



The corresponding inverse of functions of multidual complex variables in Clifford analysis

Ji Eun Kim

Department of Mathematics, Pusan National University, Busan 46241, Republic of Korea.

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Abstract

We aim to investigate the differentiability of multidual functions and the notion of the hyperholomorphicity to multidual-valued functions. Also, we provide the basic statements which extend holomorphic functions to the higher multidual generalized Clifford analysis. ©2016 All rights reserved.

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1. Introduction

In 1873, Algebra of dual numbers was introduced by Clifford [3] who informed that dual numbers form an algebra but not a field because dual numbers have no inverse elements. Kotelnikov [14] developed and generalized applications to mechanics in their principle of transference by dual vectors and dual quaternions. The mathematical study of dual functions with values in dual numbers has been done. In 2011, Ercan et al. [5] obtained generalized Euler's and De Moivre's formulas for functions with dual quaternion variable. Recently, Messelmi [16] generalized a theory of dual functions and gave the notion of holomorphic dual functions. Furthermore, the concept of multicomplex numbers has been introduced by a generator i , such that $i^n = -1$. Messelmi [17] contributed to the development of multidual numbers, by generalizing the dual numbers and their functions in higher dimensions. Moreover, he obtained conditions of differentiable multidual functions and hyperholomorphic functions to multidual variables. Wang et al. [18] investigated the geometric structure of unit dual quaternion and derived the exponential form of unit dual quaternion and its approximate logarithmic mapping.

Email address: jeunkim@pusan.ac.kr (Ji Eun Kim)

Brackx et al. [2] investigated a new multi-dimensional integral transform within the Clifford analysis setting and introduced it earlier as an operator exponential. Agarwal et al. [1] have studied results in the evaluation of generalized integrals and the solution of differential and integral equations using the computations of image formulas for special functions and various fractional calculus operators. Kumar et al. [15] have researched two fractional integral formulas involving the products of the multivariable H -function and a general class of polynomials.

Kajiwara et al. [6] studied regenerations of complex numbers, specially quaternions, and spaces of functions in infinite dimensional complex analysis and Clifford analysis. Deavours [4] gave the definitions and notions of quaternions and researched properties and theories of functions of quaternionic variables. Kim et al. [7, 8] obtained some results for the differentiability and the regularity of functions on the ternary quaternion and the reduced quaternion field and on dual split quaternions in Clifford analysis. Also, Kim and Shon [9, 10] researched corresponding Cauchy-Riemann systems by the regularity of functions with values in special quaternions such as reduced quaternions, split quaternions and dual quaternions in Clifford analysis. Kim and Shon [11, 12] investigated properties of a corresponding Cauchy-Riemann system and a regularity of functions with values in special quaternions defined by the corresponding differential operators of special quaternions systems. Kim and Shon [13] investigated the differentiation and integration for regular functions of bicomplex numbers satisfying the commutative multiplicative rule.

Here, in this paper, we give the notion of hyperholomorphicity to multidual functions and investigate the differentiability of multidual functions. Moreover, we provide the basic statements that extend holomorphically complex functions to the wider multidual complex Clifford algebra and generalize some usual complex functions to the multidual complex algebra.

2. Hyperholomorphic function of Dual complex variables

Throughout this paper, let \mathbf{R} , \mathbf{C} and \mathbf{N} be the sets of real numbers, complex numbers and positive integers, respectively. The set of dual complex number is

$$\mathbf{DC}_n = \left\{ Z = \sum_{r=0}^n z_r \varepsilon^r \mid z_i \in \mathbf{C}, \varepsilon^{n+1} = 0, \varepsilon^r \neq 0, r = 1, 2, \dots, n \right\},$$

which is isomorphic to \mathbf{C}^n , where one new element ε with the properties $\varepsilon^{n+1} = 0$ and $\varepsilon^r \neq 0, r = 1, 2, \dots, n$. For example, using a $(n + 1) \times (n + 1)$ matrix, ε can be represented as

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & \dots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & & 0 \end{bmatrix}.$$

For two elements Z and W of \mathbf{DC}_n , the addition and the multiplication of \mathbf{DC}_n are, respectively,

$$Z + W = \sum_{r=0}^n (z_r + w_r) \varepsilon^r, \quad ZW = \sum_{r=0}^n \left(\sum_{t=0}^r z_t w_{r-t} \right) \varepsilon^r = \sum_{r=0}^n \left(\sum_{t=0}^r w_t z_{r-t} \right) \varepsilon^r.$$

Also, the power of dual complex numbers is, for $m \in \mathbf{N} := \{1, 2, \dots\}$,

$$Z^m = \sum_{r_m=0}^n \left(\sum_{r_{m-1}=0}^{r_m} \dots \sum_{r_2=0}^{r_3} \sum_{r_1=0}^{r_2} z_{r_1} z_{r_2-r_1} z_{r_3-r_2} \dots z_{r_m-r_{m-1}} \right) \varepsilon^{r_m}.$$

The conjugate of Z is

$$Z^* = \bar{z}_0 + \bar{z}_1 \varepsilon + \bar{z}_2 \varepsilon^2 + \dots + \bar{z}_n \varepsilon^n = \sum_{r=0}^n \bar{z}_r \varepsilon^r,$$

where $z_k = x_k + iy_k, \bar{z}_k = x_k - iy_k, x_k, y_k \in \mathbb{R}$ for $k = 1, 2, \dots, n$. The conjugate of Z is described by

$$\begin{cases} \text{real}(Z) = \text{real}(Z^*), \\ ZZ^* \in \mathbf{R}. \end{cases} \tag{2.1}$$

Using the relation (2.1), we get

$$ZZ^* = z_0\bar{z}_0 + \sum_{r=1}^n \left(\sum_{t=0}^r z_t\bar{z}_{r-t} \right) \varepsilon^r \in \mathbf{R},$$

which implies that

$$\sum_{t=0}^r z_t\bar{z}_{r-t} = 0, \quad r = 1, 2, \dots, n.$$

In fact, the above equation is equivalent to

$$\sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \text{real}(z_r\bar{z}_{n-r}) = \begin{cases} 0, & n : \text{odd}; \\ -\frac{1}{2}|z_{\frac{n}{2}}|^2, & n : \text{even}, \end{cases}$$

$$\sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} (x_r x_{2n+r} + x_{r+2} x_{2(n-1)+r}) = \begin{cases} 0, & n : \text{odd}; \\ -\frac{1}{2}(x_n^2 + x_{n+1}^2), & n : \text{even}, \end{cases}$$

where $\lfloor \frac{n-1}{2} \rfloor$ denotes the greatest integer less than or equal to $\frac{n-1}{2}$. That is, the above equation is equivalent to

$$\sum_{r=0}^t x_r x_{t-r} = -\sum_{r=0}^t y_r y_{t-r}.$$

This can be written in matrix form

$$\begin{bmatrix} 2x_0 & 0 & \cdots & 0 & 0 \\ x_1 & 2x_0 & 0 & \cdots & 0 \\ x_2 & x_1 & \ddots & 0 & \vdots \\ \vdots & \vdots & & 2x_0 & 0 \\ x_{n-1} & x_{n-2} & \cdots & x_1 & 2x_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = - \begin{bmatrix} 2y_0 & 0 & \cdots & 0 & 0 \\ y_1 & 2y_0 & 0 & \cdots & 0 \\ y_2 & y_1 & \ddots & 0 & \vdots \\ \vdots & \vdots & & 2y_0 & 0 \\ y_{n-1} & y_{n-2} & \cdots & y_1 & 2y_0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

From the above elements, the inverse of Z is

$$Z^{-1} = \frac{Z^*}{ZZ^*} = \frac{Z^*}{|z_0|^2} \quad (z_0 \neq 0).$$

Let $Z = \sum_{r=0}^n z_r \varepsilon_r$ and $Z_0 = \sum_{r=0}^n z_{(0,r)} \varepsilon_r$ be two multidual complex numbers such that $z_0, z_{(0,0)} \neq 0$. Then,

$$(ZZ_0)^* = Z^* Z_0^*$$

and

$$(Z + Z_0)^* = Z^* + Z_0^*.$$

Every multidual complex number Z has another representation, using matrices, as follows:

$$Z = e_1^T \mathcal{N}(Z)^T \mathcal{E},$$

where e_1 is the first element in the canonical base of \mathbf{R}^{n+1} and e_1^T is the transpose of e_1 ,

$$\mathcal{N}(Z) = \begin{bmatrix} z_0 & 0 & \cdots & 0 & 0 \\ z_1 & z_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ z_{n-1} & z_{n-2} & \cdots & z_0 & 0 \\ z_n & z_{n-1} & \cdots & z_1 & z_0 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} 1 \\ \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^n \end{bmatrix},$$

and $\mathcal{N}(Z)^T$ is the transpose of $\mathcal{N}(Z)$.

Introducing the mapping

$$\begin{cases} \mathcal{P} & : \mathbf{DC}_n \rightarrow \mathbf{R}_+, \\ \mathcal{P}(Z) & := |z_0| = \sqrt{x_0^2 + y_0^2}, \quad Z \in \mathbf{DC}_n, \end{cases}$$

we have

$$\begin{cases} \mathcal{P}(Z + W) \leq \mathcal{P}(Z) + \mathcal{P}(W), \\ \mathcal{P}(ZW) = \mathcal{P}(Z)\mathcal{P}(W), \\ \mathcal{P}(\alpha Z) = |\alpha|\mathcal{P}(Z). \end{cases}$$

Then the inverse Z^{-1} of Z is also given by

$$Z^{-1} = \frac{Z^*}{\{\mathcal{P}(Z)\}^2} \quad (z_0 \neq 0).$$

We can give the multidisk and multidisk sphere of center

$$Z_0 = \sum_{r=0}^n z_{(0,r)} \varepsilon_{(0,r)} \in \mathbf{DC}_n$$

and radius $\gamma > 0$, respectively, by

$$\begin{aligned} D(Z_0, \gamma) &:= \{Z \in \mathbf{DC}_n \mid \mathcal{P}(Z - Z_0) < \gamma\} \\ &= \{Z \in \mathbf{DC}_n \mid |z_0 - z_{(0,0)}| = \sqrt{(x_0 - x_{(0,0)})^2 + (y_0 - y_{(0,0)})^2} < \gamma\}, \\ S(Z_0, \gamma) &:= \{Z \in \mathbf{DC}_n \mid \mathcal{P}(Z - Z_0) = \gamma\} \\ &= \{Z \in \mathbf{DC}_n \mid |z_0 - z_{(0,0)}| = \sqrt{(x_0 - x_{(0,0)})^2 + (y_0 - y_{(0,0)})^2} = \gamma\}. \end{aligned}$$

We say that Ω is a multidisk subset of \mathbf{DC}_n if there exists a subset $U \subset \mathbf{C}$ such that $\Omega = U \times \mathbf{C}^n$, where U is called the generator of Ω .

We say that Ω is an open multidisk subset of \mathbf{DC}_n if the generator of Ω is an open subset of \mathbf{C} .

Let Ω be an open set in \mathbf{DC}_n . A function F is called a multidisk complex function which is given by the following form in Ω with values in \mathbf{DC}_n :

$$F : \Omega \longrightarrow \mathbf{DC}_n$$

if it satisfies

$$F(Z) = F \left(\sum_{r=0}^n z_r \varepsilon_r \right) = \sum_{r=0}^n f_r(z_0, z_1, \dots, z_n) \varepsilon_r \in \mathbf{DC}_n,$$

where for $r = 0, 1, 2, \dots, n$,

$$\begin{aligned} f_r(z_0, z_1, \dots, z_n) &= u_r(x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n) \\ &\quad + i v_r(x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n) \end{aligned}$$

are complex-valued functions, where u_r and v_r are real-valued functions.

We give the differential operators as follows:

$$D := \frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1} \varepsilon + \frac{\partial}{\partial z_2} \varepsilon^2 + \cdots + \frac{\partial}{\partial z_n} \varepsilon^n = \sum_{r=0}^n \frac{\partial}{\partial z_r} \varepsilon^r$$

and

$$D^* = \frac{\partial}{\partial \bar{z}_0} + \frac{\partial}{\partial \bar{z}_1} \varepsilon + \frac{\partial}{\partial \bar{z}_2} \varepsilon^2 + \cdots + \frac{\partial}{\partial \bar{z}_n} \varepsilon^n = \sum_{r=0}^n \frac{\partial}{\partial \bar{z}_r} \varepsilon^r,$$

where $\frac{\partial}{\partial z_r}$ ($r = 0, 1, \dots, n$) and $\frac{\partial}{\partial \bar{z}_r}$ ($r = 0, 1, \dots, n$) are usual complex differential operators in complex analysis.

Definition 2.1. Let Ω be an open set in \mathbf{DC}_n . A function F is said to be hyperholomorphic in Ω if a function F satisfies

- (i) f_r ($r = 0, 1, \dots, n$) are continuously differential functions in Ω and
- (ii) $D^*F = 0$ in Ω .

Remark 2.2. The second condition of Definition 2.1 is equivalent to the following equations:

$$0 = D^*F = \sum_{r=0}^n \left(\sum_{t=0}^r \frac{\partial f_{r-t}}{\partial \bar{z}_t} \right) \varepsilon^r,$$

that is,

$$\sum_{t=0}^r \frac{\partial f_{r-t}}{\partial \bar{z}_t} = 0 \quad (r = 0, 1, \dots, n),$$

for example,

$$\begin{aligned} \frac{\partial f_0}{\partial \bar{z}_0} &= 0, \\ \frac{\partial f_1}{\partial \bar{z}_0} &= -\frac{\partial f_0}{\partial \bar{z}_1}, \\ \frac{\partial f_2}{\partial \bar{z}_0} + \frac{\partial f_0}{\partial \bar{z}_2} &= -\frac{\partial f_1}{\partial \bar{z}_1}, \\ \frac{\partial f_3}{\partial \bar{z}_0} + \frac{\partial f_0}{\partial \bar{z}_3} &= -\frac{\partial f_2}{\partial \bar{z}_1} - \frac{\partial f_1}{\partial \bar{z}_2}, \\ \frac{\partial f_4}{\partial \bar{z}_0} + \frac{\partial f_3}{\partial \bar{z}_1} + \frac{\partial f_1}{\partial \bar{z}_3} + \frac{\partial f_0}{\partial \bar{z}_4} &= -\frac{\partial f_2}{\partial \bar{z}_2}, \\ &\vdots \end{aligned}$$

Example 2.3. Let Ω be an open set in \mathbf{DC}_n . Since a function $F(Z) = Z^m$ ($m \in \mathbf{N}$) satisfies the following equation:

$$D^*F(Z) = \sum_{t=0}^{r_m} \frac{\partial f_{r_m-t}}{\partial \bar{z}_t} = 0,$$

where

$$f_{r_m-t}(z_0, \dots, z_n) = \sum_{r_{m-1}=0}^{r_m} \cdots \sum_{r_2=0}^{r_3} \sum_{r_1=0}^{r_2} z_{r_1} z_{r_2-r_1} z_{r_3-r_2} \cdots z_{r_m-r_{m-1}},$$

$0 \leq r_1 < r_2 < \cdots < r_m \leq n$ and $F(Z) = Z^m$ is a hyperholomorphic function in Ω .

Example 2.4. Let Ω be an open set in \mathbf{DC}_n . Since a function $F(Z) = Z^{-1}$ satisfies the following equation:

$$D^*F = \sum_{t=0}^{r_m} \frac{\partial f_{r_m-t}}{\partial \bar{z}_t} \neq 0,$$

where

$$f_{r_m-t}(z_0, \dots, z_n) = \frac{1}{z_0 \bar{z}_0} (\bar{z}_0 + \bar{z}_1 \varepsilon + \bar{z}_2 \varepsilon^2 + \dots + \bar{z}_n \varepsilon^n), \quad z_0 \neq 0,$$

a function $F(Z) = Z^{-1}$ is not hyperholomorphic in Ω . Also, $F(Z) = Z^*$ is not hyperholomorphic in Ω .

Theorem 2.5. Let Ω be an open set in \mathbf{DC}_n and F be a multidual complex function in Ω , which can be written by

$$F(Z) = \sum_{r=0}^n f_r(z_0, \dots, z_n) \varepsilon^r.$$

Then if the function F is hyperholomorphic in Ω , F satisfies

$$DF(Z) = \sum_{r=0}^n \left(\sum_{t=0}^r \frac{\partial f_{r-t}}{\partial x_t} \right) \varepsilon^r = -i \sum_{r=0}^n \left(\sum_{t=0}^r \frac{\partial f_{r-t}}{\partial y_t} \right) \varepsilon^r.$$

Proof. We have

$$\begin{aligned} DF(Z) &= \sum_{r=0}^n \left(\sum_{t=0}^r \frac{\partial f_{r-t}}{\partial z_t} \right) \varepsilon^r \\ &= \frac{\partial f_0}{\partial z_0} + \left(\frac{\partial f_1}{\partial z_0} + \frac{\partial f_0}{\partial z_1} \right) \varepsilon + \left(\frac{\partial f_2}{\partial z_0} + \frac{\partial f_1}{\partial z_1} + \frac{\partial f_0}{\partial z_2} \right) \varepsilon^2 \\ &\quad + \dots + \left(\frac{\partial f_n}{\partial z_0} + \frac{\partial f_{n-1}}{\partial z_1} + \dots + \frac{\partial f_0}{\partial z_n} \right) \varepsilon^n. \end{aligned}$$

Since F is a hyperholomorphic function in Ω , we have

$$\begin{aligned} DF &= \frac{\partial f_0}{\partial x_0} + \left(\frac{\partial f_1}{\partial x_0} + \frac{\partial f_0}{\partial x_1} \right) \varepsilon + \left(\frac{\partial f_2}{\partial x_0} + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_0}{\partial x_2} \right) \varepsilon^2 + \dots \\ &\quad + \left(\frac{\partial f_n}{\partial x_0} + \frac{\partial f_{n-1}}{\partial x_1} + \dots + \frac{\partial f_0}{\partial x_n} \right) \varepsilon^n \\ &= \sum_{r=0}^n \left(\sum_{t=0}^r \frac{\partial f_{r-t}}{\partial x_t} \right) \varepsilon^r, \end{aligned}$$

and similarly, we obtain

$$DF = -i \sum_{r=0}^n \left(\sum_{t=0}^r \frac{\partial f_{r-t}}{\partial x_t} \right) \varepsilon^r.$$

□

3. The Inverse of a multidual complex function

Let Ω be an open set in \mathbf{DC}_n and F be a multidual complex function in Ω . Let $Z(F) = \{Z \mid F(Z) = 0, Z \in \Omega\}$ be a null set such that $f_r = 0$ ($r = 0, 1, 2, \dots, n$). Then $Z(F)$ is of measure 0 in Ω . For example, if $F(Z) = Z$, then $Z(F) = \{0\}$.

We call the inverse function F^{-1} of F if

$$F^{-1} : Z \rightarrow F^{-1}(Z),$$

which is defined on almost everywhere on Ω such that

$$F^{-1} = \frac{F^*}{FF^*} = \frac{F^*}{f_0\bar{f}_0} = \frac{1}{|f_0|^2} \sum_{r=0}^n \bar{f}_r \varepsilon^r \quad (f_0 \neq 0),$$

where $F^* = \sum_{r=0}^n \bar{f}_r(z_0, z_1, \dots, z_n) \varepsilon_r$ and

$$\begin{aligned} \bar{f}_r(z_0, z_1, \dots, z_n) &= u_r(x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n) \\ &\quad - i v_r(x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n), \end{aligned}$$

($r = 0, 1, 2, \dots, n$). Also, if a function F is hyperholomorphic in Ω and $Z(F) = \{0\}$, then F^{-1} is not necessarily hyperholomorphic outside $\{0\}$ (see Example 2.4).

We consider multidual complex functions defined on almost everywhere on \mathbb{R} , that is, we deal with multidual complex functions defined outside of a subset of Ω with Lebesgue measure 0.

Theorem 3.1. *Let F be a multidual complex function in Ω and suppose that F is hyperholomorphic in Ω . Then F^{-1} is a hyperholomorphic function in Ω if and only if the following formulas hold*

$$\frac{\partial f_0}{\partial z_m} \bar{f}_0^2 + \frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_m = \sum_{r=0}^{m-1} \frac{\partial}{\partial z_r} \left(\frac{\bar{f}_{m-r}}{N_f} \right) \quad (m = 1, 2, \dots, n), \tag{3.1}$$

where $N_f = |f_0|^2$ and $f_0 \neq 0$.

Proof. Suppose F^{-1} is a hyperholomorphic function in Ω . Then we have

$$\begin{aligned} D^* F^{-1} &= \sum_{r=0}^n \left(\sum_{t=0}^r \frac{\partial}{\partial z_t} \left(\frac{\bar{f}_{r-t}}{N_f} \right) \right) \varepsilon^r \\ &= \frac{1}{N_f^2} \left\{ \left(\frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_0} \bar{f}_0^2 - \frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_0 \right) \right. \\ &\quad + \left(\frac{\partial \bar{f}_1}{\partial z_0} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_0} \bar{f}_0 \bar{f}_1 - \frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_1 + \frac{\partial \bar{f}_0}{\partial z_1} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_1} \bar{f}_0^2 - \frac{\partial \bar{f}_0}{\partial z_1} f_0 \bar{f}_0 \right) \varepsilon \\ &\quad + \left(\frac{\partial \bar{f}_2}{\partial z_0} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_0} \bar{f}_0 \bar{f}_2 - \frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_2 + \frac{\partial \bar{f}_1}{\partial z_1} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_1} \bar{f}_0 \bar{f}_1 \right. \\ &\quad \left. - \frac{\partial \bar{f}_0}{\partial z_1} f_0 \bar{f}_1 + \frac{\partial \bar{f}_0}{\partial z_2} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_2} \bar{f}_0^2 - \frac{\partial \bar{f}_0}{\partial z_2} f_0 \bar{f}_0 \right) \varepsilon^2 + \dots \left. \right\} \\ &= \frac{1}{N_f^2} \left\{ \left(\frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_0} \bar{f}_0^2 - \frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_0 \right) \right. \\ &\quad + \left(\frac{\partial \bar{f}_1}{\partial z_0} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_0} \bar{f}_0 \bar{f}_1 - \frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_1 + \frac{\partial \bar{f}_0}{\partial z_1} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_1} \bar{f}_0^2 - \frac{\partial \bar{f}_0}{\partial z_1} f_0 \bar{f}_0 \right) \varepsilon \\ &\quad + \left(\frac{\partial \bar{f}_2}{\partial z_0} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_0} \bar{f}_0 \bar{f}_2 - \frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_2 + \frac{\partial \bar{f}_1}{\partial z_1} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_1} \bar{f}_0 \bar{f}_1 \right. \\ &\quad \left. - \frac{\partial \bar{f}_0}{\partial z_1} f_0 \bar{f}_1 + \frac{\partial \bar{f}_0}{\partial z_2} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_2} \bar{f}_0^2 - \frac{\partial \bar{f}_0}{\partial z_2} f_0 \bar{f}_0 \right) \varepsilon^2 + \dots \left. \right\} \\ &= \frac{1}{N_f^2} \left\{ \left(\frac{\partial \bar{f}_1}{\partial z_0} f_0 \bar{f}_0 - \frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_1 - \frac{\partial f_0}{\partial z_1} \bar{f}_0^2 \right) \varepsilon + \left(\frac{\partial \bar{f}_2}{\partial z_0} f_0 \bar{f}_0 - \frac{\partial \bar{f}_0}{\partial z_0} f_0 \bar{f}_2 \right. \right. \\ &\quad \left. \left. + \frac{\partial \bar{f}_1}{\partial z_1} f_0 \bar{f}_0 - \frac{\partial f_0}{\partial z_1} \bar{f}_0 \bar{f}_1 - \frac{\partial \bar{f}_0}{\partial z_1} f_0 \bar{f}_1 - \frac{\partial f_0}{\partial z_2} \bar{f}_0^2 \right) \varepsilon^2 + \dots \right\}, \end{aligned}$$

the following equations are satisfied:

$$\begin{aligned} \frac{\partial \overline{f_0}}{\partial z_0} f_0 \overline{f_1} &= \frac{\partial \overline{f_1}}{\partial z_0} f_0 \overline{f_0} - \frac{\partial f_0}{\partial z_1} \overline{f_0}^2, \\ \frac{\partial \overline{f_0}}{\partial z_0} f_0 \overline{f_2} + \frac{\partial \overline{f_0}}{\partial z_1} f_0 \overline{f_1} + \frac{\partial f_0}{\partial z_1} \overline{f_0} \overline{f_1} &= \frac{\partial \overline{f_2}}{\partial z_0} f_0 \overline{f_0} + \frac{\partial \overline{f_1}}{\partial z_1} f_0 \overline{f_0} - \frac{\partial f_0}{\partial z_2} \overline{f_0}^2, \\ \frac{\partial \overline{f_0}}{\partial z_0} f_0 \overline{f_3} + \frac{\partial \overline{f_0}}{\partial z_1} f_0 \overline{f_2} + \frac{\partial f_0}{\partial z_1} \overline{f_0} \overline{f_2} + \frac{\partial f_0}{\partial z_2} \overline{f_0} \overline{f_1} + \frac{\partial \overline{f_0}}{\partial z_2} f_0 \overline{f_1} &= \frac{\partial \overline{f_3}}{\partial z_0} f_0 \overline{f_0} + \frac{\partial \overline{f_2}}{\partial z_1} f_0 \overline{f_0} + \frac{\partial \overline{f_1}}{\partial z_2} f_0 \overline{f_0} - \frac{\partial f_0}{\partial z_3} \overline{f_0}^2, \\ &\vdots \\ \frac{\partial \overline{f_0}}{\partial z_0} f_0 \overline{f_m} + \sum_{r=0}^{m-1} \left(\frac{\partial \overline{f_0}}{\partial z_r} f_0 \overline{f_{m-r}} + \frac{\partial f_0}{\partial z_r} \overline{f_0} \overline{f_{m-r}} \right) &= \sum_{r=0}^{m-1} \frac{\partial \overline{f_{m-r}}}{\partial z_r} f_0 \overline{f_0} - \frac{\partial f_0}{\partial z_m} \overline{f_0}^2, \end{aligned}$$

where $m = 1, 2, 3, \dots, n$. Therefore, by arranging terms of the above equations, we obtain the following result:

$$\frac{\partial f_0}{\partial z_m} \overline{f_0}^2 + \frac{\partial \overline{f_0}}{\partial z_0} f_0 \overline{f_m} = \sum_{r=0}^{m-1} \frac{\partial}{\partial z_r} \left(\frac{\overline{f_{m-r}}}{N_f} \right), \quad (m = 1, 2, \dots, n).$$

Conversely, since F is hyperholomorphic in Ω and we have the equation (3.1), we obtain

$$\begin{aligned} D^* F^{-1} &= \frac{1}{N_f^2} \left\{ \left(\frac{\partial \overline{f_1}}{\partial z_0} f_0 \overline{f_0} - \frac{\partial \overline{f_0}}{\partial z_0} f_0 \overline{f_1} - \frac{\partial f_0}{\partial z_1} \overline{f_0}^2 \right) \varepsilon \right. \\ &\quad + \left(\frac{\partial \overline{f_2}}{\partial z_0} f_0 \overline{f_0} - \frac{\partial \overline{f_0}}{\partial z_0} f_0 \overline{f_2} + \frac{\partial \overline{f_1}}{\partial z_1} f_0 \overline{f_0} - \frac{\partial f_0}{\partial z_1} \overline{f_0} \overline{f_1} - \frac{\partial \overline{f_0}}{\partial z_1} f_0 \overline{f_1} - \frac{\partial f_0}{\partial z_2} \overline{f_0}^2 \right) \varepsilon^2 \\ &\quad \left. + \dots + \left(\frac{\partial \overline{f_0}}{\partial z_0} f_0 \overline{f_m} + \sum_{r=1}^{m-1} \left(\frac{\partial \overline{f_0}}{\partial z_r} f_0 \overline{f_{m-r}} + \frac{\partial f_0}{\partial z_r} \overline{f_0} \overline{f_{m-r}} \right) - \sum_{r=0}^{m-1} \frac{\partial \overline{f_{m-r}}}{\partial z_r} f_0 \overline{f_0} + \frac{\partial f_0}{\partial z_m} \overline{f_0}^2 \right) \varepsilon^n \right\} = 0. \end{aligned}$$

Therefore, the inverse function F^{-1} is hyperholomorphic in Ω . □

Definition 3.2. Let F be a multidual complex function in Ω . F is called a weak hypermeromorphic function on any almost everywhere on \mathbb{R} if there exist a hyperholomorphic function $G(Z) = \sum_{r=0}^n g_r(z_0, \dots, z_n) \varepsilon^r$ and one inverse function $H(Z) = \sum_{r=0}^n h_r(z_0, \dots, z_n) \varepsilon^r$ of a multidual complex function which is hyperholomorphic on almost everywhere on \mathbb{R} such that $F(Z) = G(Z)H(Z)$, where g_r and h_r are complex-valued functions defined on Ω .

Definition 3.3. Let P be a multidual complex function in Ω . P is called a hypermeromorphic function on any almost everywhere on \mathbb{R} if there exists a weak-hypermeromorphic function Q such that $P + Q$ and PQ are weak-hypermeromorphic.

Theorem 3.4. Let Ω be an open set in \mathbf{DC}_n and F be a hyperholomorphic function in Ω . If F is a hypermeromorphic function in Ω , then there exists a multidual complex function H in Ω which satisfies the following equation:

$$\frac{\partial h_0}{\partial z_m} \overline{h_0}^2 + \frac{\partial \overline{h_0}}{\partial z_0} h_0 \overline{h_m} = \sum_{r=0}^{m-1} \frac{\partial}{\partial z_r} \left(\frac{\overline{h_{m-r}}}{N_h} \right) \quad (m = 1, 2, \dots, n), \tag{3.2}$$

where $N_h = |h_0|^2$ and $h_0 \neq 0$.

Proof. Since F is a hypermeromorphic function in Ω , by Definitions 3.2 and 3.3, there is the inverse function of one hyperholomorphic function which is hyperholomorphic on almost everywhere on \mathbb{R} . We put the inverse

function to a function H . Then the inverse function H^{-1} is hyperholomorphic in Ω . By Theorem 3.1, we have

$$\begin{aligned}
 D^*H^{-1} &= \frac{1}{N_h^2} \left\{ \left(\frac{\partial \bar{h}_1}{\partial \bar{z}_0} h_0 \bar{h}_0 - \frac{\partial \bar{h}_0}{\partial \bar{z}_0} h_0 \bar{h}_1 - \frac{\partial h_0}{\partial \bar{z}_1} \bar{h}_0^{-2} \right) \varepsilon \right. \\
 &\quad \left. + \cdots + \left(\frac{\partial \bar{h}_0}{\partial \bar{z}_0} h_0 \bar{h}_m + \sum_{r=1}^{m-1} \left(\frac{\partial \bar{h}_0}{\partial \bar{z}_r} h_0 \bar{h}_{m-r} + \frac{\partial h_0}{\partial \bar{z}_r} \bar{h}_0 \bar{h}_{m-r} \right) - \sum_{r=0}^{m-1} \frac{\partial \bar{h}_{m-r}}{\partial \bar{z}_r} h_0 \bar{h}_0 + \frac{\partial h_0}{\partial \bar{z}_m} \bar{h}_0^{-2} \right) \varepsilon^n \right\} \\
 &= 0.
 \end{aligned}$$

Therefore, we obtain the equations (3.2). □

Example 3.5. Let Ω be an open set in \mathbf{DC}_n . Since a function $F(Z) = Z$ satisfies the following equation

$$D^*F = \sum_{t=0}^r \frac{\partial z_{r-t}}{\partial \bar{z}_t} = 0,$$

the function $F(Z) = Z$ is hyperholomorphic in Ω . Moreover, we have

$$\frac{\partial f_0}{\partial \bar{z}_m} \bar{f}_0^{-2} + \frac{\partial \bar{f}_0}{\partial \bar{z}_0} f_0 \bar{f}_m = z_0 \bar{z}_1, \quad \frac{\partial}{\partial \bar{z}_0} \left(\frac{\bar{f}_1}{N_f} \right) = -\frac{\bar{z}_1}{z_0}, \quad (m = 1).$$

Since F doesn't satisfy the equation (3.1), we can also find that F^{-1} is not hyperholomorphic in Ω .

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