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Exact solutions and dynamics of generalized AKNS equations associated with the nonisospectral depending on exponential function

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Abstract

No matter constructing or solving nonlinear evolution equations (NLEEs), it is important and interesting in the field of nonlinear science. In this paper, generalized Ablowitz–Kaup–Newell–Segur (AKNS) equations are constructed and solved exactly. To be specific, the famous AKNS spectral problem is first generalized by embedding a nonisospectral parameter whose varying with time obeys the exponential function of spectral parameter. Based on the generalized AKNS spectral problem and its corresponding time evolution equation, we then derive a generalized AKNS equation with infinite number of terms. Furthermore, exact solutions of the generalized AKNS equations are formulated through the inverse scattering transform method. Finally, in the case of reflectionless potentials, the obtained exact solutions are reduced to explicit n-soliton solutions. It is shown that the dynamical evolutions of such soliton solutions possess not only time-varying speeds and amplitudes but also singular points in the process of propagations.

Keywords: Exact solution, n-soliton solution, dynamical evolution, generalized AKNS equations, inverse scattering transform.
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1. Introduction

Recently, constructing NLEEs is relatively active because of searching for as many as meaningful NLEEs and studying their properties are of both theoretical and practical value [34, 35, 45, 48, 49]. In general, there

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are two sets of NLEEs called the isospectral hierarchy and nonisospectral hierarchy respectively. Starting from a proper linear spectral problem, we can derive a hierarchy of isospectral NLEEs often describing solitary waves in lossless and uniform media if the associated spectral parameter is independent of time. While nonisospectral NLEEs which describe the solitary waves in a certain type of nonuniform media are usually resulted from the spectral problem with a time-dependent spectral parameter. In 1974, Ablowitz et al. [2] derived a hierarchy of isospectral NLEEs from the zero curvature equation

$$M_t - N_x + [M, N] = 0, \tag{1.1}$$

which is namely the compatibility condition of the following spectral problem, that is, the famous AKNS spectral problem

$$\varphi_x = M\varphi, \quad M = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1.2}$$

and its time evolution equation

$$\varphi_t = N\varphi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{1.3}$$

where the potentials $q = q(x, t)$, $r = r(x, t)$ and their derivatives of any order with respect to x and t are smooth functions which vanish as x tends to infinity, the spectral parameter k is independent of x and t (that is, $k_t = 0$), and A, B, and C are undetermined functions of x, t, q, r and k. It should be noted that the isospectral AKNS hierarchy [14]

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -q \\ r \end{pmatrix}, \quad n = 0, 1, 2, \cdots, \tag{1.4}$$

derived from Eqs. (1.1)–(1.3) includes two nontrivial equations in the cases of $n = 1, 2$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -q_{xx} + 2q^2 r \\ r_{xx} - 2qr^2 \end{pmatrix}, \tag{1.5}$$

and

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q_{xxx} - 6qrq_x \\ r_{xxx} - 6qrr_x \end{pmatrix}. \tag{1.6}$$

In Eqs. (1.4), the operator L is defined as follows

$$L = \sigma\partial + 2\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r, q), \quad \partial = \frac{\partial}{\partial x}, \quad \partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} \mathrm{d}x - \int_{x}^{+\infty} \mathrm{d}x\right), \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.7}$$

If we set $q = u$ and $r = -1$, then Eq. (1.6) reduces to the celebrated Korteweg-de Vries (KdV) equation $u_t = u_{xxx} + 6uu_x$. When $q = v$ and $r = \mp v$, Eq. (1.6) is converted into the modified KdV (mKdV) equation $v_t = v_{xxx} + 6v^2 v_x$. Subsequently, Celogero and Degasperis [8–10] and Li [24] developed different methods to construct hierarchies of nonisospectral NLEEs under the case of spectral parameter k being dependent on time t (that is, $k_t \neq 0$). For example, letting $ik_t = \frac{1}{2}(2ik)^n$ and using Eqs. (1.1)–(1.3) we can construct the following nonisospectral AKNS hierarchy [14]

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad n = 0, 1, 2, \cdots, \tag{1.8}$$

in which the following two pair of nonisospectral equations ($n = 1, 2$) are included

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q + xq_x \\ r + xr_x \end{pmatrix}, \tag{1.9}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -2q_x - xq_{xx} + 2q\partial^{-1}qr + 2xq^2r \\ 2r_x + xr_{xx} - 2r\partial^{-1}qr - 2xqr^2 \end{pmatrix}. \tag{1.10}$$

Due to appearance of nonisospectral NLEEs, the types of integrable equations were extremely enriched. In the past two decades, constructing nonisospectral NLEEs has attached much attention like those in [13, 25, 50] and has become one of the most important and significant research directions in nonlinear science.

On the other hand, it is well known that solving NLEEs plays an important role in the study of nonlinear physical phenomena in many fields such as fluid dynamics, plasma physics and nonlinear optics. Many mathematicians and physicists have done a lot of meaningful work, for instance, those in [4, 16, 18, 21, 22, 26, 27, 37, 38]. Since the pioneer work of Gardner, Green, Kruskal and Miura [18] exactly solving the initial-value problem of the KdV equation in 1965, the inverse scattering transform (IST) method has developed to a systematic method for solving NLEEs [1, 3, 6, 7, 11, 12, 14, 15, 17, 19, 20, 23, 28, 36, 39, 46, 47]. One of the advantages over other methods is that the IST can solve a whole hierarchy of NLEEs associated with the same spectral problem. As early as in 1976, the IST was extended to nonisospectral NLEEs for the first time by Chen and Liu to the nonlinear Schrödinger (NLS) equation with a linear external potential [15], Hirota and Satsuma to the KdV equation in nonuniform media [20], and Calogero and Degasperis to the KdV model [7]. In the framework of the IST with time-varying spectral parameter, Serkin, Hasegawa and Belyaeva [29–33] found nonautonomous solitons which interact elastically and generally move with varying amplitudes, speeds, and spectra adapted both to the external potentials and to the dispersion and nonlinearity variations.

The aim of this paper is to introduce such a new nonisospectral parameter k satisfying

$$ik_t = \frac{1}{2}e^{2ik}, \tag{1.11}$$

that we construct the following new and more general AKNS equations

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \sum_{j=0}^{+\infty} \frac{1}{j!} L^j \begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{1.12}$$

associated with the AKNS spectral problem (1.2). Obviously, the generalized AKNS equations (1.12) cannot be contained by the nonisospectral AKNS hierarchy (1.8) and (1.12) are more general than Eqs. (1.9) and (1.10). In this work, we shall extend the IST with the new nonisospectral parameter k in Eq. (1.11) to the generalized AKNS equations (1.12). It is shown from computer running that the dynamical evolutions of Eqs. (1.12) can possess not only time-varying speeds and amplitudes but also singular points in the process of propagations.

The rest of the paper is organized as follows. In Section 2, we derive the generalized AKNS equations (1.12) by embedding the nonisospectral parameter k determined by Eq. (1.11) to the AKNS spectral problem (1.2). In Section 3, following the steps of IST method we exactly solve the AKNS equations (1.12). Also, the uniform formulae of exact solutions of the AKNS equations (1.12) are obtained. In the special case of reflectionless potentials, the obtained exact solutions are reduced to explicit n-soliton solutions. For the cases when $n = 1, 2$, the dynamical evolutions of one-soliton solutions and two-soliton solutions possessing not only time-varying speeds and amplitudes but also singular points in the process of propagations are shown by figures. In Section 4, we conclude this paper.

2. Derivation of the generalized AKNS equations

To begin with, we substitute the matrixes M and N of Eqs. (1.2) and (1.3) into Eq. (1.1), then Eq. (1.1) is reduced to

$$A_x = qC - rB - ik_t, \tag{2.1}$$

$$q_t = B_x + 2ikB + 2qA, \tag{2.2}$$

$$r_t = C_x - 2ikC - 2rA. \tag{2.3}$$

Integrating (2.1) with respective to x and using (1.4) we have

$$A = \partial^{-1}(r, q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}e^{2ik}x + A_0, \tag{2.4}$$

where A_0 is an arbitrary function of k and t. For convenience, we employ Taylor series expansion formula of exponential function

$$e^{2ik} = \sum_{j=0}^{+\infty} \frac{1}{j!}(2ik)^j. \tag{2.5}$$

Then Eqs. (2.2) and (2.3) can be rewritten as follows

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -B \\ C \end{pmatrix} - 2ik\begin{pmatrix} -B \\ C \end{pmatrix} + \sum_{j=0}^{+\infty} \frac{1}{j!}(2ik)^j \begin{pmatrix} -xq \\ xr \end{pmatrix}. \tag{2.6}$$

We next suppose that

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{s=1}^{+\infty} \begin{pmatrix} -b_s \\ c_s \end{pmatrix}(2ik)^{s-1}, \tag{2.7}$$

with the asymptotic condition

$$\begin{pmatrix} -b_n \\ c_n \end{pmatrix} = \frac{1}{n!}\begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad n \to +\infty, \tag{2.8}$$

and substitute Eq. (2.7) into Eq. (2.6). Comparing the coefficients of $2ik$ in Eq. (2.6) yields

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{2.9}$$

$$\begin{pmatrix} -b_{s-1} \\ c_{s-1} \end{pmatrix} = L\begin{pmatrix} -b_s \\ c_s \end{pmatrix} + \frac{1}{(s-1)!}\begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad s = 2, 3, \cdots. \tag{2.10}$$

Finally, with the help of Eq. (2.10) we have

$$\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \sum_{j=1}^{+\infty} \frac{1}{j!} L^{j-1}\begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{2.11}$$

and hence obtain the generalized AKNS equations (1.1) by the substitution of Eq. (2.11) into Eq. (2.9).

3. Exact solutions and soliton dynamics

We first determine in this section the time dependence of scattering data for the AKNS spectral problem (1.2) with the nonisospectral k satisfying Eq. (1.11). With the help of the determined scattering data, we then construct exact solutions of the generalized AKNS hierarchy (1.1). Finally, we reduce the obtained exact solutions to soliton solutions and simulate the soliton dynamics by figures.

3.1. The time dependence of the scattering data

Theorem 3.1. *The scattering data*

$$\left\{\kappa_j(t), c_j(t), R(t,k) = \frac{b(k,t)}{a(k,t)}, j = 1, 2, \cdots, n\right\}, \tag{3.1}$$

$$\left\{\bar{\kappa}_m(t), \bar{c}_m(t), \bar{R}(k,t) = \frac{\bar{b}(k,t)}{\bar{a}(k,t)}, m = 1, 2, \cdots, \bar{n}\right\}, \tag{3.2}$$

for the spectral problem (1.2) *possess the following time dependence*

$$\kappa_j(t) = \frac{i}{2}\ln(e^{-2i\kappa_j(0)} - t), \tag{3.3}$$

$$c_j^2(t) = c_j^2(0)e^{-2i\kappa_j(0) - 2\int_0^t A_0(\kappa_j(w), w)\mathrm{d}w}(e^{-2i\kappa_j(0)} - t)^{-1}, \tag{3.4}$$

$$a(k,t) = a(k,0), \quad b(k,t) = b(k,0)e^{-2ik(0) - 2\int_0^t A_0(k(w), w)\mathrm{d}w}(e^{-2ik(0)} - t)^{-1}, \tag{3.5}$$

$$\bar{\kappa}_m(t) = \frac{i}{2}\ln(e^{-2i\bar{\kappa}_m(0)} - t), \tag{3.6}$$

$$\bar{c}_m^2(t) = \bar{c}_m^2(0)e^{2i\bar{\kappa}_m(0) + 2\int_0^t A_0(\bar{\kappa}_m(w), w)\mathrm{d}w}(e^{-2i\bar{\kappa}_m(0)} - t), \tag{3.7}$$

$$\bar{a}(k,t) = \bar{a}(k,0), \quad \bar{b}(k,t) = \bar{b}(k,0)e^{2i\bar{k}(0) + 2\int_0^t A_0(\bar{k}(w), w)\mathrm{d}w}(e^{-2i\bar{k}(0)} - t), \tag{3.8}$$

where $c_j^2(0)$, $\bar{c}_m^2(0)$, $\kappa_j(0)$, $\bar{\kappa}_m(0)$, $R(k,0) = b(k,0)/a(k,0)$ *and* $\bar{R}(k,0) = \bar{b}(k,0)/\bar{a}(k,0)$ *are the scattering data of* (1.2) *with the nonisospectral* k *satisfying Eq.* (1.4) *in the case of* $(q(x,0), r(x,0))^T$.

Proof. It is easy to see that if $\phi(x,k)$ is a solution of Eq. (1.2) with the nonisospectral k satisfying Eq. (1.4) then $P(x,k) = \phi_t(x,k) - N\phi(x,k)$ is also a solution of Eq. (1.2). Therefore, $P(x,k)$ can be represented by $\phi(x,k)$ and $\widetilde{\phi}(x,k)$ which also satisfies Eq. (1.2) but is independent of $\phi(x,k)$, that is, there exist two functions $\alpha(k,t)$ and $\beta(k,t)$ such that

$$\phi_t(x,k) - N\phi(x,k) = \alpha(k,t)\phi(x,k) + \beta(k,t)\widetilde{\phi}(x,k). \tag{3.9}$$

Firstly, we consider the discrete spectral $k = \kappa_j(\mathrm{Im}\kappa_j > 0)$. Since $\phi(x,\kappa_j)$ decays exponentially while $\widetilde{\phi}(x,\kappa_j)$ must increases exponentially as $x \to +\infty$, we then have $\beta(k,t) = 0$. Thus, Eq. (3.9) is simplified as:

$$\phi_t(x,\kappa_j) - N\phi(x,\kappa_j) = \alpha(\kappa_j,t)\phi(x,\kappa_j). \tag{3.10}$$

Left-multiplying (3.10) by the inner product $(\phi_2(x,\kappa_j), \phi_1(x,\kappa_j))$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_1(x,\kappa_j)\phi_2(x,\kappa_j) - (C\phi_1^2(x,\kappa_j) + B\phi_2^2(x,\kappa_j)) = 2\alpha(\kappa_j,t)\phi_1(x,\kappa_j)\phi_2(x,\kappa_j). \tag{3.11}$$

Presuming $\phi(x,\kappa_j)$ to be the normalization eigenfunction and noting that

$$2\int_{-\infty}^{\infty} c_j^2(t)\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)\mathrm{d}x = 1, \tag{3.12}$$

we have

$$\alpha(\kappa_j,t) = -c_j^2(t)\int_{-\infty}^{\infty}[C\phi_1^2(x,\kappa_j) + B\phi_2^2(x,\kappa_j)]\mathrm{d}x. \tag{3.13}$$

For convenience, we rewrite Eq. (3.13) as:

$$\alpha(\kappa_j,t) = -c_j^2(t)((\phi_2^2(x,\kappa_j), \phi_1^2(x,\kappa_j))^T, (B,C)^T), \tag{3.14}$$

where the following inner product had be used

$$(f(x), g(x)) = \int_{-\infty}^{\infty}(f_1(x)g_1(x) + f_2(x)g_2(x))\mathrm{d}x, \tag{3.15}$$

for arbitrary two vectors $f(x) = (f_1(x), f_2(x))^T$ and $g(x) = (g_1(x), g_2(x))^T$.

Using Eq. (1.2), we have

$$\phi_{1x}(x,\kappa_j) + i\kappa_j\phi_1(x,\kappa_j) = q(x)\phi_2(x,\kappa_j), \tag{3.16}$$

$$\varphi_{2x}(x,\kappa_j) - i\kappa_j\varphi_2(x,\kappa_j) = r(x)\varphi_1(x,\kappa_j), \tag{3.17}$$

and hence obtain

$$[\varphi_1(x,\kappa_j)\varphi_2(x,\kappa_j)]_x = q(x)\varphi_2^2(x,\kappa_j) + r(x)\varphi_1^2(x,\kappa_j). \tag{3.18}$$

Integrating Eq. (3.18) with respect to x from $-\infty$ to $+\infty$ yields

$$\int_{-\infty}^{\infty}[q(x)\phi_2^2(x,\kappa_j) + r(x)\phi_1^2(x,\kappa_j)]\mathrm{d}x = \int_{-\infty}^{\infty}[\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)]_x\mathrm{d}x = 0. \tag{3.19}$$

In the other hand, we rewrite Eq. (2.7) as

$$\begin{pmatrix} B \\ C \end{pmatrix} = \lim_{n\to+\infty}\sum_{s=1}^{n}\sum_{j=s}^{n}\frac{1}{j!}\bar{L}^{j-s}\begin{pmatrix} xq \\ xr \end{pmatrix}(2i\kappa_j)^{s-1}, \quad \bar{L} = \sigma\partial - 2\begin{pmatrix} q \\ r \end{pmatrix}\partial^{-1}(-r,q), \tag{3.20}$$

and then from Eq. (3.14), we obtain

$$\begin{aligned}\alpha(\kappa_j,t) &= -c_j^2(t)\left((\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j))^T, \lim_{n\to+\infty}\sum_{s=1}^{n}\sum_{j=s}^{n}\frac{1}{j!}\bar{L}^{j-s}\begin{pmatrix} xq \\ xr \end{pmatrix}(2i\kappa_j)^{s-1}\right) \\ &= \frac{1}{2}\lim_{n\to+\infty}\sum_{l=0}^{n-1}\frac{1}{l!}(2i\kappa_j)^l = \frac{1}{2}e^{2i\kappa_j},\end{aligned} \tag{3.21}$$

through using the following results

$$\bar{L}^{*j-s}(\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j))^T = (2i\kappa_j(t))^{j-s}(\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j))^T, \tag{3.22}$$

$$\left((\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j))^T, \begin{pmatrix} xq \\ xr \end{pmatrix}\right) = \int_{-\infty}^{\infty}x[\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)]_x\mathrm{d}x = -\frac{1}{2c_j^2(t)}, \tag{3.23}$$

where $\bar{L}^*$ is the conjugation operator of $\bar{L}$ [23]

$$\bar{L}^* = -\sigma\partial + 2\begin{pmatrix} -r \\ q \end{pmatrix}\partial^{-1}(q,r), \quad \bar{L} = \sigma L\sigma.$$

In view of Eq. (3.21), we simplify Eq. (3.10) as

$$\phi_t(x,\kappa_j) - N\phi(x,\kappa_j) = \frac{1}{2}e^{2i\kappa_j}\phi(x,\kappa_j). \tag{3.24}$$

Noting that

$$N \to \begin{pmatrix} -\frac{1}{2}e^{2i\kappa_j}x + A_0 & 0 \\ 0 & \frac{1}{2}e^{2i\kappa_j}x - A_0 \end{pmatrix}, \quad \phi(x,\kappa_j) \to c_j(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{i\kappa_j x}, \tag{3.25}$$

$$\phi_t(x,\kappa_j) \to c_{jt}(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{i\kappa_j x} + i\kappa_{jt}xc_j(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{i\kappa_j x}, \quad \kappa_{jt} = -\frac{i}{2}e^{2i\kappa_j}, \tag{3.26}$$

as $x \to +\infty$, then from Eqs. (3.24)–(3.26) we have

$$c_{jt}(t) = \frac{1}{2}e^{2i\kappa_j}c_j(t) - A_0. \tag{3.27}$$

Similarly, we have

$$\bar{c}_{jt}(t) = -\frac{1}{2}e^{2i\bar{\kappa}_j}\bar{c}_j(t) + A_0. \tag{3.28}$$

Secondly, we consider k as a real continuous spectral and take a solution $\varphi(x,k)$ of Eq. (1.2) with the nonisospectral k determined in Eq. (1.4), then $Q(x,k)=\varphi_t(x,k)-N\varphi(x,k)$ is also a solution of Eq. (1.2) and therefore can be represented linearly by $\varphi(x,k)$ and $\bar{\varphi}(x,k)$ which also satisfies Eq. (1.2) but is independent of $\varphi(x,k)$, that is, there exist two functions $\omega(k,t)$ and $\vartheta(k,t)$ such that

$$\varphi_t(x,k)-N\varphi(x,k)=\omega(k,t)\varphi(x,k)+\vartheta(k,t)\bar{\varphi}(x,k). \tag{3.29}$$

Using the asymptotical properties

$$\varphi_t(x,k)\to -ik_tx\begin{pmatrix}1\\0\end{pmatrix}e^{-ikx},\quad \varphi(x,k)\to\begin{pmatrix}1\\0\end{pmatrix}e^{-ikx},\quad \bar{\varphi}(x,k)\to\begin{pmatrix}0\\-1\end{pmatrix}e^{ikx}, \tag{3.30}$$

as $x\to-\infty$, from Eqs. (3.29) and (1.4) we obtain

$$\vartheta(k,t)=0,\quad \omega(k,t)=-A_0. \tag{3.31}$$

Substituting the Jost relationship $\varphi(x,k)=a(k,t)\bar{\phi}(x,k)+b(k,t)\phi(x,k)$ into Eq. (3.29) yields

$$\begin{aligned}&[a(k,t)\bar{\phi}(x,k)+b(k,t)\phi(x,k)]_t-N[a(k,t)\bar{\phi}(x,k)+b(k,t)\phi(x,k)]\\&\quad=-A_0[a(k,t)\bar{\phi}(x,k)+b(k,t)\phi(x,k)].\end{aligned} \tag{3.32}$$

Letting $x\to+\infty$ and using

$$\phi(x,k)\to\begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{ikx},\quad \bar{\phi}(x,k)\to\begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-ikx}, \tag{3.33}$$

from Eq. (3.32) we derive

$$a_t(k,t)=0,\quad b_t(k,t)=-2A_0b(k,t). \tag{3.34}$$

Similarly, we have

$$\bar{a}_t(k,t)=0,\quad \bar{b}_t(k,t)=2A_0\bar{b}(k,t). \tag{3.35}$$

Finally, solving Eqs. (3.26)–(3.28), (3.34) and (3.35) yields Eqs. (3.3)–(3.8). We therefore finish the proof. □

3.2. Exact solutions and soliton dynamics

According to Theorem 3.1, we have the following Theorem.

Theorem 3.2. *Given the scattering data for the spectral problem* (1.2) *with the nonisospectral k determined in Eq.* (1.11)*, the generalized AKNS equations* (1.12) *have exact solutions as follows:*

$$q(x,t)=-2K_1(t,x,x), \tag{3.36}$$

$$r(x,t)=\frac{K_{2x}(t,x,x)}{K_1(t,x,x)}, \tag{3.37}$$

where $K(t,x,y)=(K_1(t,x,y),K_2(t,x,y))^T$ satisfies the Gel'fand-Levitan-Marchenko (GLM) integral equation:

$$\begin{aligned}&K(t,x,y)-\begin{pmatrix}1\\0\end{pmatrix}\bar{F}(t,x+y)+\begin{pmatrix}0\\1\end{pmatrix}\int_x^\infty F(t,z+x)\bar{F}(t,z+y)\mathrm{d}z\\&\quad+\int_x^\infty K(t,x,s)\int_x^\infty F(t,z+s)\bar{F}(t,z+y)\mathrm{d}z\mathrm{d}s=0,\end{aligned} \tag{3.38}$$

with

$$F(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(t,k)e^{ikx}\mathrm{d}k+\sum_{j=1}^{n}c_j^2e^{i\kappa_jx}, \tag{3.39}$$

$$\bar{F}(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{R}(t,k)e^{ikx}\mathrm{d}k+\sum_{j=1}^{\bar{n}}\bar{c}_j^2e^{i\bar{\kappa}_jx}. \tag{3.40}$$

To give explicit form of solutions (3.36) and (3.37), in what follows we consider the reflectionless potentials $q(x,t)$ and $r(x,t)$, that is, $R(t,k) = \bar{R}(t,k) = 0$. In this case, the GLM integral equation (3.38) can be solved exactly. For convenience, we use $K(t,x,y) = (K_1(t,x,y), K_2(t,x,y))^T$ to rewrite Eq. (3.38) as:

$$K_1(t,x,y) - \bar{F}_d(t,x+y) + \int_x^{\infty} K_1(t,x,s)\int_x^{\infty} F_d(t,z+s)\bar{F}_d(t,z+y)\mathrm{d}z\mathrm{d}s = 0, \tag{3.41}$$

$$K_2(t,x,y) + \int_x^{\infty} F_d(t,z+x)\bar{F}_d(t,z+y)\mathrm{d}z + \int_x^{\infty} K_2(t,x,s)\int_x^{\infty} F_d(t,z+s)\bar{F}_d(t,z+y)\mathrm{d}z\mathrm{d}s = 0. \tag{3.42}$$

Using Eqs. (3.39) and (3.40), we get

$$\int_x^{\infty} F_d(t,s+z)\bar{F}_d(t,z+y)\mathrm{d}z = -\sum_{j=1}^{n}\sum_{m=1}^{\bar{n}} \frac{ic_j^2(t)\bar{c}_m^2(t)}{\kappa_j - \bar{\kappa}_m} e^{\kappa_j(x+s)-i\bar{\kappa}_m(x+y)}. \tag{3.43}$$

Supposing that

$$K_1(x,y,t) = \sum_{p=1}^{\bar{n}} \bar{c}_p(t) g_p(t,x) e^{-i\bar{\kappa}_p y}, \tag{3.44}$$

$$K_2(x,y,t) = \sum_{p=1}^{\bar{n}} \bar{c}_p(t) h_p(t,x) e^{-i\bar{\kappa}_p y}, \tag{3.45}$$

and substituting Eqs. (3.44) and (3.45) into Eqs. (3.41) and (3.42) yields

$$g_m(t,x) + \bar{c}_m(t)e^{-i\bar{\kappa}_m x} + \sum_{j=1}^{n}\sum_{p=1}^{\bar{n}} \frac{c_j^2(t)\bar{c}_m(t)\bar{c}_p(t)}{(\kappa_j - \bar{\kappa}_m)(\kappa_j - \bar{\kappa}_p)} e^{i(2\kappa_j - \bar{\kappa}_m - \bar{\kappa}_p)x} g_p(x,t) = 0, \tag{3.46}$$

$$h_m(x,t) - \sum_{j=1}^{n} \frac{c_j^2(t)\bar{c}_m(t)e^{i(2\kappa_j - \bar{\kappa}_m)x}}{(\kappa_j - \bar{\kappa}_m)} + \sum_{j=1}^{n}\sum_{p=1}^{\bar{n}} \frac{c_j^2(t)\bar{c}_m(t)\bar{c}_p(t)}{(\kappa_j - \bar{\kappa}_m)(\kappa_j - \bar{\kappa}_p)} e^{i(2\kappa_j - \bar{\kappa}_m - \bar{\kappa}_p)x} h_p(x,t) = 0. \tag{3.47}$$

Inducing the following vectors

$$g(x,t) = (g_1(x,t), g_2(x,t), \cdots, g_{\bar{n}}(x,t))^T, \tag{3.48}$$
$$h(x,t) = (h_1(x,t), h_2(x,t), \cdots, h_{\bar{n}}(x,t))^T, \tag{3.49}$$
$$\Lambda = (c_1(t)e^{-i\kappa_1 x}, c_2(t)e^{-i\kappa_2 x}, \cdots, c_n(t)e^{-i\kappa_n x})^T, \tag{3.50}$$
$$\bar{\Lambda} = (\bar{c}_1(t)e^{-i\bar{\kappa}_1 x}, \bar{c}_2(t)e^{-i\bar{\kappa}_2 x}, \cdots, \bar{c}_{\bar{n}}(t)e^{-i\bar{\kappa}_{\bar{n}} x})^T, \tag{3.51}$$

we can write (3.38) in the matrix from

$$W(x,t)g(x,t) = -\bar{\Lambda}(x,t), \tag{3.52}$$

$$W(x,t)h(x,t) = iP(x,t)\Lambda(x,t). \tag{3.53}$$

If $W^{-1}(x,t)$ exists, then

$$g(x,t) = -W^{-1}(x,t)\bar{\Lambda}(x,t), \tag{3.54}$$
$$h(x,t) = iW^{-1}(x,t)P(x,t)\Lambda(x,t), \tag{3.55}$$

in which

$$W(x,t) = E + P(x,t)P^T(x,t), \quad P(x,t) = \left(\frac{c_j(t)\bar{c}_m(t)}{\kappa_j - \bar{\kappa}_m} e^{i(\kappa_j - \bar{\kappa}_m)x}\right)_{\bar{n}\times n}, \tag{3.56}$$

and E is a $\bar{n}\times\bar{n}$ unit matrix. Substituting Eqs. (3.54) and (3.55) into Eqs. (3.44) and (3.45), we have

$$K_1(x,y,t) = -\bar{\Lambda}^T(y,t)W^{-1}(x,t)\bar{\Lambda}(x,t), \tag{3.57}$$

$$K_2(x,y,t) = i\mathrm{tr}(W^{-1}(x,t)P(x,t)\Lambda(y,t)\bar{\Lambda}^T(y,t)), \tag{3.58}$$

where $\mathrm{tr}(\cdot)$ means the trace of a given matrix.

Substituting Eqs. (3.57) and (3.58) into Eqs. (3.36) and (3.37), we obtain n-soliton solutions of the generalized AKNS hierarchy (1.5)

$$q(x,t) = 2\mathrm{tr}(W^{-1}(x,t)\bar{\Lambda}(x,t)\bar{\Lambda}^T(x,t)), \tag{3.59}$$

$$r(x,t) = -\frac{\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{tr}(W^{-1}(x,t)P(x,t)\frac{\mathrm{d}}{\mathrm{d}x}P^T(x,t))}{\mathrm{tr}(W^{-1}(x,t)\bar{\Lambda}(x,t)\bar{\Lambda}^T(x,t))}. \tag{3.60}$$

Particularly, when $n = \bar{n} = 1$, Eqs. (3.59) and (3.60) give the one-soliton solutions which are simplified as follows

$$q = \frac{2\bar{c}_1^2(0)e^{2i\bar{\kappa}_1(0)+2\int_0^t A_0(\bar{\kappa}_1(w),w)\mathrm{d}w}(e^{-2i\bar{\kappa}_1(0)}-t)^{x+1}}{1-4\frac{c_1^2(0)\bar{c}_1^2(0)(e^{-2i\bar{\kappa}_1(0)}-t)e^{-2i\kappa_1(0)+2i\bar{\kappa}_1(0)-2\int_0^t[A_0(\kappa_1(w),w)-A_0(\bar{\kappa}_1(w),w)]\mathrm{d}w+x[\ln(e^{-2i\bar{\kappa}_1(0)}-t)-\ln(e^{-2i\kappa_1(0)}-t)]}}{(e^{-2i\kappa_1(0)}-t)[\ln(e^{-2i\kappa_1(0)}-t)-\ln(e^{-2i\bar{\kappa}_1(0)}-t)]^2}}, \tag{3.61}$$

$$r = \frac{2c_1^2(0)e^{-2i\kappa_1(0)-2\int_0^t A_0(\kappa_1(w),w)\mathrm{d}w}(e^{-2i\kappa_1(0)}-t)^{-x-1}}{1-4\frac{c_1^2(0)\bar{c}_1^2(0)(e^{-2i\bar{\kappa}_1(0)}-t)e^{-2i\kappa_1(0)+2i\bar{\kappa}_1(0)-2\int_0^t[A_0(\kappa_1(w),w)-A_0(\bar{\kappa}_1(w),w)]\mathrm{d}w+x[\ln(e^{-2i\bar{\kappa}_1(0)}-t)-\ln(e^{-2i\kappa_1(0)}-t)]}}{(e^{-2i\kappa_1(0)}-t)[\ln(e^{-2i\kappa_1(0)}-t)-\ln(e^{-2i\bar{\kappa}_1(0)}-t)]^2}}. \tag{3.62}$$

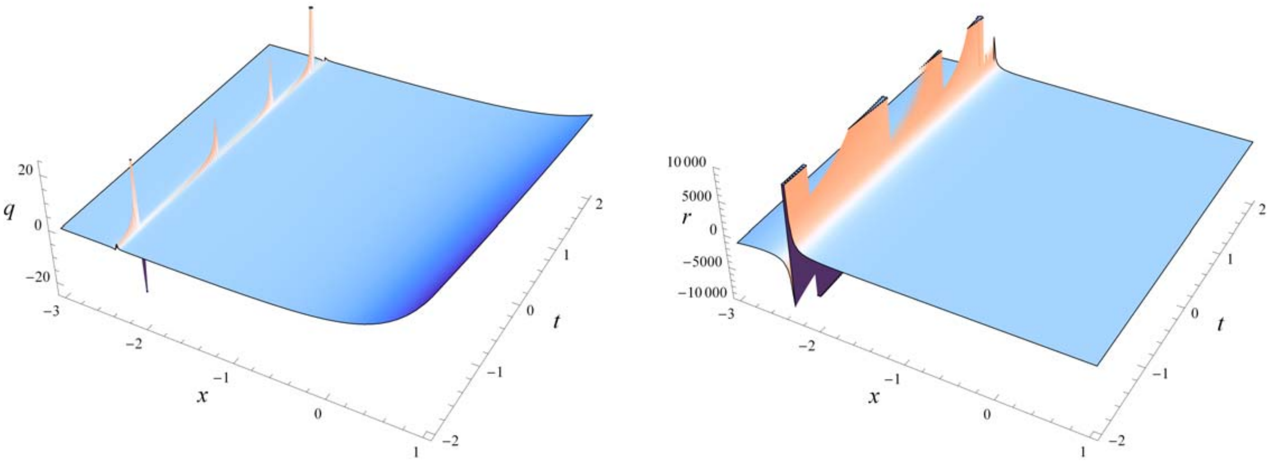

Figure 1: Spatial structures of bright and dark one-solitons determined by solutions (3.61) and (3.62).

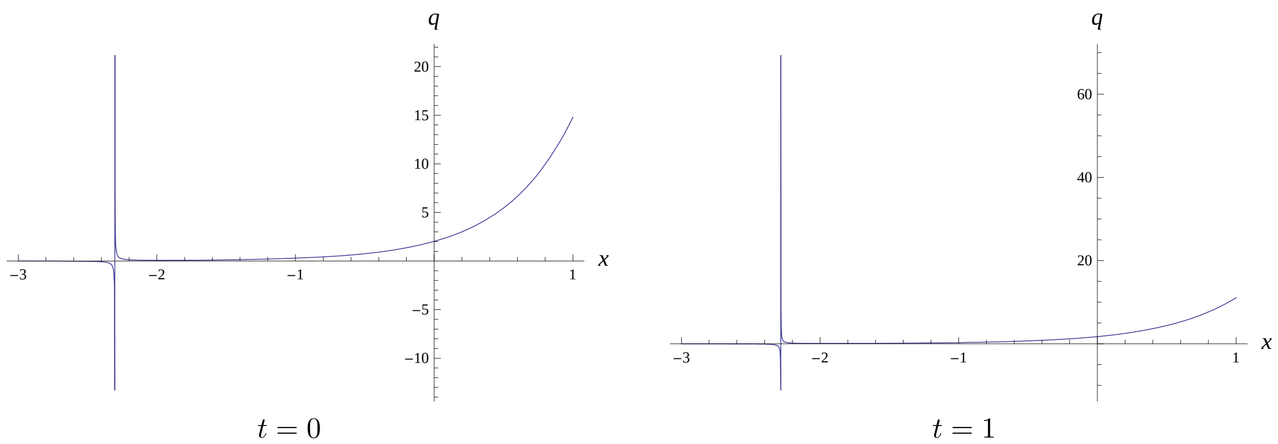

Figure 2: Dynamical evolutions of one-soliton determined by solution (3.61).

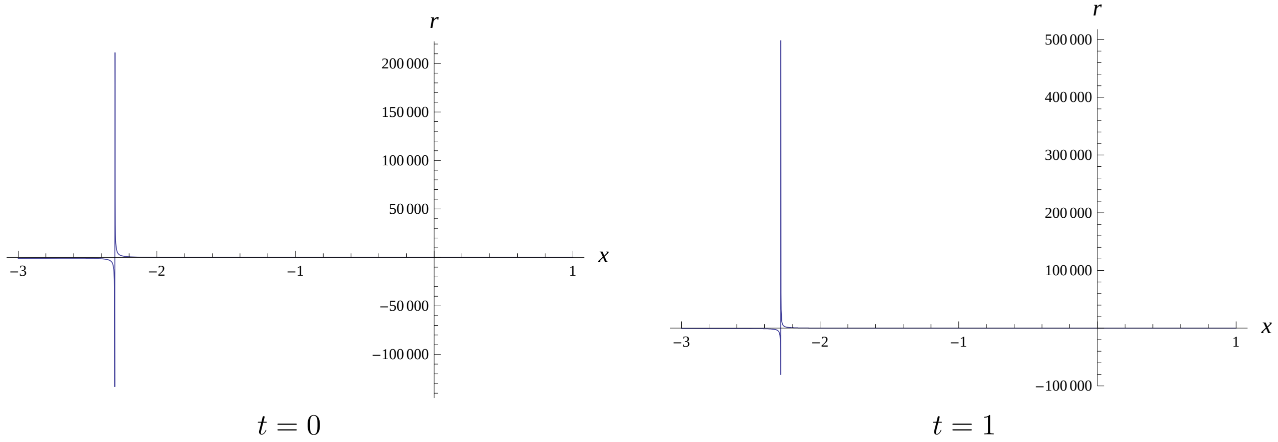

Figure 3: Dynamical evolutions of one-soliton determined by solution (3.62).

We can see that solutions (3.61) and (3.62) possess singularity. In Fig. 1, the spatial structures of singular bright and dark one-solitons determined by solutions (3.61) and (3.62) are shown by selecting the parameters as $c_1(0) = 0.1$, $\bar{c}_1(0) = 1$, $\kappa_1(0) = 2i$, $\bar{\kappa}_1(0) = i$, $A_0(\kappa_1(t), t) = 0$, $A_0(\bar{\kappa}_1(t), t) = 0$. Figs. 2 and 3 are used to describe the corresponding dynamical evolutions of these bright and dark one-solitons at times $t = 0$ and $t = 1$. It can be seen from Figs. 1–3 that the bright and dark one-solitons determined by solutions (3.61) and (3.62) possess time-varying amplitudes and singular points in the process of propagations.

In Fig. 4, the spatial structures of singular bright and dark two-solitons determined by solutions (3.59) and (3.60) are shown by selecting the parameters as $c_1(0) = 1$, $\bar{c}_1(0) = 0.2$, $c_2(0) = 2$, $\bar{c}_2(0) = 0.3$, $\kappa_1(0) = 0.8i$, $\bar{\kappa}_1(0) = i$, $\kappa_2(0) = 1.5i$, $\bar{\kappa}_2(0) = 2i$, $A_0(\kappa_1(t), t) = 0$, $A_0(\bar{\kappa}_1(t), t) = 0$, $A_0(\kappa_2(t), t) = 0$, $A_0(\bar{\kappa}_2(t), t) = 0$. We use Figs. 5 and 6 to describe the corresponding dynamical evolutions of these bright and dark two-solitons at times $t = 0$ and $t = 1$. Figs. 4–6 show that the bright and dark two-solitons determined by solutions (3.59) and (3.60) possess not only singular points but also time-varying velocities and amplitudes in the process of propagations.

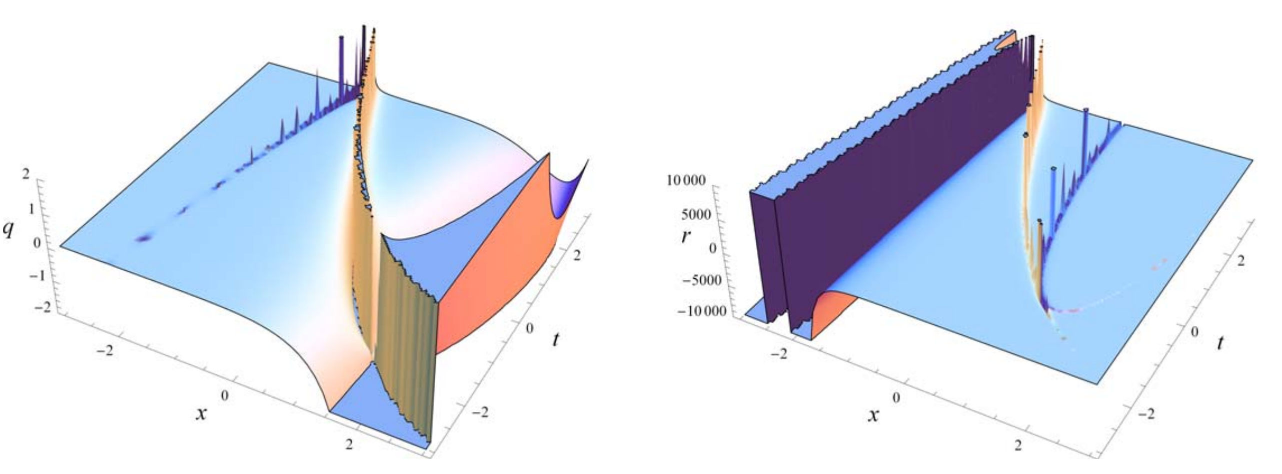

Figure 4: Spatial structures of bright and dark two-solitons determined by solutions (3.59) and (3.60).

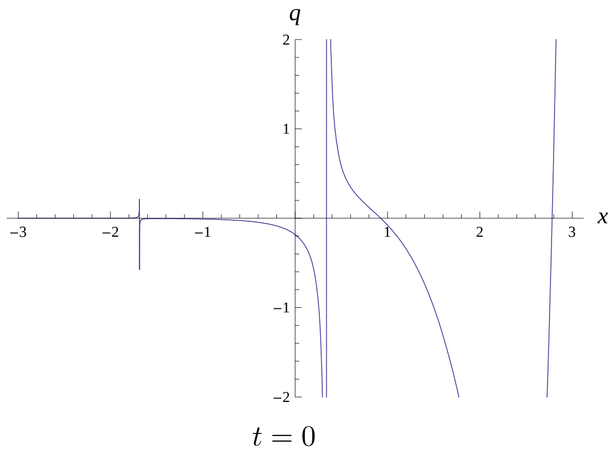

Figure 5: Dynamical evolutions of of two-soliton determined by solution (3.59).

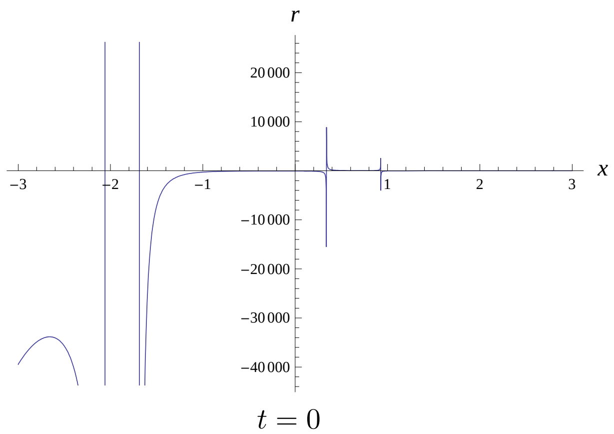

Figure 6: Dynamical evolutions of two-soliton determined by solution (3.60).

4. Conclusions

In summary, we have generalized the AKNS spectral problem (1.2) by embedding a nonisospectral parameter which varies with time obeying the exponential function of spectral parameter k determined in Eq. (1.11). Starting from the generalized AKNS spectral problem (1.2) and its corresponding time evolution equation (1.3), together with (1.11), we constructed a generalized AKNS equations (1.12) with infinite number of terms. In order to solve the generalized AKNS equations (1.12), the IST method is employed. As a result, exact solutions (3.36) and (3.37) are formulated and then reduced to explicit n-soliton solutions (3.59) and (3.60) in the case of reflectionless potentials. This paper shows that the dynamical evolutions of one-soliton solutions and two-soliton solutions determined by solutions (3.59) and (3.60) possess time-varying speeds, amplitudes and singular points in the process of propagations. To the best of our knowledge, the nonisospectral parameter k satisfying Eq. (1.11), the generalized AKNS equations (1.12) and the n-soliton solutions (3.59) and (3.60) have not been reported in literatures. Recently, fractional-order differential calculus and its applications have attached much attention [5, 40–44]. How to construct hierarchies of fractional-order NLEEs and their exact solutions in the framework of IST method is worthy of study.

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