \psi_{11} & \cdots & 0 & \frac{1}{2}\lambda_1\psi_{11} & \cdots & \frac{1}{2}\lambda_1^N\psi_{11} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \psi_{1N} & \frac{1}{2}\lambda_N\psi_{1N} & \cdots & \frac{1}{2}\lambda_N^N\psi_{1N} \\ \psi_{21} & \cdots & 0 & -\frac{1}{2}\lambda_1\psi_{21} & \cdots & -\frac{1}{2}\lambda_1^N\psi_{21} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \psi_{2N} & -\frac{1}{2}\lambda_N\psi_{2N} & \cdots & -\frac{1}{2}\lambda_N^N\psi_{2N} \end{pmatrix}$$

$$= ((\prod_{j=1}^{N} \psi_{1j}\psi_{2j})(\prod_{k=1}^{N} \lambda_k) \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{N-1} \\ & \cdots & \cdots & \\ 1 & \lambda_N & \cdots & \lambda_N^{N-1} \end{vmatrix}.$$

That is, $\tilde{F}_{m+1}, 1 \leq m \leq N, \bar{F}_j, 1 \leq j \leq N$ are functionally independent in some region of R^{4N}.

4. The symmetry of the discrete integrable system

In this section we use the symmetry theory to find the solution to Eq. (2.7). The vector fields of the general form are

$$\nu_1 = \tau(t)\partial_t + \phi_n(t, p^{(n)})\partial_{p^{(n)}}, \nu_2 = \tau(t)\partial_t + \psi_n(t, q^{(n)})\partial_{q^{(n)}}.$$

So as to obtain the Lie algebra of local Lie point symmetries of Eq. (2.7), we present the first prolongation of ν_1, ν_2, that is,

$$Pr^{(n)}\nu_1 = \tau(t)\partial_t + \sum_{i=-1}^{1} \phi_i(t, p^{(i)})\partial_{p^{(i)}} + \phi^{(1)}\partial_{\dot{p}}, \quad \phi^{(1)} = D_t\phi(t, p) - [D_t\tau(t)]\dot{p},$$

$$Pr^{(n)}\nu_2 = \tau_1(t)\partial_t + \sum_{i=-1}^{1} \psi_i(t, q^{(i)})\partial_{q^{(i)}} + \psi^{(1)}\partial_{\dot{q}}, \ \psi^{(1)} = -D_t\psi(t, q) + [D_t\tau_1(t)]\dot{q},$$

where D_t stands for the total derivative operator. Substituting $Pr^{(n)}\nu$ into Eq. (2.7), we have

$$\varphi_{(-1)}\frac{p^2}{p^{2(-1)}} - \varphi\frac{2p}{p^{(-1)}} + \varphi_t + (\varphi_p - \dot{\tau})(\frac{p^2}{p^{(-1)}} - q^{(1)}) + \psi\frac{2q}{p^{(-1)}} - \psi_{(1)} - \psi_t - (\psi_q - \dot{\tau}_1)(q^{(1)} - \frac{q^2}{p^{(-1)}}) = 0. \tag{4.1}$$

Substituting the second derivative $\partial_p\partial_{q^{(-1)}}$ and $\partial_q\partial_{q^{(1)}}$ into Eq. (4.1), we infer

$$-\varphi_{pp}\frac{1}{p^{2(-1)}} = 0, \quad \psi_{qq} = 0.$$

It is easy to get that

$$\varphi_{pp} = 0, \quad \psi_{qq} = 0,$$

which infer the following result

$$\varphi = ap, \quad \psi = bq. \tag{4.2}$$

Substituting Eq. (4.2) into Eq. (4.1), we gain

$$a_{(-1)}\frac{p^2}{p^{(-1)}} - 2a\frac{p^2}{p^{(-1)}} + (2a - \dot{\tau})(\frac{p^2}{p^{(-1)}} - q^{(1)}) + 2b\frac{q^2}{p^{(-1)}} - b_{(1)}q^{(1)} - (2b - \dot{\tau}_1)(q^{(1)} - \frac{q^2}{p^{(-1)}}) = 0, \tag{4.3}$$

inferring the special coefficient relation of the Eq. (4.3), we have

$$\begin{cases} \dot{\tau} = a_{(-1)}, \\ \dot{\tau}_1 = 4b, \\ b_{(1)} = 2b - 2a + a_{(-1)}, \end{cases}$$

where the coefficients $a_{(1)}, b_{(1)}$ are constants.

Since the solution of the Eq. (2.7) could be obtained on the basis of the symmetry theory with the help of the infinitesimal generator, we can rewrite Eq. (2.7) into the following form.

$$\begin{cases} p_t = \dfrac{p^2}{p^{(-1)}} - q^{(1)}, \\ q_t = q^{(1)} - \dfrac{q^2}{p^{(-1)}}, \\ prX_1[p] = \tau(t)p_t + \varphi = 0, \\ prX_2[q] = \tau_1(t)q_t + \psi = 0. \end{cases} \tag{4.4}$$

If we take some appropriate initial values

$$a = 3b, \quad b_{(1)} = -2b, \quad a_{(-1)} = 2b, \quad \dot{\tau}_1 = 4b, \quad \dot{\tau} = 2b, \tag{4.5}$$

and set the step length $p - p^{(-1)} = q^{(1)} - q = h$, we obtain the solution of Eq. (2.7)

$$\begin{cases} q = (\dfrac{ap}{\tau} + \dfrac{p^2}{p-h}) - h, \\ \dfrac{b}{\tau_1}(\dfrac{ap}{\tau} + \dfrac{p^2}{p-h} - h) = \dfrac{(\frac{ap}{\tau} - \frac{p^2}{p-h} - h)^2}{p-h} - (\dfrac{ap}{\tau} + \dfrac{p^2}{p-h}), \end{cases} \tag{4.6}$$

according to Eqs. (4.4) and (4.5).

5. Conclusion

In this paper, we prove that Eqs. (3.8) and (3.9) are an integrable symplectic map through the binary nonlinearization method, and we know that $\tilde{F}_{m+1}, 1 \leq m \leq N, \bar{F}_j, 1 \leq j \leq N$ are functionally independent in some region of R^{4N}. According to the symmetry theory, we gain the seed symmetry and the infinitesimal generator by the seed symmetry and the recursion operator. And we gain the infinitesimal generator of the discrete lattice equation based on the Lie point symmetry theory.

The explanation of the p and q with the time variable t is given in Figs. 1–3. According to the figures, by adjusting the step size, we find that the longer the step length, the greater the minimum, and the smaller the maximum. Meanwhile, the convergence rate of the graphics is becoming slower.

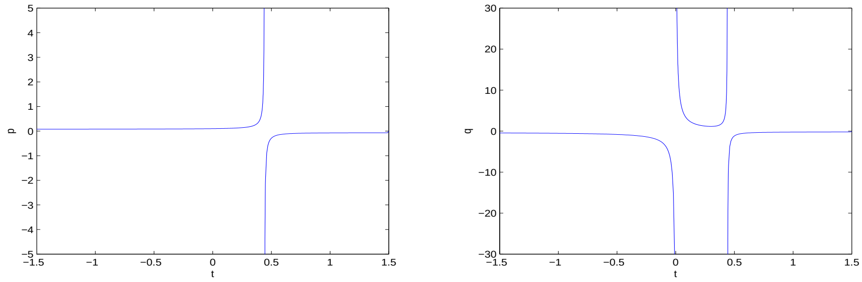

Figure 1: the left figure indicates the solution of the q and the right is the solution of the p, based on the Eq. (4.6), with the step length $h = 0.1$.

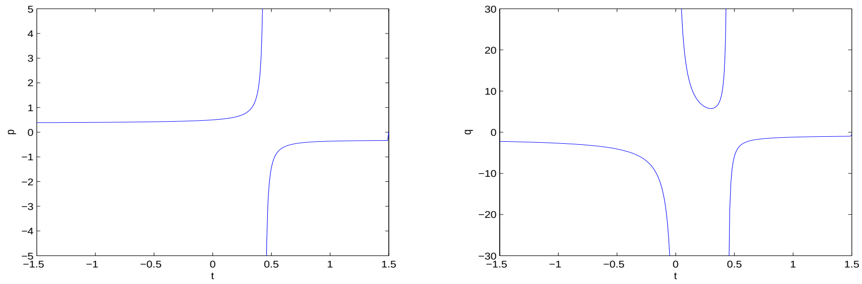

Figure 2: the left figure indicates the solution of the q and the right is the solution of the p, based on the Eq. (4.6), with the step length $h = 0.5$.

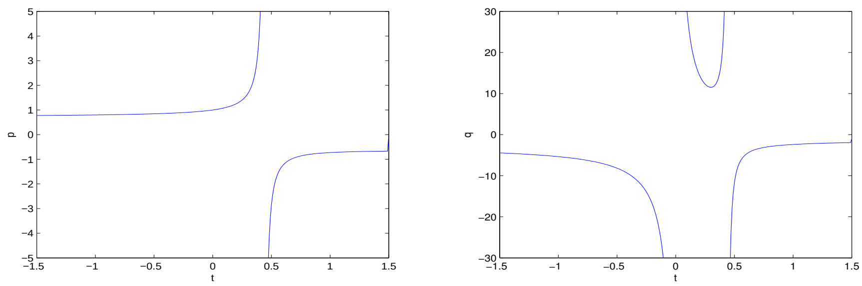

Figure 3: the left figure indicates the solution of the q and the right is the solution of the p, based on the Eq. (4.6), with the step length $h = 1$.

Our plan of the future work is that using Bäcklund transformation or Dârboux transformation to research these discrete integrable systems. Moreover, we would try to create the potential of the spectral problem r_t to get more explicit equation. It can be better to research the essence of nature.

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