Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4257–4262
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Spectral analysis of a selfadjoint matrix-valued discrete operator on the whole axis

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Communicated by M. Eslamian

Abstract

The spectral analysis of matrix-valued difference equations of second order having polynomial-type Jost solutions, was first used by Aygar and Bairamov. They investigated this problem on semi-axis. The main aim of this paper is to extend similar results to the whole axis. We find polynomial-type Jost solutions of a second order matrix selfadjoint difference equation to the whole axis. Then, we obtain the analytical properties and asymptotic behaviors of these Jost solutions. Furthermore, we investigate continuous spectrum and eigenvalues of the operator L generated by a matrix-valued difference expression of second order. Finally, we get that the operator L has a finite number of real eigenvalues.

Keywords: Difference equations, discrete operator, Jost solution, eigenvalue, continuous spectrum.
2010 MSC: 39A05, 39A70, 39A10, 47A05.

1. Introduction

The problems of spectral theory of differential equations have been intensively investigated by several authors [4, 8, 13–16, 18]. In [13], the author showed that the Sturm–Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+ := [0, \infty) \tag{1.1}$$

has a bounded solution satisfying the condition

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \geq 0\},$$

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Received 2016-04-19

which is called Jost solution of the equation (1.1), where λ is a spectral parameter and q is a real-valued function. The modeling of certain problems in engineering, physics, economics, control theory and other areas has led to a rapid development of the theory of difference equations. The spectral analysis of discrete equations has also been applied to the solutions of classes of nonlinear discrete equations and Toda lattices [9, 17]. Furthermore, there are a lot of studies about the spectral analysis of selfadjoint and nonselfadjoint difference equations [1–3, 5–7, 11, 19]. All of the above mentioned papers deal with difference equations with scalar coefficients except [5, 19]. But spectral analysis of selfadjoint matrix-valued difference equations with polynomial-type Jost solutions on the whole axis has not been used yet.

Let $\mathbb{C}_\upsilon$ is a υ-dimensional ($\upsilon < \infty$) Euclidian space and denote by $\ell_2(\mathbb{R}, \mathbb{C}_\upsilon)$ the Hilbert space of all matrix sequences $Y = \{Y_n\}$ ($Y_n \in \mathbb{C}_\upsilon$, $n \in \mathbb{Z}$) such that $\sum_{-\infty}^{\infty} ||Y_n||^2_{\mathbb{C}_\upsilon} < \infty$ with the inner product $\langle Y, Z\rangle = \sum_{-\infty}^{\infty} (Y_n, Z_n)_{\mathbb{C}_\upsilon}$, where $||.||_{\mathbb{C}_\upsilon}$ and $(.,.)_{\mathbb{C}_\upsilon}$ express the matrix norm and inner product in $\mathbb{C}_\upsilon$, respectively. We introduce the difference operator L generated in $\ell_2(\mathbb{R}, \mathbb{C}_\upsilon)$ by the matrix difference equation

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = \lambda y_n, \quad n \in \mathbb{Z}, \tag{1.2}$$

where $\{A_n\}$ and $\{B_n\}$ are linear operators (matrices) acting in $\mathbb{C}_\upsilon$, $n \in \mathbb{Z}$ and λ is a spectral parameter. Throughout the paper, we will assume that $A_n = A_n^\star$, $B_n = B_n^\star$ ($n \in \mathbb{Z}$) and $\det A_n \neq 0$, where $\star$ denotes the adjoint operator. Further, we can obtain the following Jacobi matrix by using the operator L

$$(J)_{ij} = \begin{cases} B_i & \text{if} \quad i = j, \\ A_{i-1} & \text{if} \quad i = j+1, \\ A_i & \text{if} \quad i = j-1, \\ 0 & \text{otherwise,} \end{cases}$$

where 0 is the zero matrix in $\mathbb{C}_\upsilon$ and $i, j \in \mathbb{Z}$. It is clear that L is a selfadjoint operator which a is discrete analogue of the matrix-valued Sturm–Liouville operator generated in $L_2(-\infty, \infty)$. So, L is called matrix-valued discrete operator. The paper is organized as follows: In Section 2, we get the polynomial-type Jost solutions of (1.2), and investigate analytical properties and asymptotic behaviors of them. In Section 3, using the properties of the Jost solutions, we obtain eigenvalues and continuous spectrum of L. Moreover, we prove that the operator L has a finite number of real eigenvalues, under the condition

$$\sum_{-\infty}^{\infty} |n| \left(||I - A_n|| + ||B_n||\right) < \infty, \tag{1.3}$$

where I denotes the identity matrix in $\mathbb{C}_\upsilon$.

2. Jost Functions

Suppose that the sequences of matrices $\{A_n\}$ and $\{B_n\}, n \in \mathbb{Z}$ satisfy (1.3). Let $E(z) := \{E_n(z)\}$ and $F(z) := \{F_n(z)\}$, $n \in \mathbb{Z}$ denote the matrix solutions of the equation

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = (z + z^{-1})Y_n, \quad n \in \mathbb{Z}, \tag{2.1}$$

under the conditions

$$\lim_{n\to\infty} Y_n(z)z^{-n} = I, \quad z \in D_0 := \{z : |z| = 1\},$$

and

$$\lim_{n\to-\infty} Y_n(z)z^{n} = I, \quad z \in D_0,$$

respectively. The solutions $E(z)$ and $F(z)$ are bounded, and are called the Jost solutions of (2.1).

Theorem 2.1. *Assume* (1.3). *Then, for* $z \in D_0$ *and* $n \in \mathbb{Z}$, (2.1) *has the solutions* $E_n(z)$ *and* $F_n(z)$ *having representations*

$$E_n(z) = T_n z^n \left[I + \sum_{m=1}^{\infty} K_{nm} z^m \right], \tag{2.2}$$

and

$$F_n(z) = R_n z^{-n} \left[I + \sum_{-\infty}^{m=-1} M_{nm} z^{-m} \right], \tag{2.3}$$

respectively, where T_n, R_n, K_{nm} *and* M_{nm} *are expressed in terms of* $\{A_n\}$ *and* $\{B_n\}$ *by*

$$T_n = \prod_{p=n}^{\infty} A_p^{-1}, \tag{2.4}$$

$$K_{n1} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p, \tag{2.5}$$

$$K_{n2} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p1} + \sum_{p=n+1}^{\infty} T_p^{-1} \left(I - A_p^2\right) T_p, \tag{2.6}$$

$$K_{n,m+2} = \sum_{p=n+1}^{\infty} T_p^{-1} \left(I - A_p^2\right) T_p K_{p+1,m} - \sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p,m+1} + K_{n+1,m}, \tag{2.7}$$

for $m \in \mathbb{Z}^+$,

$$R_n = \prod_{-\infty}^{p=n-1} A_p^{-1}, \tag{2.8}$$

$$M_{n,-1} = -\sum_{-\infty}^{p=n-1} R_p^{-1} B_p R_p, \tag{2.9}$$

$$M_{n,-2} = -\sum_{-\infty}^{p=n-1} R_p^{-1} B_p R_p M_{p,-1} + \sum_{-\infty}^{p=n-1} R_p^{-1} \left(I - A_{p-1}^2\right) R_p, \tag{2.10}$$

$$M_{n,m-2} = \sum_{-\infty}^{p=n-1} R_p^{-1} \left(I - A_{p-1}^2\right) R_p M_{p-1,m} - \sum_{-\infty}^{p=n-1} R_p^{-1} B_p R_p M_{p,m-1} + M_{n-1,m}, \tag{2.11}$$

for $m \in \mathbb{Z}^-$.

Proof. If we put $E(z)$ and $F(z)$ into (2.1), then we have

$$\begin{aligned} A_{n-1} T_{n-1} z^{n-1} \left[I + \sum_{m=1}^{\infty} K_{n-1,m} z^m \right] &+ B_n T_n z^n \left[I + \sum_{m=1}^{\infty} K_{nm} z^m \right] \\ &+ A_n \left\{ T_{n+1} z^{n+1} \left[I + \sum_{m=1}^{\infty} K_{n+1,m} z^m \right] \right\} \\ &= T_n z^{n+1} \left[I + \sum_{m=1}^{\infty} K_{nm} z^m \right] + T_n z^{n-1} \left[I + \sum_{m=1}^{\infty} K_{nm} z^m \right] \end{aligned}$$

and

$$A_{n-1} R_{n-1} z^{-n+1} \left[I + \sum_{-\infty}^{m=-1} M_{n-1,m} z^{-m} \right] + B_n R_n z^{-n} \left[I + \sum_{-\infty}^{m=-1} M_{nm} z^{-m} \right]$$

$$
\begin{aligned}
&+ A_n R_{n+1} z^{-n-1} \left[I + \sum_{-\infty}^{m=-1} M_{n+1,m} z^{-m} \right] \\
&= R_n z^{-n+1} \left[I + \sum_{-\infty}^{m=-1} M_{nm} z^{-m} \right] + R_n z^{-n-1} \left[I + \sum_{-\infty}^{m=-1} M_{nm} z^{-m} \right],
\end{aligned}
$$

respectively. Using these equations, we get T_n, R_n as convergent products and K_{nm}, M_{nm} as convergent series, under the condition (1.3). $\square$

Theorem 2.2. *Under the assumption* (1.3)*, the following inequalities hold*

$$
||K_{nm}|| \leq C_1 \sum_{p=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(||I - A_p|| + ||B_p|| \right), m \in \mathbb{Z}^+, \tag{2.12}
$$

$$
||M_{nm}|| \leq C_2 \sum_{-\infty}^{p=n+\left[\left|\frac{m}{2}\right|\right]} \left(||I - A_p|| + ||B_p|| \right), m \in \mathbb{Z}^-, \tag{2.13}
$$

where $\left[\left|\frac{m}{2}\right|\right]$ *is the integer part of* $\frac{m}{2}$*, while* C_1 *and* C_2 *are positive constants.*

Proof. Using (2.4)–(2.7) and (2.8)–(2.11), we can get the proof by mathematical induction. $\square$

Corollary 2.3. *It follows from* (2.2)*,* (2.3) *and Theorem* 2.2 *that* $E_n(z)$ *and* $F_n(z)$ *have analytic continuation from* D_0 *to* $\{z : |z| < 1\} \setminus \{0\}$*.*

Theorem 2.4. *Assume* (1.3)*. Then the Jost solutions satisfy the following asymptotic equations for* $z \in D := \{z : |z| \leq 1\} \setminus \{0\}$

$$
E_n(z) = z^n \left[I + o(1) \right], \quad n \to \infty \tag{2.14}
$$

$$
F_n(z) = z^{-n} \left[I + o(1) \right], \quad n \to -\infty. \tag{2.15}
$$

Proof. The proof of (2.14) was given in [5]. If we use (2.8) and (2.13), we can write

$$
\lim_{n \to -\infty} R_n = I \tag{2.16}
$$

and

$$
\sum_{-\infty}^{m=-1} M_{nm} z^{-m} = o(1), \quad z \in D, \quad n \to -\infty. \tag{2.17}
$$

Using (2.3), (2.16), and (2.17), we get (2.15) for $z \in D$. $\square$

3. Continuous and Discrete Spectra of L

Theorem 3.1. *If the condition* (1.3) *is satisfied, then* $\sigma_c(L) = [-2, 2]$*, where* $\sigma_c(L)$ *denotes the continuous spectrum of* L*.*

Proof. Let us introduce the difference operators L_0 and L_1 generated in $\ell_2(\mathbb{R}, \mathbb{C}_\upsilon)$ by the difference expressions

$$
l_0(Y) = Y_{n-1} + Y_{n+1}
$$

and

$$
l_1(Y) = (A_{n-1} - I)Y_{n-1} + B_n Y_n + (A_n - I)Y_{n+1},
$$

respectively. We can also define the following Jacobi matrices

$$(J_0)_{ij} = \begin{cases} I & i = j+1, \quad i = j-1, \\ 0 & \text{otherwise}, \end{cases}$$

$$(J_1)_{ij} = \begin{cases} B_i & i = j, \\ A_i - I & i = j-1, \\ A_{i-1} - I & i = j+1, \\ 0 & \text{otherwise}, \end{cases}$$

corresponding to the operators L_0 and L_1, respectively, where $i, j \in \mathbb{Z}$. It is obvious that $L = L_0 + L_1$ and L_0 is a selfadjoint operator. Also, it is known that $\sigma(L_0) = \sigma_{\mathrm{c}}(L_0) = [-2, 2]$, (see [5]). It follows from (1.3) that the operator L_1 is compact in $\ell_2(\mathbb{R}, \mathbb{C}_v)$ (see [12]). Then, using the Weyl theorem (see [10]) on compact perturbation, we obtain $\sigma_{\mathrm{c}}(L) = \sigma_{\mathrm{c}}(L_0) = [-2, 2]$. This completes the proof. □

Now, we denote the solution of the equation

$$U_{n-1}A_{n-1} + U_n B_n + U_{n+1}A_n = (z + z^{-1})U_n, \quad n \in \mathbb{Z},$$

satisfying the condition

$$\lim_{n \to -\infty} U_n(z) z^n = I, \quad z \in D_0,$$

by $G(z) := \{G_n(z)\}$. It is clear that $G(z)$ is the adjoint matrix of $F(z)$. Let us introduce $f(z) := \det W[E(z), G(z)]$, where $W[E(z), G(z)]$ denotes the wronskian of the solutions $E(z)$ and $G(z)$ which is defined by $W[E(z), G(z)] = G_{n-1}A_{n-1}E_n - G_n A_{n-1} E_{n-1}$. The set of all eigenvalues of L we denote by

$$\sigma_{\mathrm{d}}(L) = \left\{ \lambda \in \mathbb{C} : \lambda = z + z^{-1}, \quad z \in (-1, 0) \cup (0, 1), \quad f(z) = 0 \right\}.$$

Since $\sigma_{\mathrm{d}}(L)$ and $\sigma_{\mathrm{c}}(L)$ are disjoined sets, we can get

$$\sigma_{\mathrm{d}}(L) \subset (-\infty, -2) \cup (2, \infty). \tag{3.1}$$

Definition 3.2. The multiplicity of a zero of the function f is called the multiplicity of the corresponding eigenvalue of L.

Theorem 3.3. *Under the condition* (1.3)*, the operator L has a finite number of real eigenvalues.*

Proof. Since the operator L is selfadjoint, its eigenvalues are real. To complete the proof, we have to show that the function f has finitely many zeros. Using (3.1), we get that the limit points of the set of all eigenvalues of L could not be different from ± 2, $\pm\infty$. Since $\lambda = z + z^{-1}$, the limit points of the set of all eigenvalues of L could be $\pm\infty$ only in the case of $z = 0$. But it contradicts the fact that the operator L is bounded, so we cannot consider 0 as a zero of the function f. On the other hand, for $z = \pm 1$, the limit points of the set of all eigenvalues could be ± 2. But from operator theory and Theorem 3.1, the eigenvalues of selfadjoint operators are not the elements of its continuous spectrum. Because of this reason, we also cannot consider $z = \pm 1$ as zeros of the function f, i.e., the set of all eigenvalues of the operator L has not any limit points. Finally, the set of zeros of the function f in D is finite, by the Bolzano–Weierstrass theorem. □

Acknowledgment

The authors thank the referees for their contributions.

References

[1] M. Adıvar, E. Bairamov, *Spectral properties of non-selfadjoint difference operators*, J. Math. Anal. Appl., **261** (2001), 461–478. 1

[2] R. P. Agarwal, *Difference equations and inequalities*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, (2000).

[3] R. P. Agarwal, P. J. Y. Wong, *Advanced topics in difference equations*, Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, (1997). 1

[4] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, Translated by N. Kemmer, Hafner Publishing Co., New York, (1965). 1

[5] Y. Aygar, B. Bairamov, *Jost solution and the spectral properties of the matrix-valued difference operators*, Appl. Math. Comput., **218** (2012), 9676–9681. 1, 2, 3

[6] E. Bairamov, Y. Aygar, T. Koprubas, *The spectrum of eigenparameter-dependent discrete Sturm-Liouville equations*, J. Comput. Appl. Math., **235** (2011), 4519–4523.

[7] E. Bairamov, A. O. Çelebi, *Spectrum and spectral expansion for the non-selfadjoint discrete Dirac operators*, Quart. J. Math. Oxford Ser., **50** (1999), 371–384. 1

[8] E. Bairamov, E. Kir, *Spectral properties of a finite system of Sturm-Liouville differential operators*, Indian J. Pure Appl. Math., **35** (2004), 249–256. 1

[9] Y. M. Berezanskiĭ, *Integration of nonlinear difference equations by the method of the inverse spectral problem*, Dokl. Akad. Nauk SSSR, **281** (1985), 16–19. 1

[10] I. M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators*, Daniel Davey & Co., Inc., New York, (1966). 3

[11] G. S. Guseĭnov, *The inverse problem of scattering theory for a second order difference equation on the whole real line*, Dokl. Akad. Nauk SSSR, **230** (1976), 1045–1048. 1

[12] L. A. Lusternik, V. J. Sobolev, *Elements of functional analysis*, Hindustan Publishing Corp., Delhi, (1974). 3

[13] V. A. Marchenko, *Sturm-Liouville operators and applications*, Birkhäuser Verlag, Basel, (1986). 1

[14] M. A. Naimark, *Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint differential operator of the second order on a semi-axis*, Amer. Math. Soc. Transl., **16** (1960), 103–193.

[15] M. Olgun, C. Coskun, *Non-selfadjoint matrix Sturm-Liouville operators with spectral singularities*, Appl. Math. Comput., **216** (2010), 2271–2275.

[16] J. Schwartz, *Some non-selfadjoint operators*, Comm. Pure Appl. Math., **13** (1960), 609–639. 1

[17] M. Toda, *Theory of nonlinear lattices*, Springer-Verlag, Berlin-New York, (1981). 1

[18] H. Weyl, *Über beschränkte quadratische Formen, deren Differenz vollständig ist.*, Rend. Circ. Mat. Palermo, **27** (1909), 373–392. 1

[19] S. Yardimci, *A note on the spectral singularities of non-selfadjoint matrix-valued difference operators*, J. Comput. Appl. Math., **234** (2010), 3039–3042. 1