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Fixed point theorems for generalized multivalued nonlinear $\mathcal{F}$-contractions

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Abstract

In this paper, we introduce certain new concepts of α-η-lower semi-continuous and α-η-upper semi-continuous mappings. By using these concepts, we prove some fixed point results for generalized multivalued nonlinear $\mathcal{F}$-contractions in metric spaces and ordered metric spaces. As an application of our results we deduce Suzuki-Wardowski type fixed point results and fixed point results for orbitally lower semi-continuous mappings in complete metric spaces. Our results generalize and extend many recent fixed point theorems including the main results of Minak et al. [G. Minak, M. Olgun, I. Altun, Carpathian J. Math., **31** (2015), 241–248], Altun et al. [I. Altun, G. Mınak, M. Olgun, Nonlinear Anal. Model. Control, **21** (2016), 201–210] and Olgun et al. [M. Olgun, G. Minak, I. Altun, J. Nonlinear Convex Anal., **17** (2016), 579–587].

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1. Introduction and preliminaries

Let $(\mathcal{X}, d)$ be a metric space. $2^{\mathcal{X}}$ denotes the family of all nonempty subsets of $\mathcal{X}$, $C(\mathcal{X})$ denotes the family of all nonempty, closed subsets of $\mathcal{X}$, $CB(\mathcal{X})$ denotes the family of all nonempty, closed, and bounded subsets of $\mathcal{X}$ and $K(\mathcal{X})$ denotes the family of all nonempty compact subsets of $\mathcal{X}$. It is clear that, $K(\mathcal{X}) \subseteq CB(\mathcal{X}) \subseteq C(\mathcal{X}) \subseteq P(\mathcal{X})$. For $\mathcal{A}, \mathcal{B} \in C(\mathcal{X})$, let

$$H(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{x \in \mathcal{A}} D(x, \mathcal{B}), \sup_{y \in \mathcal{B}} D(y, \mathcal{A}) \right\},$$

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where $D(x,\mathcal{B}) = \inf\{d(x,y) : y \in \mathcal{B}\}$. Then H is called generalized Pompeiu-Hausdorff distance on $C(\mathcal{X})$. It is well-known that H is a metric on $CB(\mathcal{X})$, which is called Pompeiu-Hausdorff metric induced by d. For more details see [3],[11].

An interesting generalization of the Banach contraction principle to multivalued mappings is known as Nadler's fixed point theorem [25]. After this, many authors extended Nadler's fixed point theorem in many directions (see [10, 12, 24, 29] and references therein). In 2012, Samet et al. [28] defined α-admissible mappings. This notion is generalized by many authors (see [20, 21]). Salimi et al. [27] generalized this idea by introducing the function η and established fixed point theorems. Next, Asl et al. [8] extended these concepts to multivalued mappings by introducing the notion of α^*-admissible mappings as follows:

Definition 1.1 ([8])**.** Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be a function, then $\mathcal{T}$ is an α_*-admissible mapping, if

$$\alpha(y,z) \geq 1 \text{ implies that } \alpha_*(\mathcal{T}y, \mathcal{T}z) \geq 1, \quad y, z \in \mathcal{X},$$

where

$$\alpha_*(\mathcal{A}, \mathcal{B}) = \inf_{y \in \mathcal{A}, z \in \mathcal{B}} \alpha(y,z).$$

Hussain et al. [19] modified the notion of α_*-admissible as follows:

Definition 1.2 ([19])**.** Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions where η is bounded, then $\mathcal{T}$ is an α_*-admissible mapping with respect to η, if

$$\alpha(y,z) \geq \eta(y,z) \text{ implies that } \alpha_*(\mathcal{T}y, \mathcal{T}z) \geq \eta_*(\mathcal{T}y, \mathcal{T}z), \quad y, z \in \mathcal{X},$$

where

$$\alpha_*(\mathcal{A}, \mathcal{B}) = \inf_{y \in \mathcal{A}, z \in \mathcal{B}} \alpha(y,z), \quad \eta_*(\mathcal{A}, \mathcal{B}) = \sup_{y \in \mathcal{A}, z \in \mathcal{B}} \eta(y,z).$$

Further, Ali et al. [4] generalized Definition 1.2 in the following way.

Definition 1.3 ([4])**.** Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha, \eta : X \times X \to \mathbb{R}_+$ be two functions. We say that $\mathcal{T}$ is generalized α_*-admissible mapping with respect to η, if

$$\alpha(y,z) \geq \eta(y,z) \text{ implies that } \alpha(u,v) \geq \eta(u,v), \quad \text{for all } u \in \mathrm{T}y, \ v \in \mathrm{T}z.$$

In 2014, Hussain et al. [16] introduced the notion of α-η continuous mappings as follows:

Definition 1.4 ([16])**.** Let $(\mathcal{X}, d)$ be a metric space, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be functions. Then $\mathcal{T}$ is an α-η-continuous mapping on $\mathcal{X}$, if for given $z \in X$ and sequence $\{z_n\}$ with

$$z_n \to z \text{ as } n \to \infty, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}), \quad \text{for all } n \in \mathbb{N} \Rightarrow \ \mathcal{T}z_n \to \mathcal{T}z.$$

After that Hussain et al. [15] generalized Definition 1.4 to multivalued maps.

Definition 1.5 ([15])**.** Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. We say that $\mathcal{T}$ is α-η continuous multivalued mapping, if for given $z \in X$ and sequence $\{z_n\}$ with $z_n \to z$ as $n \to \infty$, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$, for all $n \in \mathbb{N}$ we have $\mathcal{T}z_n \to \mathcal{T}z$. That is, $\lim_{n\to\infty} d(z_n, z) = 0$ and $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ implies $\lim_{n\to\infty} H(\mathcal{T}z_n, \mathcal{T}z) = 0$.

Recently, Wardowski [31] defined $\mathcal{F}$-contraction and proved a fixed point result as a generalization of the Banach contraction principle for this contraction. This idea has been extended in many directions (see [1, 14, 17] and references therein). Hussain et al. [18] broadened this idea to α-$\mathcal{GF}$-contraction with respect to a general family of functions $\mathcal{G}$. Following Wardowski and Hussain, we denote by $\mathfrak{F}$, the set of all functions $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

($\mathcal{F}_1$) $\mathcal{F}$ is strictly increasing;

($\mathcal{F}_2$) for all sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$, if and only if $\lim_{n\to\infty} \mathcal{F}(\alpha_n) = -\infty$;

($\mathcal{F}_3$) there exists $0 < k < 1$ such that $\lim_{\alpha\to 0^+} \alpha^k \mathcal{F}(\alpha) = 0$,

$\mathfrak{F}_*$, if $\mathcal{F}$ also satisfies the following:

($\mathcal{F}_4$) $\mathcal{F}(\inf A) = \inf \mathcal{F}(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$,

$\mathfrak{G}$, the set of all functions $\mathcal{G} : \mathbb{R}^{+^4} \to \mathbb{R}^+$ satisfying:

($\mathcal{G}$) for all $t_1, t_t, t_3, t_4 \in \mathbb{R}^+$ with $t_1t_2t_3t_4 = 0$ there exists $\tau > 0$ such that $\mathcal{G}(t_1, t_2, t_3, t_4) = \tau$.

On unifying the concepts of Wardowski's and Nadlers, Altun et al. [5] gave the concept of multivalued $\mathcal{F}$-contractions and established some fixed point results. On the other side, Minak et al. [23], extended the results of Wardowski as follows:

Theorem 1.6 ([23]). *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If there exists $\tau > 0$ such that for any $z \in \mathcal{X}$ with $d(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}^z_\sigma$ satisfying*

$$\tau + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

where

$$\mathcal{F}^z_\sigma = \{y \in \mathcal{T}z : \mathcal{F}(d(z, y)) \leq \mathcal{F}(D(z, \mathcal{T}z)) + \sigma\},$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$ and $z \to d(z, \mathcal{T}z)$ is lower semi-continuous.

Theorem 1.7 ([23]). *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$. If there exists $\tau > 0$ such that for any $z \in \mathcal{X}$ with $d(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}^z_\sigma$ satisfying*

$$\tau + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$ and $z \to d(z, \mathcal{T}z)$ is lower semi-continuous.

Minak et al. [23] also showed that $\mathcal{F}^z_\sigma \neq \emptyset$ in both cases when $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{F} \in \mathfrak{F}_*$. The aim of the present paper is to introduce the concept of α-η-semicontinuous multivalued mappings and to prove fixed point theorem for multivalued nonlinear $\mathcal{F}$-contractions that generalize the results of Altun et al. [6], Minak et al. [23], Olgun et al. [26] and Hussain et al. [18]. The following lemmas will be used in the sequel.

Lemma 1.8 ([3]). *Let $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ be a multivalued function, then the following statements are equivalent.*

1. *$\mathcal{T}$ is lower semi-continuous.*
2. *$V \subset \mathcal{Y} \Rightarrow \mathcal{T}^{-1}[int(V)]$ is open in $\mathcal{X}$,*

where $int(V)$ denotes the interior of V.

Lemma 1.9 ([3]). *Let $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ be a multivalued function, then the following statements are equivalent.*

1. *$\mathcal{T}$ is upper semi-continuous.*
2. *$V \subset \mathcal{Y} \Rightarrow \mathcal{T}^{-1}[\overline{V}]$ is closed in $\mathcal{X}$,*

where $\overline{V}$ denotes the closure of V.

2. Fixed point results for modified α-η-$\mathcal{GF}$-contraction

We begin this section with the following definitions.

Definition 2.1. Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. We say that $\mathcal{T}$ is α-η lower semi-continuous multivalued mapping on $\mathcal{X}$, if for given $z \in X$ and sequence $\{z_n\}$ with

$$\lim_{n\to\infty} d(z_n, z) = 0, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}), \quad \text{for all } n \in \mathbb{N},$$

implies

$$\lim_{n\to\infty} \inf D(z_n, \mathcal{T}z_n) \geq D(z, \mathcal{T}z).$$

Definition 2.2. Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. We say that $\mathcal{T}$ is α-η upper semi-continuous multivalued mapping on $\mathcal{X}$, if for given $z \in X$ and sequence $\{z_n\}$ with

$$\lim_{n\to\infty} d(z_n, z) = 0, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}), \quad \text{for all } n \in \mathbb{N},$$

implies

$$\lim_{n\to\infty} \sup D(z_n, \mathcal{T}z_n) \leq D(z, \mathcal{T}z).$$

Lemma 2.3. *Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Then $\mathcal{T}$ is α-η continuous, if and only if it is α-η upper semi-continuous and α-η lower semi-continuous.*

Proof. Suppose that $\mathcal{T}$ is α-η upper semi-continuous and α-η lower semi-continuous. Then there exists a sequence $\{z_n\}$ in $\mathcal{X}$ and $z \in \mathcal{X}$ with

$$\lim_{n\to\infty} d(z_n, z) = 0, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}), \quad \text{for all } n \in \mathbb{N},$$

implies

$$\lim_{n\to\infty} \inf D(z_n, \mathcal{T}z_n) \geq D(z, \mathcal{T}z), \tag{2.1}$$

and

$$\lim_{n\to\infty} \sup D(z_n, \mathcal{T}z_n) \leq D(z, \mathcal{T}z). \tag{2.2}$$

From (2.1) and (2.2), we get that $D(z_n, \mathcal{T}z_n) \to D(z, \mathcal{T}z)$ as $n \to \infty$. This is possible only when $\mathcal{T}z_n \to \mathcal{T}z$. Consequently, $\mathcal{T}$ is α-η continuous.

Conversely, suppose that $\mathcal{T}$ is α-η continuous. Then there exists a sequence $\{z_n\}$ in $\mathcal{X}$ and $z \in \mathcal{X}$ with $z_n \to z$ as $n \to \infty$ and $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ for all $n \in \mathbb{N}$ implies $\mathcal{T}z_n \to \mathcal{T}z$ as $n \to \infty$. This implies that $D(z_n, \mathcal{T}z_n) \to D(z, \mathcal{T}z)$ as $n \to \infty$ or $\lim_{n\to\infty} D(z_n, \mathcal{T}z_n) = D(z, \mathcal{T}z)$. From here it follows that $\lim_{n\to\infty} \inf D(z_n, \mathcal{T}z_n) \geq D(z, \mathcal{T}z)$ and $\lim_{n\to\infty} \sup D(z_n, \mathcal{T}z_n) \leq D(z, \mathcal{T}z)$. Hence $\mathcal{T}$ is α-η upper semi-continuous and α-η lower semi-continuous. □

Remark 2.4. As semi-continuity is a weaker property than continuity, an α-η upper semi-continuous and α-η lower semi-continuous mapping need not to be α-η continuous mapping, as shown in the examples below.

Example 2.5. Let $\mathcal{X} = \mathbb{R}$ with usual metric d. Then $(\mathcal{X}, d)$ is a metric space. Define $\mathcal{T}_1 : \mathcal{X} \to 2^{\mathcal{X}}$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ by

$$\mathcal{T}_1 z = \begin{cases} \{0\} & \text{if } z \neq 0, \\ [-1, 1] & \text{if } z = 0, \end{cases}$$

$$\alpha(y, z) = \begin{cases} 1 & \text{if } z, y \neq 0, \\ 0 & \text{if } z = y = 0, \end{cases}$$

and $\eta(z, y) = \frac{1}{2}$, for all $z, y \in \mathcal{X}$.

Firstly, we show that $\mathcal{T}_1$ is not lower semi-continuous multivalued map. For this, let $V = [-1, 1] \subset 2^{\mathcal{X}}$, then $\mathcal{T}_1^{-1}(\text{int}(V)) = \mathcal{T}_1^{-1}((-1,1)) = \{0\}$ which is not open in $\mathbb{R}$, so by Lemma 1.8, $\mathcal{T}_1$ is not lower semi-continuous. But $\mathcal{T}_1$ is α-η lower semi-continuous multivalued map. Indeed, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ for sequence z_n of non-zero real numbers. Here arises two cases:

Case I. $z_n \to z = 0$.
If $z_n \to 0$, then $\mathcal{T}_1 z_n = \{0\}$ and $\mathcal{T}_1 z = [-1,1]$ such that $D(z_n, \mathcal{T}_1 z_n) = D(z_n, \{0\}) = z_n$ and $D(z, \mathcal{T}_1 z) = D(0, [-1,1]) = 0$. This implies that

$$\lim_{n\to\infty} \inf D(z_n, \mathcal{T} z_n) = \lim_{n\to\infty} \inf z_n = z = 0 = D(z, \mathcal{T} z).$$

Case II. $z_n \to z \neq 0$.
If $z_n \to z$, then $\mathcal{T}_1 z_n = \{0\}$ and $\mathcal{T}_1 z = \{0\}$ such that $D(z_n, \mathcal{T}_1 z_n) = D(z_n, \{0\}) = z_n$ and $D(z, \mathcal{T}_1 z) = z$. This implies that

$$\lim_{n\to\infty} \inf D(z_n, \mathcal{T}_1 z_n) = \lim_{n\to\infty} \inf z_n = z = D(z, \mathcal{T}_1 z).$$

On the other hand, in Case I we have

$$\lim_{n\to\infty} H(\mathcal{T}_1 z_n, \mathcal{T}_1 z) = 1.$$

Hence $\mathcal{T}_1$ is not α-η-continuous multivalued map.

Example 2.6. Consider $\mathcal{X}$ the same as in Example 2.5. Define $\mathcal{T}_2 : \mathcal{X} \to 2^{\mathcal{X}}$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ by

$$\mathcal{T}_2 z = \begin{cases} [-1,1] & \text{if } \ z \neq 0, \\ \{0\} & \text{if } \ z = 0, \end{cases}$$

$$\alpha(z,y) = \begin{cases} 0 & \text{if } \ z, y \neq 0, \\ 2 & \text{if } \ z = y = 0, \end{cases}$$

and $\eta(z,y) = \frac{1}{4}$, for all $z, y \in \mathcal{X}$.

Firstly, we show that $\mathcal{T}_2$ is not upper semi-continuous multivalued map. For this, let $V = [-1,1] \subset 2^{\mathcal{X}}$, then $\mathcal{T}_2^{-1}(\overline{V}) = \mathcal{T}_2^{-1}([-1,1]) = \mathbb{R}\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$, which is not closed in $\mathbb{R}$, so by Lemma 1.9, $\mathcal{T}_2$ is not upper semi-continuous. But $\mathcal{T}_2$ is α-η upper semi-continuous multivalued map. Indeed, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ for sequence $z_n = 0$ for all $n \in \mathbb{N}$. Then z_n approaches to $z = 0$ only. Therefore, If $z_n \to 0$, then $\mathcal{T}_2 z_n = \{0\}$ and $\mathcal{T}_2 z = \{0\}$. This implies that

$$\lim_{n\to\infty} \sup D(z_n, \mathcal{T}_2 z_n) = 0 = D(z, \mathcal{T}_2 z).$$

On the other hand,

$$\lim_{n\to\infty} H(\mathcal{T}_2 z_n, \mathcal{T}_2 z) = 1.$$

Hence $\mathcal{T}_2$ is not α-η-continuous multivalued map.

Remark 2.7. Let $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued map on a metric space $(\mathcal{X}, d)$. Let $f : \mathcal{X} \to \mathbb{R}$, defined by $f(z) = D(z, \mathcal{T} z)$, for all $z \in \mathcal{X}$, be a lower semi-continuous mapping. Take $\alpha(z,y) = \eta(z,y)$, for all $z, y \in \mathcal{X}$, then for $z \in \mathcal{X}$ and a sequence $\{z_n\}$ with

$$\lim_{n\to\infty} d(z_n, z) = 0, \quad \alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1}) \ \text{ for all } n \in \mathbb{N},$$

we have

$$\lim_{n\to\infty} \inf f(z_n) \geq f(z),$$

and so

$$\lim_{n\to\infty} \inf D(z_n, \mathcal{T} z_n) \geq D(z, \mathcal{T} z).$$

This shows that $\mathcal{T}$ is α-η lower semi-continuous mapping. But if $\mathcal{T}$ is α-η lower semi-continuous mapping, then f needs to be lower semi-continuous as shown in Example 2.12. Similarly, if $f : \mathcal{X} \to \mathbb{R}$ is upper semi-continuous mapping then, $\mathcal{T}$ is α-η upper semi-continuous mapping but not conversely.

Theorem 2.8. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{G} \in \mathfrak{G}$ fulfilling the following assertions:*

(1) *if for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying*

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y));$$

(2) *$\mathcal{T}$ is generalized α_*-admissible mapping with respect to η;*

(3) *$\mathcal{T}$ is α-η lower semi-continuous mapping;*

(4) *there exists $z_0 \in \mathcal{X}$ and $y_0 \in \mathcal{T}z_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$.*

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$.

Proof. Let $z_0 \in \mathcal{X}$, since $\mathcal{T}z \in K(\mathcal{X})$ for every $z \in \mathcal{X}$, the set $\mathcal{F}_\sigma^z$ is non-empty for any $\sigma > 0$, then there exists $z_1 \in \mathcal{F}_\sigma^{z_0}$ and by hypothesis $\alpha(z_0, z_1) \geq \eta(z_0, z_1)$. Assume that $z_1 \notin \mathcal{T}z_1$, otherwise z_1 is the fixed point of $\mathcal{T}$. Then, since $\mathcal{T}z_1$ is closed, $D(z_1, \mathcal{T}z_1) > 0$, so from condition (1), we have

$$\mathcal{G}(D(z_0, \mathcal{T}z_0), D(z_1, \mathcal{T}z_1), D(z_0, \mathcal{T}z_1), D(z_1, \mathcal{T}z_0)) + \mathcal{F}(D(z_1, \mathcal{T}z_1)) \leq \mathcal{F}(d(z_0, z_1)). \tag{2.3}$$

Now for $z_1 \in \mathcal{X}$ there exists $z_2 \in \mathcal{F}_\sigma^{z_1}$ with $z_2 \notin \mathcal{T}z_2$, otherwise z_2 is the fixed point of $\mathcal{T}$, since $\mathcal{T}z_2$ is closed, so, $D(z_2, \mathcal{T}z_2) > 0$. Since $\mathcal{T}$ is generalized α_*-admissible mapping with respect to η, then $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$. Again by using condition (1), we get

$$\mathcal{G}(D(z_1, \mathcal{T}z_1), D(z_2, \mathcal{T}z_2), D(z_1, \mathcal{T}z_2), D(z_2, \mathcal{T}z_1)) + \mathcal{F}(D(z_2, \mathcal{T}z_2)) \leq \mathcal{F}(d(z_1, z_2)).$$

On continuing recursively, we get a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $\mathcal{X}$ such that $z_{n+1} \in \mathcal{F}_\sigma^{z_n}$, $z_{n+1} \notin \mathcal{T}z_{n+1}$, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ and

$$\mathcal{G}(D(z_n, \mathcal{T}z_n), D(z_{n+1}, \mathcal{T}z_{n+1}), D(z_n, \mathcal{T}z_{n+1}), D(z_{n+1}, \mathcal{T}z_n)) + \mathcal{F}(D(z_{n+1}, \mathcal{T}z_{n+1})) \leq \mathcal{F}(d(z_n, z_{n+1})).$$

As $z_{n+1} \in \mathcal{T}z_n$, this implies that

$$\mathcal{G}(D(z_n, \mathcal{T}z_n), D(z_{n+1}, \mathcal{T}z_{n+1}), D(z_n, \mathcal{T}z_{n+1}), 0) + \mathcal{F}(D(z_{n+1}, \mathcal{T}z_{n+1})) \leq \mathcal{F}(d(z_n, z_{n+1})). \tag{2.4}$$

From $(\mathcal{G})$ there exists $\tau > 0$ such that

$$\mathcal{G}(D(z_n, \mathcal{T}z_n), D(z_{n+1}, \mathcal{T}z_{n+1}), D(z_n, \mathcal{T}z_{n+1}), 0) = \tau.$$

From equation (2.4), we get that

$$\mathcal{F}(D(z_{n+1}, \mathcal{T}z_{n+1})) \leq \mathcal{F}(d(z_n, z_{n+1})) - \tau. \tag{2.5}$$

Since $z_{n+1} \in \mathcal{F}_\sigma^{z_n}$, we have

$$\mathcal{F}(d(z_n, z_{n+1})) \leq \mathcal{F}(D(z_n, \mathcal{T}z_n)) + \sigma. \tag{2.6}$$

Combining equations (2.5) and (2.6) gives

$$\mathcal{F}(D(z_{n+1}, \mathcal{T}z_{n+1})) \leq \mathcal{F}(D(z_n, \mathcal{T}z_n)) + \sigma - \tau. \tag{2.7}$$

Since $\mathcal{T}z_n$ and $\mathcal{T}z_{n+1}$ is compact, there exists $z_{n+1} \in \mathcal{T}z_n$ and $z_{n+2} \in \mathcal{T}z_{n+1}$ such that $d(z_n, z_{n+1}) = D(z_n, \mathcal{T}z_n)$ and $d(z_{n+1}, z_{n+2}) = D(z_{n+1}, \mathcal{T}z_{n+1})$, so equation (2.7) implies

$$\mathcal{F}(d(z_{n+1}, z_{n+2})) \leq \mathcal{F}(d(z_n, z_{n+1})) + \sigma - \tau. \tag{2.8}$$

By using equation (2.8), we get

$$\begin{aligned}\mathcal{F}(d(z_{n+1}, z_{n+2})) &\le \mathcal{F}(d(z_n, z_{n+1})) + \sigma - \tau \\ &\le \mathcal{F}(d(z_{n-1}, z_n)) + 2\sigma - 2\tau \\ &\vdots \\ &\le \mathcal{F}(d(z_0, z_1)) + n\sigma - n\tau \\ &= \mathcal{F}(d(z_0, z_1)) - n(\tau - \sigma).\end{aligned} \tag{2.9}$$

By letting limit as $n \to \infty$ in equation (2.9), we get $\lim_{n\to\infty} \mathcal{F}(d(z_{n+1}, z_{n+2})) = -\infty$, so by ($\mathcal{F}2$), we obtain

$$\lim_{n\to\infty} d(z_{n+1}, z_{n+2}) = 0. \tag{2.10}$$

Now from ($\mathcal{F}3$), there exists $0 < k < 1$ such that

$$\lim_{n\to\infty} [d(z_{n+1}, z_{n+2})]^k \mathcal{F}(d(z_{n+1}, z_{n+2})) = 0. \tag{2.11}$$

By equation (2.9), we get

$$\lim_{n\to\infty} [d(z_{n+1}, z_{n+2})]^k [\mathcal{F}(d(z_{n+1}, z_{n+2})) - d(z_0, z_1)] \le -n(\tau - \sigma)[d(z_{n+1}, z_{n+2})]^k \le 0. \tag{2.12}$$

By taking limit as $n \to \infty$ in equation (2.12) and applying equations (2.10) and (2.11), we have

$$\lim_{n\to\infty} n[d(z_{n+1}, z_{n+2})]^k = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n[d(z_{n+1}, z_{n+2})]^k \le 1$, or $d(z_{n+1}, z_{n+2}) \le \frac{1}{n^{1/k}}$, for all $n > n_1$. Next, for $m > n > n_1$ we have

$$d(z_n, z_m) \le \sum_{i=n}^{m-1} d(z_i, z_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{1/k}},$$

since $0 < k < 1$, $\sum_{i=n}^{m-1} \frac{1}{i^{1/k}}$ converges. Therefore, $d(z_n, z_m) \to 0$ as $m, n \to \infty$. Thus, $\{z_n\}$ is a Cauchy sequence. Since $\mathcal{X}$ is complete, there exists $z^* \in \mathcal{X}$ such that $z_n \to z^*$ as $n \to \infty$. From equations (2.7) and (2.10), we have

$$\lim_{n\to\infty} D(z_n, \mathcal{T}z_n) = 0.$$

Since $\mathcal{T}$ is α-η lower semi-continuous mapping, then

$$0 \le D(z, Tz) \le \lim_{n\to\infty} \inf D(z_n, \mathcal{T}z_n) = 0.$$

Thus, $\mathcal{T}$ has a fixed point. □

Theorem 2.9. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}_*$ and $\mathcal{G} \in \mathfrak{G}$ satisfy all assertions of Theorem 2.8. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Proof. Let $z_0 \in \mathcal{X}$, since $\mathcal{T}z \in C(\mathcal{X})$ for every $z \in \mathcal{X}$ and $\mathcal{F} \in \mathfrak{F}_*$, the set $\mathcal{F}_\sigma^z$ is non-empty for any $\sigma > 0$, then there exists $z_1 \in \mathcal{F}_\sigma^{z_0}$ and by hypothesis $\alpha(z_0, z_1) \ge \eta(z_0, z_1)$. Assume that $z_1 \notin \mathcal{T}z_1$, otherwise z_1 is the fixed point of $\mathcal{T}$. Then, since $\mathcal{T}z_1$ is closed, $D(z_1, \mathcal{T}z_1) > 0$, so from condition (1) of Theorem 2.8, we have

$$\mathcal{G}(D(z_0, \mathcal{T}z_0), D(z_1, \mathcal{T}z_1), D(z_0, \mathcal{T}z_1), D(z_1, \mathcal{T}z_0)) + \mathcal{F}(D(z_1, \mathcal{T}z_1)) \le \mathcal{F}(d(z_0, z_1)).$$

Now for $z_1 \in \mathcal{X}$ there exists $z_2 \in \mathcal{F}_\sigma^{z_1}$ with $z_2 \notin \mathcal{T}z_2$, otherwise z_2 is the fixed point of $\mathcal{T}$, since

$\mathcal{T}z_2$ is closed, so, $D(z_2,\mathcal{T}z_2)>0$. Since $\mathcal{T}$ is generalized α_*-admissible mapping with respect to η, then $\alpha(z_1,z_2)\geq\eta(z_1,z_2)$. Again by using condition (1) of Theorem 2.8, we get

$$\mathcal{G}(D(z_1,\mathcal{T}z_1),D(z_2,\mathcal{T}z_2),D(z_1,\mathcal{T}z_2),D(z_2,\mathcal{T}z_1))+\mathcal{F}(D(z_2,\mathcal{T}z_2))\leq\mathcal{F}(d(z_1,z_2)).$$

On continuing recursively, we get a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that $z_{n+1}\in\mathcal{F}_\sigma^{z_n}$, $z_{n+1}\notin\mathcal{T}z_{n+1}$, $\alpha(z_n,z_{n+1})\geq\eta(z_n,z_{n+1})$ and

$$\begin{aligned}&\mathcal{G}(D(z_n,\mathcal{T}z_n),D(z_{n+1},\mathcal{T}z_{n+1}),D(z_n,\mathcal{T}z_{n+1}),D(z_{n+1},\mathcal{T}z_n))+\mathcal{F}(D(z_{n+1},\mathcal{T}z_{n+1}))\\&\leq\mathcal{F}(d(z_n,z_{n+1})).\end{aligned}$$

The rest of the proof can be completed as the proof of Theorem 2.8. □

Corollary 2.10. *Let $(\mathcal{X},d)$ be a complete metric space and $\alpha,\eta:\mathcal{X}\times\mathcal{X}\to\mathbb{R}_+$ be two functions. Let $\mathcal{T}:\mathcal{X}\to K(\mathcal{X})$ and $\mathcal{F}\in\mathfrak{F}$ fulfill the conditions (2)-(4) of Theorem 2.8 and if for any $z\in\mathcal{X}$ with $D(z,\mathcal{T}z)>0$, there exists $y\in\mathcal{F}_\sigma^z$ with $\alpha(z,y)\geq\eta(z,y)$ satisfying*

$$\tau+\mathcal{F}(D(y,\mathcal{T}y))\leq\mathcal{F}(d(z,y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma<\tau$.

Proof. Define $\mathcal{G}_L:\mathbb{R}^{+^4}\to\mathbb{R}^+$ by $\mathcal{G}(t_1,t_2,t_3,t_4)=L\min\{t_1,t_2,t_3,t_4\}+\tau$, where $L\in\mathbb{R}^+$ and $\tau>0$. Then $\mathcal{G}_L\in\mathfrak{G}$ (see Example 2.1 of [18]). Therefore, the result follows by taking $\mathcal{G}=\mathcal{G}_L$ in Theorem 2.8. □

Corollary 2.11. *Let $(\mathcal{X},d)$ be a complete metric space and $\alpha,\eta:\mathcal{X}\times\mathcal{X}\to\mathbb{R}_+$ be two functions. Let $\mathcal{T}:\mathcal{X}\to C(\mathcal{X})$ and $\mathcal{F}\in\mathfrak{F}_*$ satisfy all conditions of Corollary 2.10. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Proof. By defining same $\mathcal{G}_L$ as in Corollary 2.10 and using Theorem 2.9, we get the required result. □

Example 2.12. Let $\mathcal{X}=\left\{\frac{1}{2^{n-1}}:n\in\mathbb{N}\right\}\cup\{0\}$ with usual metric d. Then $(\mathcal{X},d)$ is a metric space. Define $\mathcal{T}:\mathcal{X}\to K(\mathcal{X})$, $\alpha,\eta:\mathcal{X}\times\mathcal{X}\to\mathbb{R}_+$, $\mathcal{G}:\mathbb{R}^4\to\mathbb{R}^+$ and $\mathcal{F}:\mathbb{R}^+\to\mathbb{R}$ by

$$\mathcal{T}z=\begin{cases}\left\{\frac{1}{2^n}\right\} & \text{if } z=\frac{1}{2^{n-1}},\\ \{0\} & \text{if } z=0,\end{cases}$$

$$\alpha(z,y)=\begin{cases}2 & \text{if } z=\frac{1}{2^{n-1}},\\ \frac{1}{2} & \text{if } z=0,\end{cases}$$

$\eta(z,y)=1$, for all $z,y\in\mathcal{X}$, $\mathcal{G}(t_1,t_2,t_3,t_4)=\tau$, where $\tau>0$ and $\mathcal{F}(r)=\ln(r)$. Then

$$D(z,\mathcal{T}z)=\begin{cases}\frac{1}{2^n} & \text{if } z=\frac{1}{2^{n-1}},\\ 0 & \text{if } z=0.\end{cases}$$

Let $D(z,\mathcal{T}z)>0$, then $z=\frac{1}{2^{n-1}}$, so, $\mathcal{T}z=\left\{\frac{1}{2^n}\right\}$. Thus for $y=\frac{1}{2^n}\in\mathcal{T}z$, we have

$$\mathcal{F}(d(z,y))-\mathcal{F}(D(z,\mathcal{T}z))=\mathcal{F}\left(\frac{1}{2^n}\right)-\mathcal{F}\left(\frac{1}{2^n}\right)=0.$$

Therefore, $y\in\mathcal{F}_\sigma^z$ for $\sigma>0$ with $\alpha(z,y)\geq\eta(z,y)$ and

$$\begin{aligned}\mathcal{F}(D(y,\mathcal{T}y))-\mathcal{F}(d(z,y))&=\mathcal{F}\left(\frac{1}{2^{n+1}}\right)-\mathcal{F}\left(\frac{1}{2^n}\right)\\&=\ln\left(\frac{1}{2^{n+1}}\right)-\ln\left(\frac{1}{2^n}\right)\\&=\ln\left(\frac{2^n}{2^{n+1}}\right)=\ln\left(\frac{1}{2}\right)\\&=-\ln 2.\end{aligned}$$

Hence $\tau+\mathcal{F}(D(y,\mathcal{T}y))\leq\mathcal{F}(d(z,y))$ is satisfied for $0<\sigma<\tau\leq\ln 2$.

Since $\alpha(z,y) \geq \eta(z,y)$ when $z,y \in \left\{\frac{1}{2^{n-1}} : n \in \mathbb{N}\right\}$, this implies that $\alpha(u,v) = 2 > 1 = \eta(u,v)$ for all $u \in \mathcal{T}z$ and $v \in \mathcal{T}y$. Hence $\mathcal{T}$ is generalized α_*-admissible mapping with respect to η.

Next, let $\lim_{n\to\infty} d(z_n, z) = 0$ and $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$, for all $n \in \mathbb{N}$, then $z_n \in \left\{\frac{1}{2^{n-1}} : n \in \mathbb{N}\right\}$. This implies that $\mathcal{T}z_n = \left\{\frac{1}{2^n}\right\}$ and $D(z_n, \mathcal{T}z_n) = \frac{1}{2^n}$, for all $n \in \mathbb{N}$. Here arises two cases:

Case I. $z_n \to z = 0$.
Then $\mathcal{T}z = \{0\}$ and $D(z, \mathcal{T}z) = 0$. Thus

$$\begin{aligned}\lim_{n\to\infty} \inf D(z_n, \mathcal{T}z_n) &= \lim_{n\to\infty} \inf\left(\frac{1}{2^n}\right)\\ &\geq 0 = D(z, \mathcal{T}z).\end{aligned}$$

Case II. $z_n \to z = \frac{1}{2^{n-1}}$.
Then $\mathcal{T}z = \left\{\frac{1}{2^n}\right\}$ and $D\left(z, \mathcal{T}z\right) = \frac{1}{2^n}$. Thus

$$\begin{aligned}\lim_{n\to\infty} \inf D(z_n, \mathcal{T}z_n) &= \lim_{n\to\infty} \inf(\frac{1}{2^n})\\ &= \frac{1}{2^n} = D(z, \mathcal{T}z).\end{aligned}$$

Hence $\mathcal{T}$ is α-η lower semi-continuous mapping. Thus, all conditions of Corollary 2.10 (and Theorem 2.8) hold and 0 is a fixed point of $\mathcal{T}$.

On the other hand, define $f : \mathcal{X} \to \mathbb{R}$, by $f(z) = D(z, \mathcal{T}z)$, for all $z \in \mathcal{X}$. Then

$$\lim_{z\to 1} \inf f(z) = 0 \ngeq \frac{1}{2} = f(1).$$

Hence f is not lower semi-continuous mapping at $z = 1$. That is, Theorems 1.6 and 1.7 can not be applied for this example.

Example 2.13. Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned}S_1 &= 1,\\ S_2 &= 1+2,\\ &\vdots\\ S_n &= 1+2+3+...+n = \frac{n(n+1)}{2},\\ &\vdots\end{aligned}$$

Let $\mathcal{X} = \{S_n : n \in \mathbb{N}\}$ with usual metric d. Then $(\mathcal{X}, d)$ is a metric space. Define $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$, $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, $\mathcal{G} : \mathbb{R}^4 \to \mathbb{R}^+$ and $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathcal{T}z = \left\{\begin{array}{cc} \{S_{n-1}, S_{n+1}\} & \text{if } \ z = S_n, \ n > 2,\\ \{z\} & \text{otherwise,}\end{array}\right.$$

$$\alpha(z,y) = \left\{\begin{array}{ll} 3 & \text{if } \ z \in \{S_n : n \geq 2\},\\ 1 & \text{otherwiswe,}\end{array}\right.$$

$\eta(z,y) = 2$, for all $z, y \in \mathcal{X}$, $\mathcal{G}(t_1,t_2,t_3,t_4) = L\min\{t_1,t_2,t_3,t_4\} + \tau$, where $\tau = \frac{1}{e^n}$, $n \in \mathbb{N}$, $L \in \mathbb{R}^+$ and $\mathcal{F}(r) = \ln(r)$. Then

$$D(z, \mathcal{T}z) = \left\{\begin{array}{ll} |n| & \text{if } \ z = S_n, \ n > 2,\\ 0 & \text{otherwise.}\end{array}\right.$$

Let $D(z,\mathcal{T}z)>0$, then $z=S_n, n>2$, so, $\mathcal{T}z=\{S_{n-1},S_{n+1}\}$. Thus for $y=S_{n-1}\in\mathcal{T}z$, we have

$$\mathcal{F}(d(z,y))-\mathcal{F}(D(z,\mathcal{T}z))=\mathcal{F}\left(|n|\right)-\mathcal{F}\left(|n|\right)=0.$$

Therefore, $y\in\mathcal{F}^z_\sigma$ for $\sigma=\frac{1}{e^{n+1}}$, $n\in\mathbb{N}$ with $\alpha(z,y)\geq\eta(z,y)$ and

$$\begin{aligned}\mathcal{F}(D(y,\mathcal{T}y))-\mathcal{F}(d(z,y))&=\mathcal{F}\left(|n-1|\right)-\mathcal{F}\left(|n|\right)\\&=\ln\left(|n-1|\right)-\ln\left(|n|\right)\\&=\ln\left(\frac{|n-1|}{|n|}\right)\\&<-\frac{1}{e^n}.\end{aligned}$$

This implies that $\tau+\mathcal{F}(D(y,\mathcal{T}y))\leq\mathcal{F}(d(z,y))$. Since $D(z,\mathcal{T}y)=0$, we have,

$$\begin{aligned}\mathcal{G}(D(z,\mathcal{T}z),D(y,\mathcal{T}y),D(z,\mathcal{T}y),D(y,\mathcal{T}z))+\mathcal{F}(D(y,\mathcal{T}y))&=\tau+\mathcal{F}(D(y,\mathcal{T}y))\\&\leq\mathcal{F}(d(z,y)).\end{aligned}$$

Hence condition (1) of Theorem 2.8 is satisfied for $0<\sigma=\frac{1}{e^{n+1}}<\tau=\frac{1}{e^n}$.

Since $\alpha(z,y)\geq\eta(z,y)$ when $z,y\in\{S_n:n\geq2\}$, this implies that $\alpha(u,v)=3>2=\eta(u,v)$ for all $u\in\mathcal{T}z$ and $v\in\mathcal{T}y$. Hence $\mathcal{T}$ is generalized α_*-admissible mapping with respect to η.

Next, let $\lim_{n\to\infty}d(z_n,z)=0$ and $\alpha(z_n,z_{n+1})\geq\eta(z_n,z_{n+1})$, for all $n\in\mathbb{N}$, then $z_n\in\{S_n:n\in\mathbb{N},n\geq2\}$. Here arises two cases:

Case I. $z_n\in\{S_n:n>2\}$.
Then $\mathcal{T}z_n=\{S_{n-1},S_{n+1}\}$ and $D(z_n,\mathcal{T}z_n)=|n|$, for all $n\in\mathbb{N}$.

Subcase I. $z_n\to z=S_n, n>2$.
Then $\mathcal{T}z=\{S_{n-1},S_{n+1}\}$ and $D(z,\mathcal{T}z)=|n|$. Thus

$$\begin{aligned}\lim_{n\to\infty}\inf D(z_n,\mathcal{T}z_n)&=\lim_{n\to\infty}\inf\left(|n|\right)\\&=|n|=D(z,\mathcal{T}z).\end{aligned}$$

Subcase II. $z_n\to z=S_1$.
Then $\mathcal{T}z=\{S_1\}$ and $D(z,\mathcal{T}z)=0$. Thus

$$\begin{aligned}\lim_{n\to\infty}\inf D(z_n,\mathcal{T}z_n)&=\lim_{n\to\infty}\inf\left(|n|\right)\\&\geq0=D(z,\mathcal{T}z).\end{aligned}$$

Subcase III. $z_n\to z=S_2$.
Then $\mathcal{T}z=\{S_2\}$ and $D(z,\mathcal{T}z)=0$. Thus

$$\begin{aligned}\lim_{n\to\infty}\inf D(z_n,\mathcal{T}z_n)&=\lim_{n\to\infty}\inf\left(|n|\right)\\&\geq0=D(z,\mathcal{T}z).\end{aligned}$$

Case II. $z_n\in\{S_2\}$.
Then z_n approaches to S_2 only. Therefore, $\mathcal{T}z_n=\{z_n\}$ and $\mathcal{T}z=\{z\}$. This implies that

$$\lim_{n\to\infty}\inf D(z_n,\mathcal{T}z_n)=0=D(z,\mathcal{T}z).$$

Hence $\mathcal{T}$ is α-η lower semi-continuous mapping. Thus, all the conditions of Theorem 2.8 hold and $\{S_1,S_2\}$ is set of fixed points of $\mathcal{T}$.

As an application of Theorems 2.8 and 2.9, we get the following results.

Theorem 2.14. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{G} \in \mathfrak{G}$ fulfill the conditions* (2) and (4) of Theorem 2.8. If for any $y, z \in \mathcal{X}$ with $\alpha(z, y) \geq \eta(z, y)$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$ we have

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\mathcal{T}$ is α-η continuous mapping.

Proof. By Lemma 2.3, we have $\mathcal{T}$ is α-η-lower semi-continuous mapping. Also, for $z \in \mathcal{X}$ and $y \in \mathcal{F}_\sigma^z$ with $D(z, \mathcal{T}z) > 0$ we have

$$\begin{aligned}\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(D(y, \mathcal{T}y))\\ &\leq \mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y))\\ &\leq \mathcal{F}(d(z, y)).\end{aligned}$$

Thus, all the conditions of Theorem 2.8 are satisfied, so, $\mathcal{T}$ has a fixed point. □

By similar arguments of Theorem 2.14, we state the following and omit its proof.

Theorem 2.15. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}_*$ and $\mathcal{G} \in \mathfrak{G}$ satisfy all assertions of Theorem* 2.14. *Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

On considering $\mathcal{G} = \mathcal{G}_L$, as in Corollary 2.10, Theorems 2.14 and 2.15 reduce to the following corollaries.

Corollary 2.16. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ fulfill the conditions* (2) and (4) of Theorem 2.8. If for any $y, z \in \mathcal{X}$ with $\alpha(z, y) \geq \eta(z, y)$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$ we have

$$\tau + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\mathcal{T}$ is α-η continuous mapping.

Corollary 2.17. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all assertions of Corollary* 2.16. *Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Theorem 2.18. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{G} \in \mathfrak{G}$. If for $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying*

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$ and $z \to D(z, \mathcal{T}z)$ is lower semi-continuous.

Proof. Define $\alpha(z, y) = d(z, y) = \eta(z, y)$ for all $z, y \in \mathcal{X}$. Then $\alpha(u, v) = d(z, y) = \eta(u, v)$, for all $u \in \mathcal{T}z$ and $v \in \mathcal{T}y$, that is, $\mathcal{T}$ is generalized α_*-admissible mapping with respect to η. Since $z \to D(z, \mathcal{T}z)$ is lower semi-continuous, therefore by Remark 2.7, $\mathcal{T}$ is α-η-lower semi-continuous. Thus, all the conditions of Theorem 2.8 holds. Hence $\mathcal{T}$ has a fixed point in $\mathcal{X}$. □

Theorem 2.19. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}_*$ and $\mathcal{G} \in \mathfrak{G}$. If for $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying*

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$ and $z \to D(z, \mathcal{T}z)$ is lower semi-continuous.

Proof. By defining $\alpha(z, y)$ and $\eta(z, y)$ the same as in proof of Theorem 2.18 and by using Theorem 2.8, we get the required result. □

Remark 2.20. By taking $\mathcal{G} = \mathcal{G}_L$, as in Corollary 2.11, in Theorems 2.18 and 2.19, we get Theorems 1.6 and 1.7.

3. Fixed point results for α-η-$\mathcal{F}$-contraction of Hardy-Rogers type

In this section we establish certain fixed point results for α-η-$\mathcal{F}$-contraction of Hardy-Rogers type.

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ fulfill the following assertions:*

1. *$\mathcal{T}$ is generalized α_*-admissible mapping with respect to η;*
2. *$\mathcal{T}$ is α-η lower semi-continuous mapping;*
3. *there exist $z_0 \in \mathcal{X}$ and $y_0 \in \mathcal{T}z_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$;*
4. *there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying

$$\begin{aligned}\tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(&a_1 d(z, y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y)\\ &+ a_4 D(z, \mathcal{T}y) + a_5 D(y, \mathcal{T}z)),\end{aligned}$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. Let $z_0 \in \mathcal{X}$, since $\mathcal{T}z \in K(\mathcal{X})$ for every $z \in \mathcal{X}$, the set $\mathcal{F}_\sigma^z$ is non-empty for any $\sigma > 0$, then there exists $z_1 \in \mathcal{F}_\sigma^{z_0}$ and by hypothesis $\alpha(z_0, z_1) \geq \eta(z_0, z_1)$. Assume that $z_1 \notin \mathcal{T}z_1$, otherwise z_1 is the fixed point of $\mathcal{T}$. Then, since $\mathcal{T}z_1$ is closed, $D(z_1, \mathcal{T}z_1) > 0$, so, from (4), we have

$$\begin{aligned}\tau(d(z_0, z_1)) + \mathcal{F}(D(z_1, \mathcal{T}z_1)) \leq \mathcal{F}(&a_1 d(z_0, z_1) + a_2 D(z_0, \mathcal{T}z_0) + a_3 D(z_1, \mathcal{T}z_1)\\ &+ a_4 D(z_0, \mathcal{T}z_1) + a_5 D(z_1, \mathcal{T}z_0)).\end{aligned}$$

Now for $z_1 \in \mathcal{X}$ there exists $z_2 \in \mathcal{F}_\sigma^{z_1}$ with $z_2 \notin \mathcal{T}z_2$, otherwise z_2 is the fixed point of $\mathcal{T}$, since $\mathcal{T}z_2$ is closed, so, $D(z_2, \mathcal{T}z_2) > 0$. Since $\mathcal{T}$ is generalized α_*-admissible mapping with respect to η, then $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$. Again by using (4), we get

$$\begin{aligned}\tau(d(z_1, z_2)) + \mathcal{F}(D(z_2, \mathcal{T}z_2)) \leq \mathcal{F}(&a_1 d(z_1, z_2) + a_2 D(z_1, \mathcal{T}z_1) + a_3 D(z_2, \mathcal{T}z_2)\\ &+ a_4 D(z_1, \mathcal{T}z_2) + a_5 D(z_2, \mathcal{T}z_1)).\end{aligned}$$

On continuing recursively, we get a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $\mathcal{X}$ such that $z_{n+1} \in \mathcal{F}_\sigma^{z_n}$, $z_{n+1} \notin \mathcal{T}z_{n+1}$, $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ and

$$\begin{aligned}\tau(d(z_n, z_{n+1})) + \mathcal{F}(D(z_{n+1}, \mathcal{T}z_{n+1})) \leq \mathcal{F}(&a_1 d(z_n, z_{n+1}) + a_2 D(z_n, \mathcal{T}z_n) + a_3 D(z_{n+1}, \mathcal{T}z_{n+1})\\ &+ a_4 D(z_n, \mathcal{T}z_{n+1}) + a_5 D(z_{n+1}, \mathcal{T}z_n)).\end{aligned}$$

As $z_{n+1} \in \mathcal{T}z_n$, this implies that

$$\begin{aligned}\tau(d(z_n, z_{n+1})) + \mathcal{F}(D(z_{n+1}, \mathcal{T}z_{n+1})) \leq \mathcal{F}(&a_1 d(z_n, z_{n+1}) + a_2 D(z_n, \mathcal{T}z_n)\\ &+ a_3 D(z_{n+1}, \mathcal{T}z_{n+1}) + a_4 D(z_n, \mathcal{T}z_{n+1})).\end{aligned} \tag{3.1}$$

Since $z_{n+1} \in \mathcal{F}_\sigma^{z_n}$, we have

$$\mathcal{F}(d(z_n, z_{n+1})) \leq \mathcal{F}(D(z_n, \mathcal{T}z_n)) + \sigma. \tag{3.2}$$

As $\mathcal{T}z_n$ and $\mathcal{T}z_{n+1}$ is compact, there exist $z_{n+1} \in \mathcal{T}z_n$ and $z_{n+2} \in \mathcal{T}z_{n+1}$ such that $d(z_n, z_{n+1}) = D(z_n, \mathcal{T}z_n)$ and $d(z_{n+1}, z_{n+2}) = D(z_{n+1}, \mathcal{T}z_{n+1})$, so equations (3.1) and (3.2) imply

$$\begin{aligned}\tau(d(z_n, z_{n+1})) + \mathcal{F}(d(z_{n+1}, z_{n+2})) \leq \mathcal{F}(&a_1 d(z_n, z_{n+1}) + a_2 d(z_n, z_{n+1})\\ &+ a_3 d(z_{n+1}, z_{n+2}) + a_4 d(z_n, z_{n+2})),\end{aligned}$$

and

$$\mathcal{F}(d(z_n, z_{n+1})) \leq \mathcal{F}(d(z_n, z_{n+1})) + \sigma. \tag{3.3}$$

Let $d_n = d(z_n, z_{n+1})$, for $n \in \mathbb{N}$, then

$$\begin{aligned} \tau(d_n) + \mathcal{F}(d_{n+1}) &\leq \mathcal{F}((a_1 + a_2)d_n + a_3 d_{n+1} + a_4 d(z_n, z_{n+2}) \\ &\leq \mathcal{F}((a_1 + a_2 + a_4)d_n + (a_3 + a_4)d_{n+1}). \end{aligned} \tag{3.4}$$

Assume that there exists $n \in \mathbb{N}$ such that $d_{n+1} \geq d_n$, then from (3.4), we get

$$\tau(d_n) + \mathcal{F}(d_{n+1}) \leq \mathcal{F}(d_{n+1}).$$

This is a contradiction to the fact that $\tau(d_n) > 0$. Hence $d_{n+1} < d_n$ for all $n \in \mathbb{N}$. This shows that sequence $\{d_n\}$ is decreasing. Therefore, there exists $\delta \geq 0$ such that $\lim_{n\to\infty} d_n = \delta$. Now let $\delta > 0$. From (3.4), we get

$$\tau(d_n) + \mathcal{F}(d_{n+1}) \leq \mathcal{F}(d_n). \tag{3.5}$$

Combining (3.3) and (3.5) gives

$$\begin{aligned} \mathcal{F}(d_{n+1}) &\leq \mathcal{F}(d_n) + \sigma - \tau(d_n) \\ &\leq \mathcal{F}(d_{n-1}) + 2\sigma - \tau(d_n) - \tau(d_{n-1}) \\ &\vdots \\ &\leq \mathcal{F}(d_0) + n\sigma - \tau(d_n) - \tau(d_{n-1}) - \cdots - \tau(d_0). \end{aligned} \tag{3.6}$$

Let $\tau(d_{p_n}) = \min\{\tau(d_0), \tau(d_1), \cdots, \tau(d_n)\}$ for all $n \in \mathbb{N}$. From (3.6), we get

$$\mathcal{F}(d_{n+1}) \leq \mathcal{F}(d_0) + n(\sigma - \tau(d_{p_n})). \tag{3.7}$$

From (3.6), we also get

$$\mathcal{F}(D(z_{n+1}, \mathcal{T}z_{n+1})) \leq \mathcal{F}(D(z_0, \mathcal{T}z_0)) + n(\sigma - \tau(d_{p_n})).$$

Now consider the sequence $\{\tau(d_{p_n})\}$. We distinguish two cases.

Case 1. For each $n \in \mathbb{N}$, there is $m > n$ such that $\tau(d_{p_n}) > \tau(d_{p_m})$. Then we obtain a subsequence $\{d_{p_{n_k}}\}$ of $\{d_{p_n}\}$ with $\tau(d_{p_{n_k}}) > \tau(d_{p_{n_{k+1}}})$ for all k. Since $d_{p_{n_k}} \to \delta^+$, we deduce that

$$\lim_{k\to\infty} \inf \tau(d_{p_{n_k}}) > \sigma.$$

Hence $\mathcal{F}(d_{n_k}) \leq \mathcal{F}(d_0) + n(\sigma - \tau(d_{p_{n_k}}))$ for all k. Consequently, $\lim_{k\to\infty} \mathcal{F}(d_{n_k}) = -\infty$ and by ($\mathcal{F}2$), we obtain $\lim_{k\to\infty} d_{p_{n_k}} = 0$, which contradicts that $\lim_{n\to\infty} d_n > 0$.

Case 2. There is $n_0 \in \mathbb{N}$ such that $\tau(d_{p_{n_0}}) > \tau(d_{p_m})$ for all $m > n_0$. Then $\mathcal{F}(d_m) \leq \mathcal{F}(d_0) + m(\sigma - \tau(d_{p_{n_0}}))$ for all $m > n_0$. Hence $\lim_{m\to\infty} \mathcal{F}(d_m) = -\infty$, so $\lim_{m\to\infty} d_m = 0$, which contradicts that $\lim_{m\to\infty} d_m > 0$. Thus, $\lim_{n\to\infty} d_n = 0$. From ($\mathcal{F}3$), there exists $0 < r < 1$ such that

$$\lim_{n\to\infty} (d_n)^r \mathcal{F}(d_n) = 0.$$

By (3.7), we get for all $n \in \mathbb{N}$

$$(d_n)^r \mathcal{F}(d_n) - (d_n)^r \mathcal{F}(d_0) \leq (d_n)^r n(\sigma - \tau(d - p_n)) \leq 0. \tag{3.8}$$

By letting $n \to \infty$ in (3.8), we obtain

$$\lim_{n\to\infty} n(d_n)^r = 0$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n(d_n)^r \leq 1$, or, $d_n \leq \frac{1}{n^{1/r}}$, for all $n > n_1$. Rest of the proof can be completed as in Theorem 2.8.

□

Following the arguments in the proof of Theorem 3.1 and taking $\mathcal{F} \in \mathfrak{F}_*$, we obtain the following result.

Theorem 3.2. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all conditions of Theorem 3.1. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

By taking $a_1 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$ in Theorems 3.1 and 3.2 respectively, we get the following.

Corollary 3.3. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ fulfill the following assertions:*

1. *$\mathcal{T}$ is generalized α_*-admissible mapping with respect to η;*
2. *$\mathcal{T}$ is α-η lower semi-continuous mapping;*
3. *there exist $z_0 \in \mathcal{X}$ and $y_0 \in \mathcal{T}z_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$;*
4. *there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying

$$\tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)).$$

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Corollary 3.4. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all conditions of Corollary 3.3. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

By taking $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5 = 1/2$ in Theorems 3.1 and 3.2 respectively, we get the following results for $\mathcal{F}$-contraction of Chatterjea type.

Corollary 3.5. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ fulfill the following assertions:*

1. *$\mathcal{T}$ is generalized α_*-admissible mapping with respect to η;*
2. *$\mathcal{T}$ is α-η lower semi-continuous mapping;*
3. *there exist $z_0 \in \mathcal{X}$ and $y_0 \in \mathcal{T}z_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$;*
4. *there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\liminf_{t \to s^+} \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying

$$\tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}\left(\frac{D(z, \mathcal{T}y) + D(y, \mathcal{T}z)}{2}\right).$$

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Corollary 3.6. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all conditions of Corollary 3.5. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

If we choose $a_4 = a_5 = 0$ in Theorems 3.1 and 3.2 respectively, we obtain the following results for $\mathcal{F}$-contraction of Reich-type.

Corollary 3.7. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ fulfill the following assertions:*

1. *$\mathcal{T}$ is generalized α_*-admissible mapping with respect to η;*

2. *$\mathcal{T}$ is α-η lower semi-continuous mapping;*
3. *there exist $z_0 \in \mathcal{X}$ and $y_0 \in \mathcal{T}z_0$ such that $\alpha(z_0, y_0) \geq \eta(z_0, y_0)$;*
4. *there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ with $\alpha(z, y) \geq \eta(z, y)$ satisfying

$$\tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(a_1 d(z, y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y)),$$

where $a_1, a_2, a_3 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 = 1$ and $a_3 \neq 1$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Corollary 3.8. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all conditions of Corollary* 3.7. *Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

As an application of Theorems 3.1 and 3.2, we obtain the following.

Theorem 3.9. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ fulfill the conditions* (1) and (3) of Theorem 3.1 and the following assertions:

1. $\mathcal{T}$ is α-η continuous mapping;
2. there exists a function $\tau : (0, \infty) \to (0, \infty)$ such that

$$\lim_{t \to s^+} \inf \tau(t) > 0, \quad \text{for all } s \geq 0,$$

and for any $y, z \in \mathcal{X}$ with $\alpha(z, y) \geq \eta(z, y)$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$ satisfying

$$\begin{aligned}\tau(d(z, y)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(&a_1 d(z, y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y) \\ &+ a_4 D(z, \mathcal{T}y) + a_5 D(y, \mathcal{T}z)),\end{aligned}$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. By Lemma 2.3, we have $\mathcal{T}$ is α-η-lower semi continuous mapping. Also, for $z \in \mathcal{X}$ and $y \in \mathcal{F}_\sigma^z$ with $D(z, \mathcal{T}z) > 0$, we have

$$\begin{aligned}\mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(&a_1 d(z, y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y) \\ &+ a_4 D(z, \mathcal{T}y) + a_5 D(y, \mathcal{T}z)) - \tau(d(z, y)).\end{aligned}$$

Thus, all conditions of Theorem 3.1 are satisfied. Hence $\mathcal{T}$ has a fixed point. □

By similar arguments of Theorem 3.9 and using Theorem 3.2, we state the following theorem.

Theorem 3.10. *Let $(\mathcal{X}, d)$ be a complete metric space and $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be two functions. Let $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all conditions of Theorem* 3.9. *Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Theorem 3.11. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying

$$\begin{aligned}\tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(&a_1 d(z, y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y) \\ &+ a_4 D(z, \mathcal{T}y) + a_5 D(y, \mathcal{T}z)),\end{aligned}$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$, then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, \mathcal{T}z)$ is lower semi-continuous.

Proof. Define $\alpha(z,y) = d(z,y) = \eta(z,y)$ for all $z, y \in \mathcal{X}$. Then by using Remark 2.7 and Theorem 3.1, we get the required result. □

Theorem 3.12. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all assertions of Theorem 3.11. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Proof. Define $\alpha(z,y) = d(z,y) = \eta(z,y)$ for all $z, y \in \mathcal{X}$. Then by using Remark 2.7 and Theorem 3.2, we get the required result. □

By taking $a_1 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$ in Theorems 3.11 and 3.12, we get the following corollaries.

Corollary 3.13 (Theorem 11 of [6]). *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If there exist $\sigma > 0$ and a function $\tau : (0,\infty) \to (\sigma, \infty)$ such that*

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying

$$\tau(d(z,y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z,y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, \mathcal{T}z)$ is lower semi-continuous.

Corollary 3.14 (Theorem 10 of [6]). *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all assertions of Corollary 3.13. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

By taking $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5 = 1/2$ in Theorems 3.11 and 3.12, we get the following.

Corollary 3.15. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If there exist $\sigma > 0$ and a function $\tau : (0,\infty) \to (\sigma, \infty)$ such that*

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying

$$\tau(d(z,y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}\left(\frac{D(z, \mathcal{T}y) + D(y, \mathcal{T}z)}{2}\right),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, \mathcal{T}z)$ is lower semi-continuous.

Corollary 3.16. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all assertions of Corollary 3.15. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

By choosing $a_4 = a_5 = 0$ in Theorems 3.11 and 3.12, we get the following.

Corollary 3.17. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If there exist $\sigma > 0$ and a function $\tau : (0,\infty) \to (\sigma, \infty)$ such that*

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying

$$\tau(d(z,y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}\left(a_1 d(z,y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y)\right),$$

where $a_1, a_2, a_3 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 = 1$ and $a_3 \neq 1$, then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, \mathcal{T}z)$ is lower semi-continuous.

Corollary 3.18. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all assertions of Corollary* 3.17*. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

Remark 3.19. Corollary 3.13 is a generalization of Theorem 2.3 of [26]. In fact, if τ is a constant, then $\mathcal{T}$ is a multivalued $\mathcal{F}$-contraction and every multivalued $\mathcal{F}$-contraction is multivalued nonexpansive and every multivalued nonexpansive map is upper semi-continuous, then $\mathcal{T}$ is upper semi-continuous. Therefore, the function $z \to D(z, \mathcal{T}z)$ is lower semi-continuous. On the other hand for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$ and $y \in \mathcal{F}_\sigma^z$, we have

$$\tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \tau(d(z, y)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)).$$

Hence $\mathcal{T}$ satisfies all conditions of Corollary 3.13. Similarly, Corollary 3.14 generalizes Theorem 2.5 of [26].

Remark 3.20. If we take $\mathcal{T}$, a single self-mapping on $\mathcal{X}$, Theorems 3.11 and 3.12 reduce to Theorem 1 of [30].

4. Fixed point results in partially ordered metric space

Let $(\mathcal{X}, d, \preceq)$ be a partially ordered metric space and $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ be a multivalued mapping. For $A, B \in 2^{\mathcal{X}}$, $A \preceq B$ implies that $a \preceq b$ for all $a \in A$ and $b \in B$. We say that $\mathcal{T}$ is monotone increasing, if $\mathcal{T}y \preceq \mathcal{T}z$, for all $y, z \in \mathcal{X}$, for which $y \preceq z$. There are many applications in differential and integral equations of monotone mappings in ordered metric spaces (see [2, 7, 16, 17] and references therein). In this section, from Sections 2 and 3, we derive the following new results in partially ordered metric spaces.

Theorem 4.1. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{G} \in \mathfrak{G}$ fulfill the following assertions:*

1. *if for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ with $z \preceq y$ satisfying*
$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y));$$
2. *$\mathcal{T}$ is monotone increasing;*
3. *there exist $z_0 \in \mathcal{X}$ and $y_0 \in \mathcal{T}z_0$ such that $z_0 \preceq y_0$;*
4. *for given $z \in X$ and sequence $\{z_n\}$ with $z_n \to z$ as $n \to \infty$ and $z_n \preceq z_{n+1}$ for all $n \in \mathbb{N}$, we have*
$$\lim_{n\to\infty} \inf D(z_n, \mathcal{T}z_n) \geq D(z, \mathcal{T}z),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$.

Proof. Define $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\alpha(z, y) = \begin{cases} 2 & z \preceq y, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta(z, y) = \begin{cases} 1 & z \preceq y, \\ 0 & \text{otherwise}, \end{cases}$$

then for $z, y \in \mathcal{X}$ with $z \preceq y$, $\alpha(y, z) \geq \eta(y, z)$ implies $u \preceq v$ for all $u \in \mathcal{T}z$ and $v \in \mathcal{T}y$. Hence $\alpha(u, v) = 2 > 1 = \eta(u, v)$, for all $u \in \mathcal{T}z$ and $v \in \mathcal{T}y$ and $\alpha(u, v) = \eta(u, v) = 0$ otherwise. This shows that $\mathcal{T}$ is generalized α_*-admissible mapping with respect to η. Thus, all the conditions of Theorem 2.8 are satisfied and $\mathcal{T}$ has a fixed point. □

By similar arguments as in Theorem 4.1, we state the following.

Theorem 4.2. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}_*$ and $\mathcal{G} \in \mathfrak{G}$ fulfill all conditions of Theorem* 4.1*. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$.*

Theorem 4.3. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{G} \in \mathfrak{G}$ fulfill the conditions* (2) *and* (3) *of Theorem* 4.1 *and the following assertions:*

1. *If for any $z, y \in \mathcal{X}$ with $z \preceq y$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$ satisfying*

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y));$$

2. *for given $z \in X$ and sequence $\{z_n\}$ with $z_n \to z$ as $n \to \infty$ and $z_n \preceq z_{n+1}$ for all $n \in \mathbb{N}$, we have $\mathcal{T}z_n \to \mathcal{T}z$,*

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Theorem 4.4. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}_*$ and $\mathcal{G} \in \mathfrak{G}$ fulfill all conditions of Theorem 4.3. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

By taking $\mathcal{G} = \mathcal{G}_L$, as in Corollary 2.10, Theorems 4.1–4.4 reduce to the following.

Corollary 4.5. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ satisfy conditions (2)-(4) of Theorem 4.1 and if for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ with $z \preceq y$ satisfying*

$$\tau + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$.

Corollary 4.6. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all conditions of Corollary 4.5. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$.*

Corollary 4.7. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ fulfill conditions (2)-(4) of Theorem 4.1 and if for any $z \in \mathcal{X}$ with $z \preceq y$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$ we have*

$$\tau + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Corollary 4.8. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfy all conditions of Corollary 4.7. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$.*

Theorem 4.9. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ fulfill the following assertions:*

1. *$\mathcal{T}$ is monotone increasing;*
2. *there exist $z_0 \in \mathcal{X}$ and $y_0 \in \mathcal{T}z_0$ such that $z_0 \preceq y_0$;*
3. *for given $z \in X$ and sequence $\{z_n\}$ with $z_n \to z$ as $n \to \infty$ and $z_n \preceq z_{n+1}$ for all $n \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \inf D(z_n, \mathcal{T}z_n) \geq D(z, \mathcal{T}z);$$

4. *there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\lim_{t\to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ with $z \preceq y$ satisfying

$$\tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(a_1 d(z, y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y) + a_4 D(z, \mathcal{T}y) + a_5 D(y, \mathcal{T}z)),$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Theorem 4.10. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ fulfill all conditions of Theorem 4.9. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$.*

Theorem 4.11. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$ fulfill the conditions (1) and (2) of Theorem 4.9 and the following assertions:*

1. *for given $z \in X$ and sequence $\{z_n\}$ with $z_n \to z$ as $n \to \infty$ and $z_n \preceq z_{n+1}$ for all $n \in \mathbb{N}$, we have $\mathcal{T}z_n \to \mathcal{T}z$;*
2. *there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*
$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \textit{for all } s \geq 0,$$
and for any $z, y \in \mathcal{X}$ with $z \preceq y$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$, satisfying
$$\begin{aligned}\tau(d(z, y)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(&a_1 d(z, y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y) \\ &+ a_4 D(z, \mathcal{T}y) + a_5 D(y, \mathcal{T}z)),\end{aligned}$$
where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Theorem 4.12. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ fulfill all conditions of Theorem 4.11. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$.*

5. Suzuki-Wardowski type fixed point results

In this section we establish certain fixed point results for Suzuki-Wardowski type multivalued $\mathcal{F}$-contractions.

Theorem 5.1. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If for $z, y \in \mathcal{X}$ with $\frac{1}{2}D(z, \mathcal{T}z) \leq d(z, y)$ and $D(z, \mathcal{T}z) > 0$, we have*
$$\tau + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)), \tag{5.1}$$
then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, \mathcal{T}z)$ is lower semi-continuous.

Proof. Suppose that $\mathcal{G} = \mathcal{G}_L$ as in Corollary 2.10. Let $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$ and $y \in \mathcal{F}^z_\sigma$, $\sigma < \tau$. Then $y \in \mathcal{T}z$, therefore we have $\frac{1}{2}D(z, \mathcal{T}z) \leq D(z, \mathcal{T}z) \leq d(z, y)$. So, by using (5.1), we get
$$\begin{aligned}\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(D(y, \mathcal{T}y)) &= \tau + \mathcal{F}(D(y, \mathcal{T}y)) \\ &\leq \mathcal{F}(d(z, y)).\end{aligned}$$

Thus, all conditions of Theorem 2.18 hold and $\mathcal{T}$ has a fixed point. □

Theorem 5.2. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$. If for $z, y \in \mathcal{X}$ with $\frac{1}{2}D(z, \mathcal{T}z) \leq d(z, y)$ and $D(z, \mathcal{T}z) > 0$, we have*
$$\tau + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$
then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, \mathcal{T}z)$ is lower semi-continuous.

Proof. By taking $\mathcal{G} = \mathcal{G}_L$ as in Corollary 2.10 and by using Theorem 2.19, we get the required result. □

Theorem 5.3. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If for $z, y \in \mathcal{X}$ with $\frac{1}{2}D(z, \mathcal{T}z) \leq d(z, y)$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$, we have*
$$\tau + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$
then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. Since every multivalued F-contraction is multivalued nonexpansive and every multivalued nonexpansive map is upper semi-continuous, then $\mathcal{T}$ is upper semi-continuous. Therefore, the function $z \to D(z, \mathcal{T}z)$ is lower semi-continuous (see the Proposition 4.2.6 of [3]). Also, for $z, y \in \mathcal{X}$ with $\frac{1}{2}D(z, \mathcal{T}z) \leq d(z, y)$ and $D(z, \mathcal{T}z) > 0$ we have

$$\begin{aligned} \tau + \mathcal{F}(D(y, \mathcal{T}y)) &\leq \tau + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \\ &\leq \mathcal{F}(d(z, y)). \end{aligned}$$

Thus, all conditions of Theorem 5.1 hold and $\mathcal{T}$ has a fixed point. □

By similar arguments as in Theorem 5.3, we state the following theorem and omit its proof.

Theorem 5.4. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$. If for $z, y \in \mathcal{X}$ with $\frac{1}{2} D(z, \mathcal{T}z) \leq d(z, y)$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$, we have*

$$\tau + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

By considering $\mathcal{T}$ a single-valued mapping in Theorem 5.3, we get the following.

Corollary 5.5. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ and $\mathcal{F} \in \mathfrak{F}$. If for $z, y \in \mathcal{X}$ with $\frac{1}{2} d(z, \mathcal{T}z) \leq d(z, y)$ and $d(\mathcal{T}z, \mathcal{T}y) > 0$, we have*

$$\tau + \mathcal{F}(d(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Remark 5.6. Corollary 5.5 is a generalization of the Corollary 3.1 of [18]. In fact, let Corollary 3.1 of [18] holds, then $\frac{1}{2}d(z, \mathcal{T}z) \leq d(z, \mathcal{T}z) \leq d(z, y)$. This implies that $\tau + \mathcal{F}(d(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y))$. Hence $\mathcal{T}$ satisfies all conditions of Corollary 5.5 and $\mathcal{T}$ has a fixed point.

Theorem 5.7. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}$. If there exists a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\lim_{t \to s^+} \inf \tau(t) > 0, \quad \text{for all } s \geq 0,$$

and for $z, y \in \mathcal{X}$ with $\frac{1}{2}D(z, \mathcal{T}z) \leq d(z, y)$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$, we have

$$\begin{aligned} \tau(d(z, y)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(&a_1 d(z, y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y) \\ &+ a_4 D(z, \mathcal{T}y) + a_5 D(y, \mathcal{T}z)), \end{aligned} \tag{5.2}$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$, then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. Let $\lim_{t \to s^+} \inf \tau(t) > \sigma$ for $\sigma > 0$, and for all $s \geq 0$. Also suppose that $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$ and $y \in \mathcal{F}_\sigma^z$, $\sigma < \tau$. Then $\lim_{t \to s^+} \inf \tau(t) > 0$ and $y \in \mathcal{T}z$, therefore we have $\frac{1}{2}D(z, \mathcal{T}z) \leq D(z, \mathcal{T}z) \leq d(z, y)$. So, by using (5.2), we get

$$\begin{aligned} \tau(d(z, y)) + \mathcal{F}(D(y, \mathcal{T}y)) &\leq \tau(d(z, y)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \\ &\leq \mathcal{F}(a_1 d(z, y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y) + a_4 D(z, \mathcal{T}y) + a_5 D(y, \mathcal{T}z)). \end{aligned}$$

Since $\mathcal{T}$ is continuous, then $\mathcal{T}$ is upper semi-continuous. Therefore, the function $z \to D(z, \mathcal{T}z)$ is lower semi-continuous (see the Proposition 4.2.6 of [3]). Thus, all conditions of Theorem 3.11 hold and $\mathcal{T}$ has a fixed point. □

Theorem 5.8. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ be a continuous mapping and $\mathcal{F} \in \mathfrak{F}_*$. If there exists a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\lim_{t \to s^+} \inf \tau(t) > 0, \quad \textit{for all } s \geq 0,$$

and for $z, y \in \mathcal{X}$ with $\frac{1}{2} D(z, \mathcal{T}z) \leq d(z, y)$ and $H(\mathcal{T}z, \mathcal{T}y) > 0$, we have

$$\begin{aligned}\tau(d(z,y)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(&a_1 d(z,y) + a_2 D(z, \mathcal{T}z) + a_3 D(y, \mathcal{T}y) \\ &+ a_4 D(z, \mathcal{T}y) + a_5 D(y, \mathcal{T}z)),\end{aligned}$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

Proof. By using the same arguments as in Theorem 5.7 and by using Theorem 3.12, we get the required result. □

6. Applications to orbitally lower semi-continuous mappings

Let $z_0 \in \mathcal{X}$ be any point. Then an orbit $O(z_0)$ of a mapping $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{X}}$ at a point z_0 is a set

$$O(z_0) = \{z_{n+1} : z_{n+1} \in \mathcal{T}z_n, \quad n = 0, 1, 2, ...\}.$$

Recall that a function $g : \mathcal{X} \to \mathbb{R}$ is called $\mathcal{T}$-orbitally lower semi-continuous, if for any sequence $\{z_n\}$ in $\mathcal{X}$ with $z_{n+1} \in \mathcal{T}z_n$ for all $n = 0, 1, 2, \cdots$, $g(z) \leq \lim_{n\to\infty} \inf g(z_n)$, whenever $\lim_{n\to\infty} z_n = z$ [9]. Many authors extended Nadler's fixed point theorem for lower semi-continuous mappings (see [13, 22, 23] and references therein). In this section, as an application of our results proved in Sections 1 and 2, we deduce certain fixed point theorems.

Theorem 6.1. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{G} \in \mathfrak{G}$. If for $z \in O(w), w \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}^z_\sigma$ satisfying*

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)), \tag{6.1}$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$ and $z \to D(z, \mathcal{T}z)$ is $\mathcal{T}$-orbitally lower semi-continuous.

Proof. Define $\alpha, \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ by

$$\alpha(z, y) = \begin{cases} 2 & \text{if} \quad z, y \in O(w), \\ 0 & \quad \text{otherwise}, \end{cases} \qquad \textit{and} \qquad \eta(z, y) = 1, \quad \forall z, y \in \mathcal{X}.$$

Then $\alpha(z, y) \geq \eta(z, y)$, when $z, y \in O(w)$. Since $z \to D(z, \mathcal{T}z)$ is $\mathcal{T}$-orbitally lower semi-continuous, so for any sequence $\{z_n\}$ in $\mathcal{X}$ with $z_{n+1} \in \mathcal{T}z_n$ and $\lim_{n\to\infty} d(z_n, z) = 0$, we have

$$D(z, \mathcal{T}z) \leq \lim_{n\to\infty} \inf D(z_n, \mathcal{T}z_n).$$

This implies that $\mathcal{T}$ is α-η-lower semi-continuous mapping. Now let $\alpha(z, y) \geq \eta(z, y)$, then $z, y \in O(w)$. So, for all $u \in \mathcal{T}z$ and $v \in \mathcal{T}y$ we have $u, v \in O(w)$. Therefore, $\alpha(u, v) = 2 > 1 = \eta(u, v)$. This shows that $\mathcal{T}$ is generalized α_*-admissible mapping with respect to η. Also, from equation (6.1), for any $z \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}^z_\sigma$ with $\alpha(z, y) \geq \eta(z, y)$, we have

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)).$$

Thus, all the conditions of Theorem 2.8 are satisfied and so $\mathcal{T}$ has a fixed point. □

By similar arguments as in Theorem 6.1, we state the following theorem and omit its proof.

Theorem 6.2. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}_*$ and $\mathcal{G} \in \mathfrak{G}$. If for $z \in O(w), w \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying*

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$ and $z \to D(z, \mathcal{T}z)$ is $\mathcal{T}$-orbitally lower semi-continuous.

Theorem 6.3. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}$ and $\mathcal{G} \in \mathfrak{G}$. If for $z, y \in O(w)$ with $H(\mathcal{T}z, \mathcal{T}y) > 0$ we have*

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\mathcal{T}$ is orbitally continuous.

Proof. By defining $\alpha(z, y)$, $\eta(z, y)$ the same as in the proof of Theorem 6.1 and applying Theorem 2.14, we get the required result. □

Theorem 6.4. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$, $\mathcal{F} \in \mathfrak{F}_*$ and $\mathcal{G} \in \mathfrak{G}$. If for $z, y \in O(w)$ with $H(\mathcal{T}z, \mathcal{T}y) > 0$ satisfying*

$$\mathcal{G}(D(z, \mathcal{T}z), D(y, \mathcal{T}y), D(z, \mathcal{T}y), D(y, \mathcal{T}z)) + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\mathcal{T}$ is orbitally continuous.

Proof. By defining $\alpha(z, y)$, $\eta(z, y)$ the same as in the proof of Theorem 6.1 and applying Theorem 2.15 we get the required result. □

By taking $\mathcal{G} = \mathcal{G}_L$, as in Corollary 2.11, Theorems 6.1, 6.2, 6.3 and 6.4 reduce to the following.

Corollary 6.5. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If for $z \in O(w), w \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying*

$$\tau + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$ and $z \to D(z, \mathcal{T}z)$ is $\mathcal{T}$-orbitally lower semi-continuous.

Corollary 6.6. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$. If for $z \in O(w), w \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying*

$$\tau + \mathcal{F}(D(y, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $\sigma < \tau$ and $z \to D(z, \mathcal{T}z)$ is $\mathcal{T}$-orbitally lower semi-continuous.

Corollary 6.7. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If for $z, y \in O(w)$ with $H(\mathcal{T}z, \mathcal{T}y) > 0$ satisfying*

$$\tau + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided T is orbitally continuous.

Corollary 6.8. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$. If for $z, y \in O(w)$ with $H(\mathcal{T}z, \mathcal{T}y) > 0$ satisfying*

$$\tau + \mathcal{F}(H(\mathcal{T}z, \mathcal{T}y)) \leq \mathcal{F}(d(z, y)),$$

then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided T is orbitally continuous.

Remark 6.9. If we take $\mathcal{T}$, a single mapping from $\mathcal{X}$ to $\mathcal{X}$, Theorems 6.3 and 6.4 reduce to the Theorem 4.1 of [18] and Corollaries 6.7 and 6.8 reduce to Corollary 4.1 of [18].

Theorem 6.10. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to K(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}$. If there exist $\sigma > 0$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that*

$$\lim_{t \to s^+} \inf \tau(t) > \sigma, \quad \text{for all } s \geq 0,$$

and for any $z \in O(w), w \in \mathcal{X}$ with $D(z, \mathcal{T}z) > 0$, there exists $y \in \mathcal{F}_\sigma^z$ satisfying

$$\begin{aligned}\tau(d(z,y)) + \mathcal{F}(D(y,\mathcal{T}y)) \leq \mathcal{F}(&a_1 d(z,y) + a_2 D(z,\mathcal{T}z) + a_3 D(y,\mathcal{T}y) \\ &+ a_4 D(z,\mathcal{T}y) + a_5 D(y,\mathcal{T}z)),\end{aligned}$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, +\infty)$ such that $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$, then $\mathcal{T}$ has a fixed point in $\mathcal{X}$ provided $z \to D(z, \mathcal{T}z)$ is $\mathcal{T}$-orbitally lower semi-continuous.

Proof. By defining $\alpha(z, y)$, $\eta(z, y)$ the same as in the proof of Theorem 6.1 and applying Theorem 3.11 we get the required result. □

Theorem 6.11. *Let $(\mathcal{X}, d)$ be a complete metric space, $\mathcal{T} : \mathcal{X} \to C(\mathcal{X})$ and $\mathcal{F} \in \mathfrak{F}_*$ satisfying all conditions of Theorem 6.10. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

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