Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4388–4398
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Covering properties defined by semi-open sets

Amani Sabah[a], Moiz ud Din Khan[a,*], Ljubiša D. R. Kočinac[b]

[a]Department of Mathematics, COMSATS Institute of Information Technology, Chack Shahzad, Park road, Islamabad 45550, Pakistan.

[b]Faculty of Sciences and Mathematics, University of Niš, 18000 Niš, Serbia.

Communicated by Y. J. Cho

Abstract

We study certain covering properties in topological spaces by using semi-open covers. A part of this article deals with Menger-type covering properties. The notions of s-Menger, almost s-Menger, star s-Menger, almost star s-Menger, strongly star s-Menger spaces are defined and corresponding properties are investigated.

Keywords: Semi-open set, (star) semi-compact space, semi-Lindelöf space, s-Menger space, star s-Menger space.
2010 MSC: 54D20, 54C08.

1. Introduction

Our main focus in this paper is to study various covering properties, in particular selection principles, by using semi-open covers. We will deal with variations of the following classical selection principle:

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X. Then $\mathsf{S}_{\mathsf{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$ (see [30]).

If $\mathcal{O}$ denotes the family of all open covers of a space X, then the property $\mathsf{S}_{\mathsf{fin}}(\mathcal{O}, \mathcal{O})$ is called the *Menger (covering) property*.

This property was introduced in 1924 by K. Menger under the name Menger basis property [27], and in 1925, W. Hurewicz [16] proved that a metric space has the Menger basis property, if and only if X has

*Corresponding author
Email addresses: amani.sabah@yahoo.com (Amani Sabah), moiz@comsats.edu.pk (Moiz ud Din Khan), lkocinac@gmail.com (Ljubiša D. R. Kočinac)

Received 2016-03-02

the Menger covering property. For more information about selection principles theory and its relations with other fields of mathematics see [20, 29, 31, 35].

In 1963, N. Levine [23] defined semi-open sets in topological spaces. Since then, many mathematicians generalized different concepts and explored their properties in new setting. A set A in a topological space X is *semi-open* if and only if there exists an open set $O \subset X$ such that $O \subset A \subset \mathrm{Cl}(O)$, where $\mathrm{Cl}(O)$ denotes the closure of the set O. Equivalently, A is semi-open if and only if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ ($\mathrm{Int}(A)$ is the interior of A). If A is semi-open, then its complement is called *semi-closed* [7]. Every open set is semi-open, whereas a semi-open set may not be open. The union of any number of semi-open sets is semi-open, but the intersection of two semi-open sets may not be semi-open. The intersection of an open set and a semi-open set is always semi-open. The collection of all semi-open subsets of X is denoted by $\mathsf{SO}(X)$. According to [7], the semi-closure and semi-interior were defined analogously to the closure and interior: the *semi-interior* $\mathrm{sInt}(A)$ of a set $A \subset X$ is the union of all semi-open subsets of A; the *semi-closure* $\mathrm{sCl}(A)$ of $A \subset X$ is the intersection of all semi-closed sets containing A. A set A is semi-open if and only if $\mathrm{sInt}(A) = A$, and A is semi-closed if and only if $\mathrm{sCl}(A) = A$. Note that for any subset A of X

$$\mathrm{Int}(A) \subset \mathrm{sInt}(A) \subset A \subset \mathrm{sCl}(A) \subset \mathrm{Cl}(A).$$

A subset A of a topological space X is called a *semi-regular set* if it is semi-open as well as semi-closed or equivalently, $A = \mathrm{sCl}(\mathrm{sInt}(A))$ or $A = \mathrm{sInt}(\mathrm{sCl}(A))$. The collection of all semi-regular subsets of X is denoted by $\mathsf{SR}(X)$.

A mapping $f : (X, \tau_X) \to (Y, \tau_Y)$ is called:

1. *semi-continuous* if the preimage of every open set in Y is semi-open;
2. *s-continuous* if preimage of every semi-open set in Y is open in X;
3. *irresolute* [8] if $f^{\leftarrow}(O)$ is semi-open in X for every O semi-open in Y;
4. *semi-homeomorphism* if f is a bijection and images and preimages of semi-open sets are semi-open;
5. a *quasi-irresolute* if for every semi-regular set A in Y, the set $f^{\leftarrow}(A)$ is semi-regular in X [10].

For more details on semi-open sets and semi-continuity, we refer to [2, 6–8, 23].

2. Preliminaries

Throughout this paper, a space X is an infinite topological space (X, τ) on which no separation axioms are assumed, unless otherwise stated. We use the standard topological notation and terminology as in [13] (see also [4, 5]).

Definition 2.1. A space X is called:

- *semi-compact* [11] if every cover of X by semi open sets has a finite subcover;
- *countably semi-compact* (called *semi countably compact* by Dorsett in [11]) if every countable semi-open cover of X has a finite subcover;
- *semi-Lindelöf* [14] if every cover of X by semi-open sets has a countable subcover.

Definition 2.2 ([12])**.** A space X is *semi-regular* if for each semi-closed set A and $x \notin A$, there exist disjoint semi-open sets U and V such that $x \in U$ and $A \subset V$. (For a different definition see [24]).

Lemma 2.3 ([12])**.** *The following are equivalent in a space X:*

(i) *X is a semi-regular space;*

(ii) *For each $x \in X$ and $U \in \mathsf{SO}(X)$ such that $x \in U$, there exists a $V \in \mathsf{SO}(X)$ such that $x \in V \subset \mathrm{sCl}(V) \subset U$;*

(iii) *For each $x \in X$ and each $U \in \mathsf{SO}(X)$ with $x \in U$, there is a semi-regular $V \subset X$ such that $x \in V \subset U$.*

Semi-regularity is independent of regularity in topological spaces.

Call a space X *almost countably semi-compact* if any countable semi-open cover $\mathcal{U}$ of X contains a finite subset $\mathcal{V}$ such that $X = \bigcup\{\mathrm{sCl}(V) : V \in \mathcal{V}\}$.

We close this short section by the following result.

Theorem 2.4. *A semi-regular almost countably semi-compact, semi-Lindelöf space X is semi-compact.*

Proof. Let $\mathcal{U}$ be a semi-open cover of X. Since X is semi-regular, by Lemma 2.3, for each $x \in X$, there is a $U(x) \in \mathcal{U}$ containing x and semi-open set $V(x)$ such that $x \in V(x) \subset \mathrm{sCl}(V(x)) \subset U(x)$. Setting $\mathcal{V} = \{V(x) : x \in X\}$, we get a semi-open cover of X. By semi-Lindelöfness of X, we find a countable subcover $\mathcal{W} = \{V(x_n) : n \in \mathbb{N}\}$ of $\mathcal{V}$. Since X is almost countably semi-compact, there is a finite collection $\{V(x_{n_1}), \dots, V(x_{n_k})\} \subset \mathcal{W}$ such that $\bigcup_{i=1}^{k} \mathrm{sCl}(V(x_{n_i})) = X$. Then, clearly, the set $\{U(x_{n_i}) : i = 1, 2, \dots, k\}$ is a finite subcover of $\mathcal{U}$, witnessing semi-compactness of X. □

3. Semi-Menger and related spaces

Let $s\mathcal{O}$ denote the collection of all semi-open covers of a space X. Note that the class of semi-open covers contains the class of open covers of the space X.

Definition 3.1. A space X is said to have the *semi-Menger property* (or *s-Menger property*) if it satisfies $\mathsf{S}_{\mathsf{fin}}(s\mathcal{O}, s\mathcal{O})$.

Evidently we have the following diagram:

$$\begin{array}{ccccc} \text{semi–compact} & \Rightarrow & s\text{–Menger} & \Rightarrow & \text{semi–Lindelöf} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{compact} & \Rightarrow & \text{Menger} & \Rightarrow & \text{Lindelöf} \end{array}$$

Example 3.2.

(1) Every semi-compact space (in fact, every σ-semi-compact space = a countable union of semi-compact spaces) is semi-Menger. The converse is not true. Let the real line $\mathbb{R}$ be endowed with the topology $\tau = \{\emptyset, \mathbb{R}, (-\infty, x) : x \in \mathbb{R}\}$. Then, as it is easy to see, $(\mathbb{R}, \tau)$ is a T_0 semi-Menger space, but it is not semi-compact. Another example of a (T_1) semi-Menger space which is not semi-compact is any infinite space with the cocountable topology.

(2) Every semi-Menger space is a Menger space, but the converse is not true in general.

Recall that a Hausdorff uncountable space without isolated points is called a *Luzin space* if each nowhere dense subset in it is countable [22]. The real line $\mathbb{R}$ is a Menger space, but it is not semi-Menger. The later follows from the following facts:

(i) $\mathbb{R}$ is not a Luzin space (because a T_3 Luzin space is zero-dimensional [22, Lemma 1.3]), and

(ii) an uncountable Hausdorff space is semi-Lindelöf if and only if it is a Luzin space [15].

(3) The Sorgenfrey line $\mathbb{S}$ is a (hereditarily) Lindelöf space, which is not semi-Menger (since it is not a Menger space, as it is well known). The space of irrationals with the usual metric topology also is not semi-Menger because it is not Menger.

Call a subset A of a space X *semi-Menger relative to X* if for any sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of A by sets semi-open in X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $A \subset \bigcup_{n \in \mathbb{N}} \bigcup \mathbb{V}_n$.

Example 3.3. There is a Menger subspace of $\mathbb{R}^2$ which is not semi-Menger relative to $\mathbb{R}^2$.

Let $A = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1, y = 0\}$. Let A has the subspace topology τ_1 on $\mathbb{R}^2$. Then A is a compact space, hence Menger. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the set of all open disks in the upper half-plane of radius $1/n$ which touches A in a point $a \in A$. Then $\mathcal{U}_n$ is a cover of A by sets semi-open in (X, τ_1). But if we take a finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, then all these $\mathcal{V}_n$'s can cover only countably many points in A, so that A is not s-Menger, relative to (R^2, τ_1).

Proposition 3.4. *The following statements are true:*

(1) *An irresolute image of a semi-Menger space is semi-Menger; in particular, continuous open images of semi-Menger spaces are semi-Menger (continuous open mappings are irresolute);*

(2) *an s-continuous image of a Menger space is semi-Menger;*

(3) *a semi-continuous (in particular, continuous) image of a semi-Menger space is Menger;*

(4) *a semi-regular subspace of a semi-Menger space is also semi-Menger.*

Proof. Since (1)–(3) follow by applying definitions of mappings involved in these items, we prove only (4). Let A be a semi-regular subspace of X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of A. As A is a semi-open set and semi-open sets in a semi-open subspace are semi-open in the whole space, each $\mathcal{U}_n$ is a collection of semi-open sets in X. On the other hand, since A is also semi-closed, we conclude that each $\mathcal{U}_n \cup \{X \setminus A\} = \mathcal{G}_n$ is a semi-open cover of X. Semi-Mengerness of X implies the existence of finite sets $\mathcal{W}_n \subset \mathcal{G}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a semi-open cover of X. It follows that the finite sets $\mathcal{V}_n = \mathcal{W}_n \setminus \{X \setminus A\} \subset U_n$, $n \in \mathbb{N}$, witness for $(\mathcal{U}_n : n \in \mathbb{N})$ that A is semi-Menger. □

A property which is preserved by semi-homeomorphisms is called a *semi-topological property* [7].

Remark 3.5. From the previous proposition we see that the semi-Mengerness is a semi-topological property. However, it is not the case with the Menger property.

Let $X = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ be the upper half-plane. Endow X with the following two topologies: τ_1 is the subspace topology of the usual metric topology on $\mathbb{R}^2$, and τ_2 is the Niemytzki tangent disc topology (called also the Niemytzki plane) [13]. Then (X, τ_1) is a Menger space (being a closed subspace of the Menger space $\mathbb{R}^2$), while (X, τ_2) is not (because it is not Lindelöf). On the other hand, $\mathsf{SO}(X, \tau_1) = \mathsf{SO}(X, \tau_2)$ [8]. Therefore, the mapping $id_X : (X, \tau_1) \to (X, \tau_2)$ is a semi-homeomorphism.

Call a mapping $f : X \to Y$ *s-perfect* if for each semi-closed set $A \subset X$, the set $f(A)$ is semi-closed in Y and for each $y \in Y$, its preimage $f^{\leftarrow}(y)$ is semi-compact, relative to X.

Theorem 3.6. *If f is an s-perfect mapping from a space X onto a semi-Menger space Y, then X is also semi-Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. For each n and each $y \in Y$, there is a finite subcollection $\mathcal{G}_n^y$ of $\mathcal{U}_n$ covering $f^{\leftarrow}(y)$. Set $G_n^y = \bigcup \mathcal{G}_n^y$ and $W_n^y = Y \setminus f(X \setminus G_n^y)$. Then $y \in W_n^y$, and $\mathcal{W}_n = \{W_n^y : y \in Y\}$ is a semi-open cover of Y for each $n \in \mathbb{N}$. Since Y is semi-Menger for each n, there is a finite subcollection $\mathcal{H}_n$ of $\mathcal{W}_n$ such that $Y = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{H}_n$. To each $H \in \mathcal{H}_n$, associate finitely many sets from $\mathcal{G}_n^y$ which occur in the representation of G_n^y for which $H = Y \setminus f(X \setminus G_n^y)$. In this way, for each n, we have chosen a finite subcollection $\mathcal{V}_n$ of $\mathcal{U}_n$. Evidently, $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$, so that X is semi-Menger. □

Definition 3.7. A semi-open cover $\mathcal{U}$ is a semi-ω-cover (or *sω-cover*) of a space X if X does not belong to $\mathcal{U}$, and every finite subset of X is contained in a member of $\mathcal{U}$.

The symbol $s\Omega$ denotes the family of semi-ω-covers of a space.

Theorem 3.8. *For a space X the following are equivalent:*

(1) *X is s-Menger;*

(2) *X satisfies* $\mathsf{S}_{\mathsf{fin}}(s\Omega, s\mathcal{O})$.

Proof. (1) $\Rightarrow$ (2): It follows from the fact that every semi-ω-cover of X is a semi-open cover for X.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Partition $\mathbb{N}$ into pairwise disjoint infinite subsets N_i: $\mathbb{N} = N_1 \cup N_2 \cup \ldots \cup N_m \cup \ldots$. For each n, let $\mathcal{V}_n$ be the set of all elements of the form

$$U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k},\ \ n_1 \leq \ldots \leq n_k,\ \ n_i \in N_n,\ \ U_{n_i} \in \mathcal{U}_n,\ \ i \leq k,\ \ k \in ,N$$

which are not equal to X. Then every $\mathcal{V}_n$ is a semi-ω-cover of X. Applying (2) to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$, we can choose a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{W}_n \subset \mathcal{V}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup_{W\in\mathcal{W}_n} W = X$. Suppose $\mathcal{W}_n = \{W_n^1, \ldots, W_n^{m_n}\}$. By the construction, each $W_n^i = U_n^{n_{i_1}} \cup \ldots \cup U_n^{n_{i_k}}$, so that in this way we get finite subsets of $\mathcal{U}_p$ for some $p \in \mathbb{N}$ which cover X. If there are no elements from some $\mathcal{U}_q$ chosen in this way, then we put $\mathcal{W}_q = \emptyset$. This gives that X is really semi-Menger. □

It is known that Menger's covering property can be characterized game-theoretically and Ramsey-theoretically [30]. We do not know if it is the case for the semi-Menger property.

Problem 3.9. Can semi-Mengerness be characterized game-theoretically or Ramsey-theoretically?

3.1. Almost semi-Menger spaces

In [19], the notion of almost Menger spaces was introduced, and in [17] this class of spaces was studied. We make use of this concept and define analogously spaces by the help of semi-open covers.

Definition 3.10. A space X is *almost semi-Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{sCl}(V) : V \in \mathcal{V}_n\} = X$.

Remark 3.11. If in this definition we take $\mathrm{Cl}(V)$ instead of $\mathrm{sCl}(V)$, then we get another (wider) class of spaces called *almost S-Menger*, defined similarly to the definition of S-closed spaces [34].

Recall that a space X is said to be *s-closed* [9] if for every semi-open cover $\mathcal{U}$ of X, there is a finite set $\mathcal{V} \subset \mathcal{U}$ such that $X = \bigcup\{\mathrm{sCl}(V) : V \in \mathcal{V}\}$.

Every s-closed space is s-Menger and every s-Menger space is almost s-Menger.

Example 3.12. The Stone–Čech compactification $\beta\mathbb{N}$ of the natural numbers $\mathbb{N}$ is an almost semi-Menger space. (In [34], it was shown that $\beta\mathbb{N}$ is S-closed. On the other hand, $\beta\mathbb{N}$ is extremally disconnected [13]. In extremally disconnected spaces, the semi-closure and closure of a semi-open set coincide, hence $\beta\mathbb{N}$ is s-closed and thus almost s-Menger.)

The previous example is actually a special case of the following.

Proposition 3.13. *If a space X contains a dense subset which is semi-Menger in X, then X is almost semi-Menger.*

Proof. Let A be a dense subset of X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Since A is semi-Menger in X, there are finite sets $\mathcal{V}_n$, $n \in \mathbb{N}$ such that $A \subset \bigcup_{n\in\mathbb{N}} \bigcup\{V : V \in \mathcal{V}_n\} \subset \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{sCl}(V) : V \in \mathcal{V}_n\}$. Since D is dense in X and $\mathrm{sCl}(D) = \mathrm{Cl}(D)$, we have $X = \bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{sCl}(V) : V \in \mathcal{V}_n\}$. □

The following two theorems show when an almost s-Menger space becomes s-Menger.

Theorem 3.14. *Let X be a semi-regular space. If X is an almost s-Menger space, then X is an s-Menger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Since X is a semi-regular space, by Lemma 2.3, there exists for each n a semi-open cover $\mathcal{V}_n$ of X such that $\mathcal{V}'_n = \{\mathrm{sCl}(V) : V \in \mathcal{V}_n\}$ forms a refinement of $\mathcal{U}_n$. By assumption, there exists a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and $\bigcup(\mathcal{W}'_n : n \in \mathbb{N})$ is a cover of X, where $\mathcal{W}'_n = \{\mathrm{sCl}(W) : W \in \mathcal{W}_n\}$. For every $n \in \mathbb{N}$ and every $W \in \mathcal{W}_n$ we can choose $U_W \in \mathcal{U}_n$ such that $\mathrm{sCl}(W) \subset U_W$. Let $\mathcal{U}'_n = \{U_W : W \in \mathcal{W}_n\}$. We shall prove that $\bigcup_{n\in\mathbb{N}} \mathcal{U}'_n$ is a semi-open cover of X. Let $x \in X$. There exists an $n \in \mathbb{N}$ and an $\mathrm{sCl}(W) \in \mathcal{W}'_n$ such that $x \in \mathrm{sCl}(W)$. By construction, there exists a $U_W \in \mathcal{U}'_n$ such that $\mathrm{sCl}(W) \subset U_W$. Hence, $x \in U_W$. □

Theorem 3.15. *If X is almost semi-Menger and $\mathrm{Int}(\mathrm{Cl}(A))$ is finite for any $A \subset X$, then X is semi-Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Since X is almost semi-Menger, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup\{\mathrm{sCl}(V) : V \in \mathcal{V}_n\} = X$. Since for any $A \subset X$, $\mathrm{sCl}(A) = A \cup \mathrm{int}(\mathrm{Cl}(A))$ [3], by the assumption there are finite sets F_n, $n \in \mathbb{N}$, such that $X = \bigcup_{n\in\mathbb{N}} \bigcup\{V : V \in \mathcal{V}_n\} \cup \bigcup_{n\in\mathbb{N}} F_n$. For each n let $\mathcal{W}_n$ be a set of finitely many elements of $\mathcal{U}_n$ which cover F_n. Then the sequence $(\mathcal{V}_n \cup \mathcal{W}_n : n \in \mathbb{N})$ of finite sets witnesses that X is semi-Menger. □

In [3], it was proved that for a semi-open set U, the set $\mathrm{sCl}(U)$ is also semi-open (because for any A, $\mathrm{sCl}(A) = A \cup \mathrm{int}(\mathrm{Cl}(A))$).

Theorem 3.16. *A space X is almost s-Menger if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-regular sets, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X.*

Proof. Let X be an almost s-Menger space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-regular sets. Since every semi-regular set is semi-open (as well as semi-closed), $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of semi-open covers of X. By assumption, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X, where $\mathrm{sCl}(V) = V$ for all $V \in \mathcal{V}_n$.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Let $(\mathcal{U}'_n : n \in \mathbb{N})$ be a sequence defined by $\mathcal{U}'_n = \{\mathrm{sCl}(U) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}'_n$ is a cover of X by semi-regular sets. Thus there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X. By construction, for each $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ there exists a $U_V \in \mathcal{U}_n$ such that $V = \mathrm{sCl}(U_V)$. Hence, $\bigcup_{n\in\mathbb{N}}\{\mathrm{sCl}(U_V) : V \in \mathcal{V}_n\} = X$. So, X is an almost s-Menger space. □

Theorem 3.17. *Let X be an almost s-Menger space, and Y a topological space. If $f : X \to Y$ is a quasi-irresolute surjection, then Y is an almost s-Menger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of Y by semi-regular sets. Let $\mathcal{U}'_n = \{f^{\leftarrow}(U) : U \in \mathcal{U}_n\}$ for each $n \in \mathbb{N}$. Then $(\mathcal{U}'_n : n \in \mathbb{N})$ is a sequence of semi-regular covers of X, since f is a quasi-irresolute surjection. Since X is an almost s-Menger space, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X. For each $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ we can choose a $U_V \in \mathcal{U}_n$ such that $V = f^{\leftarrow}(U_V)$. Let $\mathcal{W}_n = \{\mathrm{sCl}(U_V) = U_V : V \in \mathcal{V}_n\}$. We will show that $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a cover of Y.

If $y = f(x) \in Y$, then there exists an $n \in \mathbb{N}$ and a $V \in \mathcal{V}_n$ such that $x \in V$. Since $V = f^{\leftarrow}(U_V)$, we have $y = f(x) \in U_V \in \mathcal{W}_n$. □

Theorem 3.18. *If for each $n \in \mathbb{N}$, X^n is an almost s-Menger space for a topological space X, then X satisfies the following selection hypothesis:*

- *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $s\omega$-covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for every $F \subset X$ there exist an $n \in \mathbb{N}$ and a $V \in \mathcal{V}_n$ such that $F \subset \mathrm{sCl}(V)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $s\omega$-covers of X. Let $\mathbb{N} = N_1 \cup N_2 \cup \ldots \cup N_n \cup \ldots$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets. For every $i \in \mathbb{N}$ and every $j \in N_i$, let $\mathcal{V}_j = \{U^i : U \in \mathcal{U}_j\}$. The sequence $\{\mathcal{V}_j : j \in N_i\}$ is a sequence of semi-open covers of X^i. Since X^i is an almost semi-Menger space, for every $i \in \mathbb{N}$, we can choose a sequence $(\mathcal{W}_j : j \in N_i)$ so that for each j, $\mathcal{W}_j = \{U^i_{j_1}, U^i_{j_2}, \ldots, U^i_{j_{k(j)}}\}$ is a finite subset of $\mathcal{V}_j$ and $\bigcup_{j\in N_i}\{\mathrm{sCl}(W) : W \in \mathcal{W}_j\}$ is a cover of X^i. We shall show that $\{\mathrm{sCl}(U_{j_p}) : 1 \le p \le k(j), j \in \mathbb{N}\}$ is an $s\omega$-cover of X. Let $F = \{x_1, x_2, \ldots, x_t\}$ be a finite subset of X. Then $(x_1, x_2, \ldots, x_t) \in X^t$, so there is an $l \in N_t$ such that $(x_1, x_2, \ldots, x_t) \in \mathcal{W}_l$. So, we can find $1 \le r \le k(l)$ such that $(x_1, x_2, \ldots, x_t) \in \mathrm{sCl}(U^t_{l_{k(l)}}) = (\mathrm{sCl}(U_{l_{k(l)}}))^t$. It is clear that $F \subset \mathrm{sCl}(U_{l_{k(l)}})$. $\square$

4. Star covering properties

The method of stars is one of classical popular topological methods. It has been used, for example, to study the problem of metrization of topological spaces, and for definitions and investigations of several important classical topological notions (see [1, 25, 37]). A number of results in the literature shows that many topological properties can be defined and studied in terms of star covering properties. In particular, such a method is also used in investigation of selection principles for topological spaces. This investigation was initiated by Kočinac in [18] and then studied in many papers [19, 21, 26, 28, 32, 33, 36].

Let A be a subset of X and $\mathcal{U}$ a collection of subsets of X. Then

$$\mathrm{St}^1(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \ne \emptyset\},$$

and inductively, for each $n \in \mathbb{N}$

$$\mathrm{St}^{n+1}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap \mathrm{St}^n(A,\mathcal{U}) \ne \emptyset\}.$$

We usually write $\mathrm{St}(A,\mathcal{U})$ instead of $\mathrm{St}^1(A,\mathcal{U})$ and $\mathrm{St}(x, U)$ for $\mathrm{St}(\{x\},\mathcal{U})$.

4.1. Star semi-compact and related spaces

Definition 4.1. A space X is called:

1. *n-star semi-compact* (resp. *n-star semi-Lindelöf*) if for every semi-open cover $\mathcal{U}$ of X, there is a finite (resp. countable) subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}^n(\bigcup\mathcal{V},\mathcal{U}) = X$; 1-star semi-compact spaces are called *star semi-compact*, and 1-star semi-Lindelöf spaces are called *star semi-Lindelöf*;
2. *ω-star semi-compact* if for every semi-open cover $\mathcal{U}$ of X, there is an $n \in \mathbb{N}$ and a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}^n(\bigcup\mathcal{V},\mathcal{U}) = X$;
3. *strongly n-star semi-compact* (resp. *strongly n-star semi-Lindelöf*) if for every semi-open cover $\mathcal{U}$ of X, there is a finite (resp. countable) subset A of X such that $\mathrm{St}^n(A,\mathcal{U}) = X$; strongly 1-star semi-compact (resp. strongly 1-star Lindelöf) spaces are called *strongly star semi-compact* (resp. *strongly star-Lindelöf*).

As usually, we say that $A \subset X$ has a property $\mathcal{P}$ in the above definition if A, as a subspace of X, has that property. We say that A *has* $\mathcal{P}$, *relative to* X (or A *has* $\mathcal{P}$ *in* X) if A is covered by sets semi-open in X.

Clearly, every strongly n-star semi-compact space is n-star semi-compact. It is also easy to see that every n-star semi-compact space is strongly $(n + 1)$-star semi-compact.

Definition 4.2. A space X is *semi-DFCC* (semi-discrete finite chain condition) provided every semi-discrete family of nonempty semi-open sets is finite.

A family $\mathcal{F}$ of subsets of X is called *semi-discrete* if each point $x \in X$ has a semi-open neighborhood V which intersects at most one element of $\mathcal{F}$.

We are going now to consider relations between countable semi-compactness and star semi-compactness. For the beginning, we need the following lemma, the proof is omitted.

Lemma 4.3. *Let X be a countably semi-compact space and A a semi-closed subset of X. Then A is countably semi-compact in X.*

Theorem 4.4. *A countably semi-compact space X is strongly star semi-compact.*

Proof. Let $\mathcal{U}$ be a semi-open cover of X. Suppose, on the contrary, that for each finite subset A of X, $\mathrm{St}(A,\mathcal{U})$ is a proper subset of X. By induction construct a countably infinite set $C = \{x_1, x_2, \ldots, x_n, \ldots\} \subset X$ such that for each integer $n \geq 1$, $x_{n+1} \notin V_n := \mathrm{St}(\{x_1, x_2, \ldots, x_n\},\mathcal{U})$. Let $y \in \mathrm{sCl}(C)$. There is a (semi-open) set $U \in \mathcal{U}$ such that $y \in U$, and thus $U \cap C \neq \emptyset$. Let m be such that $x_m \in U$. Then $y \in V_m$. Thus $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ is a countable semi-open cover of $\mathrm{sCl}(C)$ by sets semi-open in X. The set $\mathrm{sCl}(C)$ is a semi-closed subset of X, hence, by the previous lemma, it is countably semi-compact in X. But, by the construction of the set C, $\mathcal{V}$ has no finite subcollection which covers C. A contradiction. Hence, the statement of the theorem is true. □

Theorem 4.5. *Let X be a star semi-compact space and $\mathcal{U}$ a point finite semi-open cover of X. Then $\mathcal{U}$ has a finite subcover.*

Proof. Let $\mathcal{U}$ be a point finite semi-open covering of X (that is each point of X belongs to at most finitely many members of $\mathcal{U}$). There exists a finite set $F = \{x_1, x_2, \ldots, x_n\} \subset X$ such that $X = \mathrm{St}(A,\mathcal{U}) = \mathrm{St}(x_1,\mathcal{U}) \cup \mathrm{St}(x_2,\mathcal{U}) \cup \ldots \cup \mathrm{St}(x_n,\mathcal{U})$. For each t, $1 \leq t \leq n$, $\mathrm{St}(x_t,\mathcal{U})$ is the union of finitely many members of $\mathcal{U}$, so that $\mathcal{U}$ is finite. □

Theorem 4.6. *If a space X is semi-*DFCC*, then it is 2-star semi-compact.*

Proof. Suppose X is not 2-star semi-compact space and that $\mathcal{U}$ is a semi-open cover of X such that for any finite subset $\mathcal{V}$ of $\mathcal{U}$,

$$\mathrm{St}^2(\bigcup \mathcal{V},\mathcal{U}) \neq X. \tag{4.1}$$

Choose a U_0 in $\mathcal{U}$ and let $\mathcal{V}_0 = \{U_0\}$. Inductively define $\mathcal{V}_k$, a semi-discrete collection of k members of $\mathcal{U}$ such that $\mathcal{V}_{k-1} \subset \mathcal{V}_k$ for $1 \leq k < n$. By (4.1), $\mathrm{St}^2(\bigcup \mathcal{V}_{n-1},\mathcal{U}) \neq X$. Pick an $x_n \in X \setminus \mathrm{St}^2(\bigcup \mathcal{V}_{n-1},\mathcal{U})$ and a $U_n \in \mathcal{U}$ such that $x_n \in U_n$. Let $\mathcal{V}_n = \mathcal{V}_{n-1} \cup \{U_n\}$.

We claim that $\mathcal{V}_n$ is semi-discrete. Let $y \in X$ and select a $V \in \mathcal{U}$ such that $y \in V$. Let us assume that there are distinct sets U, $U' \in \mathcal{V}_n$ such that $V \cap U \neq \emptyset$ and $V \cap U' \neq \emptyset$ (say $U = U_p$ and $U' = U_q$ with $p < q$). Then $x_q \in U' \subset \mathrm{St}^2(\bigcup \mathcal{V}_p,\mathcal{U}) \subset \mathrm{St}^2(\bigcup \mathcal{V}_{q-1},\mathcal{U})$. But this contradicts the choice of x_q. So, for every $y \in X$, there is a semi-open V containing y which intersects at most one element of $\mathcal{V}_n$. Hence, our claim is proved.

Let $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$; then $\mathcal{V}$ is a countably infinite semi-discrete collection of semi-open sets. Therefore X is not semi-DFCC and the result follows. □

Theorem 4.7. *Every ω-star semi-compact space X has the property that every semi-continuous real-valued function on X is bounded.*

Proof. Suppose $f : X \longrightarrow \mathbb{R}$ is a semi-continuous mapping. Define $\mathcal{U} = \{f^{\leftarrow}(k, k+2) : k \in \mathbb{Z}\}$. By semi-continuity of f, each $f^{\leftarrow}(k, k+2)$ is semi-open in X. Hence, $\mathcal{U}$ is a semi-open cover of X. By assumption, for an $n \in \mathbb{N}$ and for a finite subset $\mathcal{V}$ of $\mathcal{U}$, we have $\mathrm{St}^n(\bigcup \mathcal{V},\mathcal{U}) = X$.

Let $M = \max\{k+2 : f^{\leftarrow}(k, k+2) \in \mathcal{V}\}$ and $m = \min\{k : f^{\leftarrow}(k, k+2) \in \mathcal{V}\}$.

Let $x \in X$. Then for $1 \leq t \leq n$, there are $f^{\leftarrow}(k_t, k_t+2) \in \mathcal{U}$ such that $x \in f^{\leftarrow}(k_t, k_t+2)$ with $f^{\leftarrow}(k_t, k_t+2) \cap f^{\leftarrow}(k_{t+1}, k_{t+1}+2) \neq \emptyset$ and $f^{\leftarrow}(k_1, k_1+2) \cap \cup\mathcal{V} \neq \emptyset$. By construction, $f(\bigcup \mathcal{V}) \subset (m, M)$. By induction, we obtain $f(x) \in (m-2n, M+2n)$. Hence, $f(X) \subset (m-2n, M+2n)$. This shows that f is bounded. □

4.2. Star semi-Menger spaces

Definition 4.8 ([18])**.** $\mathsf{S}_{\mathsf{fin}}{}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}(V,\mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an element of $\mathcal{B}$.

Definition 4.9 ([18])**.** $\mathsf{SS_{fin}}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\{\mathrm{St}(F_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

The symbols $\mathsf{S_{fin}}^*(\mathcal{O},\mathcal{O})$ and $\mathsf{SS_{fin}}^*(\mathcal{O},\mathcal{O})$ denote the *star-Menger property* and *strongly star-Menger property*, respectively.

In a similar way we introduce the following definition.

Definition 4.10.

(1) A space X is said to have the *star s-Menger property* if it satisfies $\mathsf{S_{fin}}^*(s\mathcal{O},s\mathcal{O})$.

(2) X is a *strongly star s-Menger space* if it satisfies $\mathsf{SS_{fin}}^*(s\mathcal{O},s\mathcal{O})$.

It is understood that every star s-Menger space is star semi-Lindelöf, and every strongly star s-Menger space is strongly star semi-Lindelöf. Every semi-Menger space is strongly star s-Menger.

Example 4.11. There is a strongly star s-Menger space which is not semi-Menger.

Endow the real line $\mathbb{R}$ with the topology $\tau = \{\mathbb{R}, \emptyset, \{p\}\}$, where p is a point in $\mathbb{R}$. Each subset of $\mathbb{R}$ containing p is semi-open. Let $\mathcal{U} = \{\{p,x\} : x \in \mathbb{R}\}$ be a semi-open cover of $\mathbb{R}$. This cover does not contain a countable subcover, so that this space in not semi-Lindelöf and thus cannot be semi-Menger. On the other hand, if $\mathcal{U}$ is any semi-open cover, then for the finite set $F = \{p\}$ we have $\mathrm{St}(F,\mathcal{U}) = \mathbb{R}$, that is, $(\mathbb{R},\tau)$ is strongly star compact, hence strongly star s-Menger.

Theorem 4.12. *If each finite power of a space X is star s-Menger, then X satisfies $\mathsf{S_{fin}}^*(s\mathcal{O},s\Omega)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Let $\mathbb{N} = N_1 \cup N_2 \cup \ldots$ be a partition of $\mathbb{N}$ into infinitely many infinite pairwise disjoint sets. For every $k \in \mathbb{N}$ and every $t \in N_k$, let $\mathcal{W}_t = \{U_1 \times U_2 \times \ldots \times U_k : U_1, \ldots U_k \in \mathcal{U}_t\} = \mathcal{U}_t^k$. Then $(\mathcal{W}_t : t \in N_k)$ is a sequence of semi-open covers of X^k, and since X^k is a star s-Menger space, we can choose a sequence $(\mathcal{H}_t : t \in N_k)$ such that for each t, $\mathcal{H}_t$ is a finite subset of $\mathcal{W}_t$ and $\bigcup_{t\in N_k}\{\mathrm{St}(H,\mathcal{W}_t) : H \in \mathcal{H}_t\}$ is a semi-open cover of X^k. For every $t \in N_k$ and every $H \in \mathcal{H}_t$ we have $H = U_1(H) \times U_2(H) \times \ldots \times U_k(H)$, where $U_i(H) \in \mathcal{U}_t$ for every $i \le k$. Set $\mathcal{V}_t = \{U_i(H) : i \le k, H \in \mathcal{H}_t\}$. Then for each $t \in N_k$, $\mathcal{V}_t$ is a finite subset of $\mathcal{U}_t$.

We claim that $\{\mathrm{St}(\bigcup \mathcal{V}_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an $s\omega$-cover of X. Let $F = \{x_1, \ldots, x_p\}$ be a finite subset of X. Then $y = (x_1, \ldots, x_p) \in X^p$ so that there is an $n \in N_p$ such that $y \in \mathrm{St}(H,\mathcal{W}_n)$ for some $H \in \mathcal{H}_n$. But $H = U_1(H) \times U_2(H) \times \ldots \times U_p(H)$, where $U_1(H), U_2(H), \ldots, U_p(H) \in \mathcal{V}_n$. The point y belongs to some $W \in \mathcal{W}_n$ of the form $V_1 \times V_2 \times \ldots \times V_p$, $V_i \in \mathcal{U}_n$ for each $i \le p$, which meets $U_1(H) \times U_2(H) \times \ldots \times U_p(H)$. This implies that for each $i \le p$, we have $x_i \in St(U_i(H),\mathcal{U}_n) \subset \mathrm{St}(\bigcup \mathcal{V}_n,\mathcal{U}_n)$, that is, $F \subset \mathrm{St}(\bigcup \mathcal{V}_n,\mathcal{U}_n)$. Hence, X satisfies $\mathsf{S_{fin}}^*(s\mathcal{O},s\Omega)$. □

Definition 4.13. A space X is called *meta semi-compact* if every semi-open cover $\mathcal{U}$ of X has a point-finite semi-open refinement $\mathcal{V}$ (that is, every point of X belongs to at most finitely many members of $\mathcal{V}$).

Theorem 4.14. *Every strongly star s-Menger meta semi-compact space is an s-Menger space.*

Proof. Let X be a strongly star s-Menger meta semi-compact space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be a point-finite semi-open refinement of $\mathcal{U}_n$. Since X is strongly star s-Menger, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{V}_n) = X$.

As $\mathcal{V}_n$ is a point-finite refinement and each F_n is finite, elements of each F_n belongs to finitely many members of $\mathcal{V}_n$ say $V_{n_1}, V_{n_2}, V_{n_3}, \ldots, V_{n_k}$. Let $\mathcal{V}'_n = \{V_{n_1}, V_{n_2}, V_{n_3}, \ldots, V_{n_k}\}$. Then $\mathrm{St}(F_n, \mathcal{V}_n) = \bigcup \mathcal{V}'_n$ for each $n \in \mathbb{N}$. We have that $\bigcup_{n\in\mathbb{N}}(\bigcup \mathcal{V}'_n) = X$. For every $V \in \mathcal{V}'_n$ choose $U_V \in \mathcal{U}_n$ such that $V \subset U_V$. Then, for every n, $\mathcal{W}_n := \{U_V : V \in \mathcal{V}'_n\}$ is a finite subfamily of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{W}_n = X$, that is X is an s-Menger space. □

Definition 4.15. A space X is said to be *meta semi-Lindelöf* if every semi-open cover $\mathcal{U}$ of X has a point-countable semi-open refinement $\mathcal{V}$.

Theorem 4.16. *Every strongly star s-Menger meta semi-Lindelöf space is a semi-Lindelöf space.*

Proof. Let X be a strongly star s-Menger meta semi-Lindelöf space. Let $\mathcal{U}$ be a semi-open cover of X and $\mathcal{V}$ a point-countable semi-open refinement of $\mathcal{U}$. Since X is strongly star s-Menger, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{V}_n) = X$.

For every $n \in \mathbb{N}$, denote by $\mathcal{W}_n$ the collection of all members of $\mathcal{V}$ which intersects F_n. Since $\mathcal{V}$ is point-countable and F_n is finite, $\mathcal{W}_n$ is countable. So, the collection $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a countable subfamily of $\mathcal{V}$ and is a cover of X. For every $W \in \mathcal{W}$, pick a member $U_W \in \mathcal{U}$ such that $W \in U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$. Hence, X is a semi-Lindelöf space. □

Definition 4.17. A space X is an *almost star s-Menger space* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)) : n \in \mathbb{N}\}$ is a cover of X.

Theorem 4.18. *A space X is an almost star s-Menger space if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-regular sets, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)) : n \in \mathbb{N}\}$ is a cover of X.*

Proof. Since every semi-regular set is semi-open, necessity follows.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Let $\mathcal{U}'_n = \{\mathrm{sCl}(U) : U \in \mathcal{U}_n\}$. Then $\mathcal{U}'_n$ is a cover of X by semi-regular sets. Then by assumption, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and $\{\mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}'_n)) : n \in \mathbb{N}\}$ is a cover of X.

First we shall prove that $\mathrm{St}(U,\mathcal{U}_n) = \mathrm{St}(\mathrm{sCl}(U),\mathcal{U}_n)$ for all $U \in \mathcal{U}_n$. It is obvious that $\mathrm{St}(U,\mathcal{U}_n) \subset \mathrm{St}(\mathrm{sCl}(U),\mathcal{U}_n)$ since $U \subset \mathrm{sCl}(U)$. Let $x \in \mathrm{St}(\mathrm{sCl}(U),\mathcal{U}_n)$. Then there exists a $U' \in \mathcal{U}_n$ such that $x \in U'$ and $U' \cap \mathrm{sCl}(U) \neq \emptyset$. Then $U' \cap \mathrm{sCl}(U) \neq \emptyset$ implies that $x \in \mathrm{St}(U,\mathcal{U}_n)$. Hence, $\mathrm{St}(\mathrm{sCl}(U),\mathcal{U}_n) \subset \mathrm{St}(U,\mathcal{U}_n)$.

For each $V \in \mathcal{V}_n$ we can find a $U_V \in \mathcal{U}_n$ such that $V = \mathrm{sCl}(U_V)$. Let $\mathcal{V}'_n = \{U_V : V \in \mathcal{V}_n\}$ and $x \in X$. Then there exists an $n \in \mathbb{N}$ such that $x \in \mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}'_n))$. For each semi-open set V of x, we have $V \bigcap \mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}'_n) \neq \emptyset$. Then there exists $U \in \mathcal{U}_n$ such that $(V \cap \mathrm{sCl}(U) \neq \emptyset$ and $\bigcup \mathcal{V}_n \cap \mathrm{sCl}(U) \neq \emptyset)$ imply that $(V \cap U \neq \emptyset$ and $\bigcup \mathcal{V}_n \cap \mathrm{sCl}(U) \neq \emptyset)$. We have that $\bigcup \mathcal{V}'_n \cap U \neq \emptyset$, so $x \in \mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}'_n, \mathcal{U}_n))$. Hence, $\{\mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}'_n, \mathcal{U}_n)) : n \in \mathbb{N}\}$ is a cover of X. □

Theorem 4.19. *Quasi-irresolute surjective image of an almost star s-Menger space is an almost star s-Menger space.*

Proof. Let X be an almost star s-Menger space and Y a topological space. Let $f : X \to Y$ be a quasi irresolute surjection. Take a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of Y by semi-regular sets. Let $\mathcal{U}'_n = \{f^{\leftarrow}(U) : U \in \mathcal{U}_n\}$; then each $\mathcal{U}'_n$ is a cover of X by semi-regular sets since f is quasi-irresolute. Since X is an almost star s-Menger space, there exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}'_n$ and $\{\mathrm{sCl}(St(\bigcup \mathcal{V}'_n, \mathcal{U}'_n)) : n \in \mathbb{N}\}$ is a cover of X.

Let $\mathcal{V}_n = \{U \in \mathcal{U}_n : f^{\leftarrow}(U) \in \mathcal{V}'_n\}$ and $x \in X$. Then $f^{\leftarrow}(\bigcup \mathcal{V}_n) = \bigcup \mathcal{V}'_n$ and there is $n \in \mathbb{N}$ such that $x \in \mathrm{sCl}(\mathrm{St}(f^{\leftarrow}(\bigcup \mathcal{V}_n, \mathcal{U}'_n)))$. For $y = f(x) \in Y$, $y \in f(\mathrm{sCl}(\mathrm{St}(f^{\leftarrow}(\bigcup \mathcal{V}_n, \mathcal{U}'_n)))) \subset \mathrm{sCl}(f(\mathrm{St}(f^{\leftarrow}(\bigcup \mathcal{V}_n, \mathcal{U}'_n)))) = \mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n))$. Now assume that $f^{\leftarrow}(\bigcup \mathcal{V}_n) \cap f^{\leftarrow}(U) \neq \emptyset$. Then $f(f^{\leftarrow}(\bigcup \mathcal{V}_n)) \cap f(f^{\leftarrow}(U)) \neq \emptyset$, hence $\bigcup \mathcal{V}_n \cap U \neq \emptyset$. So, it is shown that Y is an almost star s-Menger space. □

References

[1] O. T. Alas, L. R. Junqueira, R. G. Wilson, *Countability and star covering properties*, Topology Appl., **158** (2011), 620–626. 4

[2] D. R. Anderson, J. A. Jensen, *Semi-continuity on topological spaces*, Atti Accad. Naz. Lincei Rend. Cl. Sci Fis. Mat. Natur., **42** (1967), 782–783. 1

[3] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24–32. 3.1

[4] S.-S. Chang, Y. J. Cho, S. M. Kang, *Nonlinear Operator Theory in Probabilistic Metric Spaces*, Nova Science Publishers, New York, (2001). 2

[5] Y. J. Cho, M. Grabiec, V. Radu, *On Nonsymmetric Topological and Probabilistic Structures*, Nova Science Publishers, New York, (2006). 2

[6] S. G. Crossley, *A note on semitopological properties*, Proc. Amer. Math. Soc., **72** (1978), 409–412. 1

[7] S. G. Crossley, S. K. Hildebrand, *Semi-closure*, Texas J. Sci., **22** (1971), 99–112. 1, 3

[8] S. G. Crossley, S. K. Hildebrand, *Semi-topological properties*, Fund. Math., **74** (1972), 233–254. 3, 1, 3.5

[9] G. Di Maio, T. Noiri, *On s-closed spaces*, Indian J. Pure Appl. Math., **18** (1987), 226–233. 3.1

[10] G. Di Maio, T. Noiri, *Weak and strong forms of irresolute functions*, Rend. Circ. Mat. Palermo(2). Suppl., **18** (1988), 255–273. 5

[11] C. Dorsett, *Semi compactness, semi separation axioms, and product spaces*, Bull. Malaysian Math. Soc., **4** (1981), 21–28. 2.1

[12] C. Dorsett, *Semi-regular spaces*, Soochow J. Math., **8** (1982), 45–53. 2.2, 2.3

[13] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, (1989). 2, 3.5, 3.12

[14] M. Ganster, *On covering properties and generalized open sets in topological spaces*, Math. Chronicle, **19** (1990), 27–33. 2.1

[15] M. Ganster, D. S. Janković, I. L. Reilly, *On compactenss with respect to semi-open sets*, Comment. Math. Univ. Carolonae, **31** (1990), 37–39. 3.2

[16] W. Hurewicz, *Über die Verallgemeinerung des Borelschen Theorems*, Math. Z., **24** (1926), 401–425. 1

[17] D. Kocev, *Almost Menger and related spaces*, Mat. Vesnik, **61** (2009), 173–180. 3.1

[18] Lj. D. R. Kočinac, *Star-Menger and related spaces*, Publ. Math. Debrecen, **55** (1999), 421–431 4, 4.8, 4.9

[19] Lj. D. R. Kočinac, *Star-Menger and related spaces*, II. Filomat, **13** (1999), 129–140. 3.1, 4

[20] Lj. D. R. Kočinac, *Selected results on selection principles*, Proceedings of the 3rd Seminar on Geometry & Topology, Azarb. Univ. Tarbiat Moallem, Tabriz, **2004** (2004), 71–104. 1

[21] Lj. D. R. Kočinac, *Star selection principles: A survey*, Khayyam J. Math., **1** (2015), 82–106. 4

[22] K. Kunen, *Luzin spaces*, Topology Proc., **1** (1976), 191–199. 3.2

[23] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41. 1, 1

[24] S. N. Maheswari, R. Prasad, *On s-regular spaces*, Glasnik Mat. Ser., **10** (1975), 347–350. 2.2

[25] M. V. Matveev, *A survey on star covering properties*, Topology Atlas, Preprint No. **330**, (1998). 4

[26] M. V. Matveev, *On the extent of* SSM *spaces*, preprint, (1998). 4

[27] K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Stzungsberischte Abt. 3a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik, **133** (1924), 421–444. 1

[28] M. Sakai, *Star versions of the Menger property*, Topology Appl., **176** (2014), 22–34. 4

[29] M. Sakai, M. Scheepers, *The combinatorics of open covers*, Recent progress in general topology, **2014** (2014), 751–799. 1

[30] M. Scheepers, *Combinatorics of open covers, I, Ramsey theory*, Topology Appl., **69** (1996), 31–62. 1, 3

[31] M. Scheepers, *Selection principles and covering properties in Topology*, Note Mat., **22** (2003), 3–41. 1

[32] Y.-K. Song, *Remarks on strongly star-Menger spaces*, Comment. Math. Univ. Carolin., **54** (2013), 97–104. 4

[33] Y.-K. Song, *Remarks on star-Menger spaces*, Houston J. Math., **40** (2014), 917–925. 4

[34] T. Thompson, *S-closed spaces*, Proc. Amer. Math. Soc., **60** (1976), 335–338. 3.11, 3.12

[35] B. Tsaban, *Some new directions in infinite-combinatoril topology*, Set Theory, Trends in Mathematics, Birkhäuser, Basel, **2006** (2006), 225–255. 1

[36] B. Tsaban, *Combinatorial aspects of selective star covering properties in* Ψ*-spaces*, Topology Appl., **192** (2015), 198–207. 4

[37] E. K. Van Douwen, G. M. Reed, A. W. Roscoe, I. J. Tree, *Star covering properties*, Topology Appl., **39** (1991), 71–103. 4