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Strong convergence results for the split common fixed point problem

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Abstract

Recently, Boikanyo [O. A. Boikanyo, Appl. Math. Comput., **265** (2015), 844–853] constructed an algorithm for demicontractive operators and obtained the strong convergence theorem for the split common fixed point problem. In this paper, we mainly consider the viscosity iteration algorithm of the algorithm Boikanyo to approximate the split common fixed point problem, and we get the generated sequence strongly converges to a solution of this problem. The main results in this paper extend and improve some results of Boikanyo [O. A. Boikanyo, Appl. Math. Comput., **265** (2015), 844–853] and Cui and Wang [H. H. Cui, F. H. Wang, Fixed Point Theory Appl., **2014** (2014), 8 pages]. The research highlights of this paper are novel algorithms and strong convergence results.

Keywords: Split common fixed point problem, demicontractive mapping, explicit viscosity algorithm, strong convergence.
2010 MSC: 47J25, 47H45.

1. Introduction

In 1994, Censor and Elfving [6] proposed the split feasibility problem (SFP), which is to find a point

$$x \in C, \quad \text{such that} \quad Ax \in Q,$$

where C is a nonempty closed convex subset of a Hilbert space H_1, Q is a nonempty closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

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To solve this problem, Censor and Elfving [6] introduced the original algorithm in the finite-dimensional space R^n in 1994,

$$x_{n+1} = A^{-1}P_Q P_{A(C)}Ax_n, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of R^n, the bounded linear operator A of R^n is a $n \times n$ matrix and P_Q is the orthogonal projection onto the sets onto Q.

But this algorithm (1.1) involves the computation of the inverse A^{-1} (assuming the existence of the inverse of A) and thus it does not become popular.

In order to overcome the disadvantage of this algorithm, Byrne [2, 3] introduced the following algorithm:

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \quad n \geq 0,$$

where $0 < \gamma < 2/\rho$ with ρ being the spectral radius of the operator A^*A and $P_C,\ P_Q$ denote the orthogonal projection onto the sets $C,\ Q$, respectively.

However, the step size of the CQ algorithm is fixed and related to spectral radius of the operator A^*A, and the orthogonal projection onto the sets C and Q is not easily calculated usually.

Based on the applications of the SFP in intensity-modulated radiation therapy, signal processing, and image reconstruction, the SFP has received more and more attention and how to approximate the solutions of the SFP are studied extensively by so many scholars, see [4, 5, 7, 10, 12, 16–18, 20, 21, 24–27].

In 2009, Censor and Segal [8] proposed the split common fixed point problem (SCFP), which is to find a point

$$x \in Fix(U), \quad \text{such that} \quad Ax \in Fix(T), \tag{1.2}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$, and $Fix(U)$ and $Fix(T)$ denote the fixed point sets of U and T.

It is obvious to see the SCFP is a particular case of SFP and closely related to SFP.

For solving this problem, the original algorithm for directed operator was introduced by Censor and Elfving [8] in the following,

$$x_{n+1} = U(x_n - \rho A^*(I - T)Ax_n), \quad n \geq 0,$$

where the step size ρ satisfies $0 < \rho < \frac{2}{\|A\|^2}$, and they proved that the sequence $\{x_n\}$ weakly converges to a solution of the SCFP (1.2) if the SCFP consists. But the disadvantage of this algorithm is the choice of the step size ρ, which depends on the norm of operator A. Then, some authors do some improvement studies. But the improvement mainly focuses on the extension of the operator, such as

In 2010, Moudafi [15] extended to demicontractive mappings.

In 2011, he [14] also extended to quasi-nonexpansive operators.

In 2011, Wang and Xu [20] extended to finitely many directed operators.

The detailed relation of the directed operator, quasi-nonexpansive operator and demicontractive operator can see Section 3. Also there are some other researchers studied the fixed point theory and its applications [28].

Until 2014, Cui and Wang [9] proposed the following algorithm, and they proved the sequence $\{x_n\}$ converges weakly to a solution of the SCFP (1.2),

$$x_{n+1} = U_\lambda(x_n - \rho_n A^*(I - T)Ax_n), \quad n \geq 0, \tag{1.3}$$

where the step size ρ_n is chosen in the following way,

$$\rho_n = \begin{cases} \frac{(1-\tau)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, & Ax_n \neq T(Ax_n), \\ 0, & \text{otherwise}. \end{cases} \tag{1.4}$$

The step size of this algorithm ρ_n does not depend on the the norm of operator A and searches automatic.

In 2015, Boikanyo [1] extended the main results of Cui and Wang [9] and constructed the following Halpern's type algorithm for demicontractive operators that converges strongly to a solution of the SCFP (1.2),

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)U_\lambda(x_n - \rho_n A^*(I - T)Ax_n), \quad n \geq 0, \tag{1.5}$$

and the step size ρ_n is chosen as (1.4).

Motivated by Boikanyo [1] and Xu [23], in this paper, we construct the viscosity algorithms of (1.5) for demicontractive operators to approximate the solution of the SCFP (1.2),

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_\lambda(x_n - \rho_n A^*(I-T)Ax_n), \quad n \geq 0, \tag{1.6}$$

and the step size ρ_n is also chosen as (1.4).

And we prove the sequence $\{x_n\}$ generated by the (1.6) strongly converges to a solution $\hat{x}$ of the SCFP (1.2), and the $\hat{x}$ solves the following variational inequality:

$$\langle \hat{x} - f(\hat{x}), \hat{x} - z\rangle \leq 0, \quad \forall z \in S,$$

where S denotes the set of all solutions of the SCFP (1.2).

2. Preliminaries

Throughout this paper, we use $x_n \rightharpoonup x$ to indicate that $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ symbolizes the sequence $\{x_n\}$ converges strongly to x. $\mathbb{N}$ indicates the set of natural numbers.

Some concepts and lemmas will be useful in proving our main results as follows:

Let H be a Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Then the following inequality holds

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

Definition 2.1. An operator $T : H \to H$ is said to be:

(i) nonexpansive if
$$\|Tx - Ty\| \leq \|x - z\|, \quad \forall x \in H;$$

(ii) quasi-nonexpansive if
$$\|Tx - z\| \leq \|x - z\|, \quad \forall x \in H, \quad \forall z \in Fix(T);$$

(iii) directed if
$$\langle z - Tx, x - Tx\rangle \leq 0, \quad \forall x, y \in H, \quad \forall z \in Fix(T); \tag{2.2}$$

(iv) $\tau-$demicontractive with $\tau < 1$ if
$$\|Tx - z\|^2 \leq \|x - z\|^2 + \tau\|x - Tx\|^2, \quad \forall x, y \in H, \quad \forall z \in Fix(T).$$

It is easy to obtain (2.2) is equivalent to

$$\|z - Tx\|^2 + \|x - Tx\|^2 - \|x - z\|^2 \leq 0, \quad \forall x, y \in H, \quad \forall z \in Fix(T).$$

Remark 2.2. The classes of k-demicontrative operators, directed operators, quasi-nonexpansive operators and nonexpansive operators are closely related. By Definition 2.1, we easily obtain the following conclusion.

(i) The nonexpansive operator is quasi-nonexpansive operator.
(ii) The quasi-nonexpansive operator is $0-$demicontrative operator.
(iii) The directed operator is $-1-$demicontrative operator.

Definition 2.3. Let $T : H \to H$ be an operator, then $I - T$ is said to be demiclosed at zero, if for any $\{x_n\}$ in H, the following implication holds

$$\left.\begin{array}{r} x_n \rightharpoonup x \\ (I-T)x_n \to 0 \end{array}\right\} \Rightarrow x = Tx.$$

Note that the nonexpansive mappings are demiclosed at zero [11].

Definition 2.4. Let C be a nonempty closed convex subset of a Hilbert space H, the metric (nearest point) projection P_C from H to C is defined as follows.

Given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

Lemma 2.5 ([19]). *Let C be a nonempty closed convex subset of a Hilbert space H, P_C is a nonexpansive mapping from H onto C and is characterized as follows.*

Given $x \in H$, there holds the inequality

$$\langle x - P_C x, x - P_C x\rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.6 ([22]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup\limits_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.7 ([9]). *Let $A : H_1 \to H_2$ be a bounded linear operator and $T : H_2 \to H_2$ a $\tau-$demicontractive operator with $\tau < 1$. If $A^{-1}Fix(T) \neq$, then*

(a) $(I - T)Ax - 0 \leftrightarrow A^*(I - T)Ax - 0$, $\forall x \in H_1$.
(b) *In addition, for $z \in A^{-1}Fix(T)$*

$$\|x - \rho A^*(I - T)Ax - z\|^2 \leq \|x - z\|^2 - \frac{(1-\tau)^2}{4} \frac{\|(I - T)Ax\|^4}{\|A^*(I - T)Ax\|^2}, \tag{2.3}$$

where $x \in H_1$, $Ax \neq T(Ax)$ and

$$\rho := \frac{1-\tau}{2} \frac{\|(I - T)Ax\|^2}{\|A^*(I - T)Ax\|^2}.$$

Lemma 2.8 (Maingé [13]). *Let $U : H_1 \to H_1$ be a $k-$demicontractive operator with $k < 1$. Denote $U_\lambda := (1-\lambda)I + \lambda U$ for $\lambda \in (0, 1-k)$. Then for any $x \in H_1$ and $z \in Fix(U)$,*

$$\|U_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda(1 - k - \lambda)\|x - Ux\|^2. \tag{2.4}$$

3. Main results

Algorithm 3.1. Choose an initial guess $x_0 \in H_1$, arbitrarily. Let f be a fixed contraction on $Fix(U)$ with coefficient α, $\lambda \in (0, 1-\tau)$. Assume that the n-th iterate x_n has been constructed. Then the $(n + 1)$-th iterate via the following formula

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_\lambda(x_n - \rho_n A^*(I - T)Ax_n), \quad n \geq 0, \tag{3.1}$$

where A^* is the adjoint of bounded linear operator A and the step size ρ_n is chosen in the following way.

$$\rho_n = \begin{cases} \frac{(1-\tau)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, & Ax_n \neq T(Ax_n), \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

Theorem 3.2. *Assume the SCFP (1.2) is consistent ($S \neq \emptyset$). If $\alpha_n \in (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by explicit algorithm (3.1) converges strongly to a point $\hat{x} \in S$, and the $\hat{x} = P_S f(\hat{x})$, i.e., $\hat{x}$ satisfies the following variational inequality:*

$$\langle \hat{x} - f(\hat{x}), \hat{x} - z\rangle \leq 0, \quad \forall z \in S. \tag{3.3}$$

Proof. The proof is divided into three steps.

Step 1. We show that the sequence $\{x_n\}$ is bounded.

Denote $y_n := x_n - \rho_n A^*(I-T)Ax_n$, take $z \in S$, it follows from (3.1) that

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(U_\lambda y_n - z)\| \\ &\leq \alpha_n \|f(x_n) - f(z)\| + (1-\alpha_n)\|U_\lambda y_n - z\| + \alpha_n\|f(z) - z\| \\ &\leq \alpha\alpha_n\|x_n - z\| + (1-\alpha_n)\|U_\lambda y_n - z\| + \alpha_n\|f(z) - z\|. \end{aligned} \tag{3.4}$$

• If $\rho_n \neq 0$, from (2.3) and (2.4), we can get

$$\begin{aligned} \|U_\lambda y_n - z\|^2 &\leq \|y_n - z\|^2 - \lambda(1-\lambda-k)\|y_n - Uy_n\|^2 \\ &= \|x_n - \rho_n A^*(I-T)Ax_n - z\|^2 - \lambda(1-\lambda-k)\|y_n - Uy_n\|^2 \\ &\leq \|x_n - z\|^2 - \frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax_n\|^4}{\|A^*(I-T)Ax_n\|^2} \\ &\quad - \lambda(1-\lambda-k)\|y_n - Uy_n\|^2. \end{aligned}$$

Thus, we get

$$\|U_\lambda y_n - z\| \leq \|x_n - z\|. \tag{3.5}$$

By applying (3.5) to (3.4), we obtain

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha\alpha_n\|x_n - z\| + (1-\alpha_n)\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &\leq [1-(1-\alpha)\alpha_n]\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &\leq \max\{\|x_n - z\|, \ \frac{1}{1-\alpha}\|f(z) - z\|\}. \end{aligned} \tag{3.6}$$

By induction, we get

$$\|x_n - z\| \leq \max\{\|x_0 - z\|, \ \frac{1}{1-\alpha}\|f(z) - z\|\}. \tag{3.7}$$

Thus, the sequence $\{x_n\}$ is bounded, so is $\{f(x_n)\}$.

• If $\rho_n = 0$, then $y_n = x_n$. From (2.4), we can get

$$\|U_\lambda x_n - z\| \leq \|x_n - z\|. \tag{3.8}$$

By applying the inequality (3.8) to (3.4), the process is similar to (3.6), we can get (3.7), i.e., the sequence $\{x_n\}$ is bounded, so is $\{f(x_n)\}$.

Step 2. We show that the following inequality holds.

For a solution $\hat{x}$ of the variational inequality (3.3),

$$\|x_{n+1} - \hat{x}\| \leq (1-\alpha_n)\|x_n - \hat{x}\|^2 + 2\alpha_n\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle. \tag{3.9}$$

• If $\rho_n = 0$, from (2.1) and (2.4), we have

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &\leq (1-\alpha_n)\|U_\lambda x_n - \hat{x}\|^2 + 2\alpha_n\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle \\ &\leq (1-\alpha_n)[\|x_n - \hat{x}\|^2 - \lambda(1-k-\lambda)\|x_n - Ux_n\|^2] \\ &\quad + 2\alpha_n\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle. \end{aligned} \tag{3.10}$$

So,

$$\|x_{n+1} - \hat{x}\|^2 \leq (1-\alpha_n)\|x_n - \hat{x}\|^2 + 2\alpha_n\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle.$$

Thus, the inequality (3.9) is obtained.

• If $\rho_n \neq 0$, from (2.1) and (2.3), we have

$$\begin{aligned}\|x_{n+1}-\hat{x}\|^2 &\leq (1-\alpha_n)\|U_\lambda y_n-\hat{x}\|^2+2\alpha_n\langle f(x_n)-\hat{x},x_{n+1}-\hat{x}\rangle\\ &\leq (1-\alpha_n)[\|x_n-\hat{x}\|^2-\frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax_n\|^4}{\|A^*(I-T)Ax_n\|^2}]\\ &\quad-\lambda(1-\alpha_n)(1-k-\lambda)\|y_n-Uy_n\|^2\\ &\quad+2\alpha_n\langle f(x_n)-\hat{x},x_{n+1}-\hat{x}\rangle.\end{aligned} \tag{3.11}$$

So,

$$\|x_{n+1}-\hat{x}\|^2 \leq (1-\alpha_n)\|x_n-\hat{x}\|^2+2\alpha_n\langle f(x_n)-\hat{x},x_{n+1}-\hat{x}\rangle.$$

Thus, the inequality (3.9) is obtained.

Step 3. We show that $x_n \to \hat{x}$.

This step of proof is divided into two cases. Denote $s_n := \|x_n-\hat{x}\|^2$.

Case 1. Assume that there is a positive integer n_0 such that the sequence $\{s_n\}$ is decreasing for all $n \geq n_0$, then the sequence $\{s_n\}$ is convergent by the monotonic bounded principle. First, we show that

$$\limsup_{n\to\infty}\langle f(\hat{x})-\hat{x},x_n-\hat{x}\rangle \leq 0. \tag{3.12}$$

• If $\rho_n = 0$, from (3.10) and the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we get

$$\lambda(1-k-\lambda)\|x_n-Ux_n\|^2 \leq s_n-s_{n+1}+\alpha_n K,$$

where K is a nonnegative real constant such that $K \geq \sup_{n\in N}\{2\langle f(x_n)-\hat{x},x_{n+1}-\hat{x}\rangle\}$.

Since the sequence $\{s_n\}$ is convergent, then

$$\|x_n-Ux_n\| \to 0, \quad \text{as} \quad n\to\infty. \tag{3.13}$$

From (3.2), the following holds clearly

$$\|(I-T)Ax_n\| \to 0, \quad \text{as} \quad n\to\infty. \tag{3.14}$$

Based on the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_{n_k} \rightharpoonup q$ such that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(\hat{x})-\hat{x},x_n-\hat{x}\rangle &= \lim_{k\to\infty}\langle f(\hat{x})-\hat{x},x_{n_k}-\hat{x}\rangle\\ &= \langle f(\hat{x})-\hat{x},q-\hat{x}\rangle.\end{aligned}$$

From (3.13) and the demiclosedness of $I-U$ at zero, we have

$$q \in Fix(U). \tag{3.15}$$

Since A is bounded linear operator, then A is of weak continuity. Thus

$$x_{n_k} \rightharpoonup q \Rightarrow Ax_{n_k} \rightharpoonup Aq, \quad \text{as} \quad k\to\infty.$$

From (3.14) and the demiclosedness of $I-T$ at zero, then

$$Aq \in Fix(T). \tag{3.16}$$

So, $q \in S$ by (3.15) and (3.16). Hence, it follows from (3.3) that

$$\limsup_{n\to\infty}\langle f(\hat{x})-\hat{x},x_n-\hat{x}\rangle = \langle f(\hat{x})-\hat{x},q-\hat{x}\rangle \leq 0.$$

• If $\rho_n \neq 0$, from (3.11) and the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we get

$$\lambda(1-k-\lambda)\|y_n - Uy_n\|^2 + \frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax_n\|^4}{\|A^*(I-T)Ax_n\|^2} \leq s_n - s_{n+1} + \alpha_n L,$$

where L is a nonnegative real constant such that $L \geq \sup_{n\in N}\{2\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle\}$.

So, we have

$$0 \leq \lambda(1-k-\lambda)\|y_n - Uy_n\|^2 \leq s_n - s_{n+1} + \alpha_n L,$$

and

$$0 \leq \frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax_n\|^4}{\|A^*(I-T)Ax_n\|^2} \leq s_n - s_{n+1} + \alpha_n L.$$

It follows from $\{s_n\}$ is convergent that,

$$\|y_n - Uy_n\| \to 0, \quad \text{as} \quad n \to \infty, \tag{3.17}$$

$$\frac{\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|} \to 0, \quad \text{as} \quad n \to \infty. \tag{3.18}$$

Moreover,

$$\begin{aligned}
\|(I-T)Ax_n\| &= \|A\| \cdot \frac{\|(I-T)Ax_n\|}{\|A\|} \\
&= \|A\| \cdot \|(I-T)Ax_n\| \frac{\|(I-T)Ax_n\|}{\|A\|\|(I-T)Ax_n\|} \\
&\leq \|A\| \cdot \|(I-T)Ax_n\| \frac{\|(I-T)Ax_n\|}{\|A^*(I-T)Ax_n\|} \\
&= \|A\| \frac{\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|}.
\end{aligned}$$

Hence,

$$\|(I-T)Ax_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{3.19}$$

So,

$$\begin{aligned}
\|x_n - y_n\| &= \rho_n \|A^*(I-T)Ax_n\| \\
&= \frac{1-\tau}{2}\frac{\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|} \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.20}$$

For $x_n \to q$, then $y_n \to q$ from (3.20).

From (3.17) and the demiclosedness of $I - U$ at zero, we have

$$q \in Fix(U). \tag{3.21}$$

From (3.19) and the demiclosedness of $I - T$ at zero, we have

$$Aq \in Fix(T). \tag{3.22}$$

So, $q \in S$ by (3.21) and (3.22).

Hence, it follows from the variational inequality (3.3) that

$$\limsup_{n\to\infty}\langle f(\hat{x}) - \hat{x}, x_n - \hat{x}\rangle = \langle f(\hat{x}) - \hat{x}, q - \hat{x}\rangle \leq 0.$$

Second, we show that

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.23}$$

For $x \in H_1$, we get $U_\lambda x - x = \lambda(Ux - x)$ by $U_\lambda := (1-\lambda)I + \lambda U$.

• If $\rho_n = 0$, then

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \alpha_n \|f(x_n) - x_n\| + (1-\alpha_n)\|x_n - U_\lambda x_n\| \\ &\le \alpha_n \|f(x_n) - x_n\| + \lambda \|x_n - Ux_n\|.\end{aligned}$$

By (3.13) and the assumption $\lim\limits_{n\to\infty} \alpha_n = 0$, (3.23) is obtained.

• If $\rho_n \neq 0$, then

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \alpha_n \|f(x_n) - x_n\| + (1-\alpha_n)\|x_n - U_\lambda y_n\| \\ &\le \alpha_n \|f(x_n) - x_n\| + \|x_n - y_n\| + \|y_n - U_\lambda y_n\| \\ &= \alpha_n \|f(x_n) - x_n\| + \|x_n - y_n\| + \lambda\|y_n - Uy_n\|.\end{aligned}$$

Combining (3.17) and (3.20), implies that (3.23) holds.

Third, we show that $x_n \to \hat{x}$. By combining (3.12) and (3.23), we get

$$\limsup_{n\to\infty} \langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x}\rangle \le 0. \tag{3.24}$$

By applying Lemma 2.6 to the (3.9), and with the assumption of $\{\alpha_n\}$ and (3.24), $x_n \to \hat{x}$ can be easily concluded.

Case 2. Assume that there is not a positive integer n_0 such that the sequence $\{s_n\}$ is decreasing for all $n \ge n_0$, that is to say, there is a subsequence $\{s_{k_i}\}$ of $\{s_k\}$ such that $s_{k_i} < s_{k_i+1}$ for all $i \in N$.

By applying Lemma 2.8, we can define a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ as $k \to \infty$ and

$$s_{m_k} \le s_{m_k+1}. \tag{3.25}$$

First, we show that

$$\limsup_{n\to\infty} \langle f(\hat{x}) - \hat{x}, x_n - \hat{x}\rangle \le 0. \tag{3.26}$$

• If $\rho_{m_k} = 0$, from (3.10), (3.25) and the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we get

$$\begin{aligned}\lambda(1-k-\lambda)\|x_{m_k} - Ux_{m_k}\|^2 &\le s_{m_k} - s_{m_k+1} + \alpha_{m_k} K \\ &\le \alpha_{m_k} K,\end{aligned}$$

where K is a nonnegative real constant such that $K \ge \sup_{m_k \in N}\{2\langle f(x_{m_k}) - \hat{x}, x_{m_k+1} - \hat{x}\rangle\}$. So

$$\|x_{m_k} - Ux_{m_k}\| \to 0, \quad \text{as} \quad k \to \infty. \tag{3.27}$$

From (3.2), then the following holds clearly.

$$\|(I-T)Ax_{m_k}\| \to 0, \quad \text{as} \quad k \to \infty.$$

Based on the boundedness of $\{x_{m_k}\}$, there exists a subsequence $\{x_{m_k(l)}\}$ of $\{x_{m_k}\}$ and $x_{m_k(l)} \rightharpoonup q$ such that

$$\begin{aligned}\limsup_{k\to\infty} \langle f(\hat{x}) - \hat{x}, x_{m_k} - \hat{x}\rangle &= \lim_{l\to\infty} \langle f(\hat{x}) - \hat{x}, x_{m_k(l)} - \hat{x}\rangle \\ &= \langle f(\hat{x}) - \hat{x}, q - \hat{x}\rangle.\end{aligned}$$

So, we have $q \in S$ by using the similar proofs in Case 1. Hence, it follows from (3.3) that

$$\limsup_{n\to\infty} \langle f(\hat{x}) - \hat{x}, x_{m_k} - \hat{x}\rangle = \langle f(\hat{x}) - \hat{x}, q - \hat{x}\rangle \le 0.$$

• If $\rho_{m_k} \neq 0$, from (3.11) and the boundedness of $\{x_{m_k}\}$ and $\{f(x_{m_k})\}$, we get

$$\lambda(1-k-\lambda)\|y_{m_k} - Uy_{m_k}\|^2 + \frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax_{m_k}\|^4}{\|A^*(I-T)Ax_n\|^2} \leq s_{m_k} - s_{m_k+1} + \alpha_{m_k}L,$$

where L is a nonnegative real constant such that $L \geq \sup_{k\in N}\{2\langle f(x_{m_k}) - \hat{x}, x_{m_k+1} - \hat{x}\rangle\}$.

It follows from (3.25) that

$$\begin{aligned} 0 \leq \lambda(1-k-\lambda)\|y_{m_k} - Uy_{m_k}\|^2 &\leq s_{m_k} - s_{m_k+1} + \alpha_{m_k}L \\ &\leq \alpha_{m_k}L, \end{aligned}$$

and

$$\begin{aligned} 0 \leq \frac{(1-\tau)^2}{4}\frac{\|(I-T)Ax_{m_k}\|^4}{\|A^*(I-T)Ax_{m_k}\|^2} &\leq s_{m_k} - s_{m_k+1} + \alpha_{m_k}L \\ &\leq \alpha_{m_k}L. \end{aligned}$$

Thus,

$$\|y_{m_k} - Uy_{m_k}\| \to 0, \quad \text{as} \quad k \to \infty, \tag{3.28}$$

$$\frac{\|(I-T)Ax_{m_k}\|^2}{\|A^*(I-T)Ax_{m_k}\|} \to 0, \quad \text{as} \quad k \to \infty.$$

Moreover,

$$\begin{aligned} \|(I-T)Ax_{m_k}\| &= \|A\| \cdot \frac{\|(I-T)Ax_{m_k}\|}{\|A\|} \\ &= \|A\| \cdot \|(I-T)Ax_{m_k}\| \frac{\|(I-T)Ax_{m_k}\|}{\|A\|\|(I-T)Ax_{m_k}\|} \\ &\leq \|A\| \cdot \|(I-T)Ax_{m_k}\| \frac{\|(I-T)Ax_{m_k}\|}{\|A^*(I-T)Ax_{m_k}\|} \\ &= \|A\| \frac{\|(I-T)Ax_{m_k}\|^2}{\|A^*(I-T)Ax_{m_k}\|}. \end{aligned}$$

Hence,

$$\|(I-T)Ax_{m_k}\| \to 0, \quad \text{as} \quad k \to \infty.$$

So that

$$\begin{aligned} \|x_{m_k} - y_{m_k}\| &= \rho_{m_k}\|A^*(I-T)Ax_{m_k}\| \\ &= \frac{1-\tau}{2}\frac{\|(I-T)Ax_{m_k}\|^2}{\|A^*(I-T)Ax_{m_k}\|} \to 0, \quad \text{as} \quad k \to \infty. \end{aligned} \tag{3.29}$$

For $x_{m_k} \to q$, then $y_{m_k} \to q$ from (3.29).

So, we have $q \in S$ by using the similar proofs in Case 1. Hence, it follows from (3.3) that

$$\limsup_{k\to\infty}\langle f(\hat{x}) - \hat{x}, x_{m_k} - \hat{x}\rangle = \langle f(\hat{x}) - \hat{x}, q - \hat{x}\rangle \leq 0$$

Second, we show that

$$\|x_{m_k+1} - x_{m_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{3.30}$$

For $x \in H_1$, we get $U_\lambda x - x = \lambda(Ux - x)$ by $U_\lambda := (1-\lambda)I + \lambda U$.

• If $\rho_{m_k} = 0$, then

$$\begin{aligned}\|x_{m_k+1} - x_{m_k}\| &\leq \alpha_{m_k}\|f(x_{m_k}) - x_{m_k}\| + (1-\alpha_{m_k})\|x_{m_k} - U_\lambda x_{m_k}\| \\ &\leq \alpha_{m_k}\|f(x_{m_k}) - x_{m_k}\| + \lambda\|x_{m_k} - Ux_{m_k}\|.\end{aligned}$$

By the assumption $\lim_{n\to\infty} \alpha_n = 0$, the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, and (3.27), the (3.30) is obtained.

• If $\rho_{m_k} \neq 0$, then

$$\begin{aligned}\|x_{m_k+1} - x_{m_k}\| &\leq \alpha_{m_k}\|f(x_{m_k}) - x_{m_k}\| + (1-\alpha_{m_k})\|x_{m_k} - U_\lambda y_{m_k}\| \\ &\leq \alpha_{m_k}\|f(x_{m_k}) - x_{m_k}\| + \|x_{m_k} - y_{m_k}\| + \|y_{m_k} - U_\lambda y_{m_k}\| \\ &= \alpha_{m_k}\|f(x_{m_k}) - x_{m_k}\| + \|x_{m_k} - y_{m_k}\| + \lambda\|y_{m_k} - Uy_{m_k}\|.\end{aligned}$$

Combining (3.28) and (3.29), implies that (3.30) holds.

Third, we show that $x_n \to \hat{x}$ as $n \to \infty$. From (3.26) and (3.30), we get

$$\limsup_{n\to\infty}\langle f(\hat{x}) - \hat{x}, x_{m_k+1} - \hat{x}\rangle \leq 0. \tag{3.31}$$

Based on the inequality $s_{m_k} \leq s_{m_k+1}$ for all $k \in N$ and (3.9), we get

$$\alpha_{m_k}s_{m_k+1} + (1-\alpha_{m_k})(s_{m_k+1} - s_{m_k}) \leq 2\alpha_{m_k}\langle f(\hat{x}) - \hat{x}, x_{m_k+1} - \hat{x}\rangle.$$

So,

$$\alpha_{m_k}s_{m_k+1} \leq 2\alpha_{m_k}\langle f(\hat{x}) - \hat{x}, x_{m_k+1} - \hat{x}\rangle,$$

that is,

$$s_{m_k+1} \leq 2\langle f(\hat{x}) - \hat{x}, x_{m_k+1} - \hat{x}\rangle.$$

Take the limit $k \to \infty$, by using (3.31), we obtain

$$s_{m_k+1} \to 0 \quad \text{as} \quad k \to \infty.$$

Thus,

$$s_k \to 0 \quad \text{as} \quad k \to \infty,$$

because $s_k \leq s_{m_k+1}$. The proof is completed. □

Remark 3.3. The main result of Theorem 3.2 is an extension of Theorem 4.1 of [1]. If we take $f(x_n) = u$ in (3.1), where $u \in H_1$ is arbitrary but fixed, this special case will be Theorem 4.1 of [1].

4. Some special cases

In this section, we consider some special cases of Theorem 3.2, base on the relations of k-demicontrative operators, directed operators, quasi-nonexpansive operators. The details can be seen in Remark 3.3. Then, the following corollaries are obtained easily.

• **Case 1:** Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive operators, $I - U$ and $I - T$ be demiclosed at zero.

Corollary 4.1. *Assume the SCFP* (1.2) *is consistent* ($S \neq \emptyset$). *If* $\alpha_n \in (0,1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Let* $\{x_n\}$ *be given by the explicit algorithm* (3.1), *and in the algorithm* (3.1), $\lambda \in (0,1)$ *and*

$$\rho_n = \begin{cases} \frac{\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, & Ax_n \neq T(Ax_n), \\ 0, & \textit{otherwise}. \end{cases}$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $\hat{x} \in S$, *and the* $\hat{x} = P_S f(\hat{x})$, *i.e.,* $\hat{x}$ *satisfies the following variational inequality* (3.3).

• **Case 2:** Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be directed operators, $I - U$ and $I - T$ be demiclosed at zero.

Corollary 4.2. *Assume the SCFP* (1.2) *is consistent* ($S \neq \emptyset$). *If* $\alpha_n \in (0, 1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Let* $\{x_n\}$ *be given by the explicit algorithm* (3.1), *and in the algorithm* (3.1), $\lambda \in (0, 2)$ *and*

$$\rho_n = \begin{cases} \frac{\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|^2}, & Ax_n \neq T(Ax_n) \\ 0, & \textit{otherwise.} \end{cases}$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $\hat{x} \in S$, *and the* $\hat{x} = P_S f(\hat{x})$, *i.e.,* $\hat{x}$ *satisfies the following variational inequality* (3.3).

• **Case 3:** Let $U : H_1 \to H_1$ be a directed operator, $T : H_2 \to H_2$ a quasi-nonexpansive operator, $I - U$ and $I - T$ be demiclosed at zero.

Corollary 4.3. *Assume the SCFP* (1.2) *is consistent* ($S \neq \emptyset$). *If* $\alpha_n \in (0, 1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Let* $\{x_n\}$ *be given by the explicit algorithm* (3.1), *and in the algorithm* (3.1), $\lambda \in (0, 1)$ *and*

$$\rho_n = \begin{cases} \frac{\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, & Ax_n \neq T(Ax_n), \\ 0, & \textit{otherwise.} \end{cases}$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $\hat{x} \in S$, *and the* $\hat{x} = P_S f(\hat{x})$, *i.e.,* $\hat{x}$ *satisfies the following variational inequality* (3.3).

• **Case 4:** Let $U : H_1 \to H_1$ be a directed operator, $T : H_2 \to H_2$ a $\tau-$demicontractive operator, $I - U$ and $I - T$ be demiclosed at zero.

Corollary 4.4. *Assume the SCFP* (1.2) *is consistent* ($S \neq \emptyset$). *If* $\alpha_n \in (0, 1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *then the sequence* $\{x_n\}$ *generated by explicit algorithm* (3.1) *converges strongly to a point* $\hat{x} \in S$, *and the* $\hat{x} = P_S f(\hat{x})$, *i.e.,* $\hat{x}$ *satisfies the following variational inequality* (3.3).

• **Case 5:** Let $U : H_1 \to H_1$ be a quasi-nonexpansive operator, $T : H_2 \to H_2$ a $\tau-$demicontractive operator, $I - U$ and $I - T$ be demiclosed at zero.

Corollary 4.5. *Assume the SCFP* (1.2) *is consistent* ($S \neq \emptyset$). *If* $\alpha_n \in (0, 1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *then the sequence* $\{x_n\}$ *generated by explicit algorithm* (3.1) *converges strongly to a point* $\hat{x} \in S$, *and the* $\hat{x} = P_S f(\hat{x})$, *i.e.,* $\hat{x}$ *satisfies the following variational inequality* (3.3).

5. Conclusions

In this paper, we proposed a novel explicit viscosity iteration algorithm (3.1) and we proved the sequence $\{x_n\}$ converges strongly to a solution of the split common fixed point problems (1.2). This main result is an extension of Theorem 4.1 of [1]. The research highlights of this paper are novel explicit algorithms and strong convergence results. The research of this aspect for SCFP can further continue.

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