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On the convergence and data dependence results for multistep Picard-Mann iteration process in the class of contractive-like operators

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Abstract

In this paper, we introduce a new iteration process and prove the convergence of this iteration process to a fixed point of contractive-like operators. We also present a data dependence result for such mappings. Our results unify and extend various results in the existing literature.

Keywords: Fixed point, contractive-like operators, convergence and data dependence.
2010 MSC: 47H10, 54H25.

1. Introduction

Fixed point theory is one of the most significant tool of modern mathematics. This deals with the conditions which guarantee that a mapping T of a set X into itself admits one or more fixed points, that is, points x of X which solve an operator equation $x = Tx$. Fixed point theory serves as an essential tool for solving problems arising in various branches of mathematical analysis. In particular, it has deep roots in nonlinear functional analysis. For instance, split feasibility problems, variational inequality problems, nonlinear optimization problems, equilibrium problems, complementarity problems, selection and matching problems, and problems of proving an existence of solution of integral and differential equations is a partial list of those problems that fall into the category of solving a fixed point problem. These problems can be modeled by the equation $Tx = x$, where T is a nonlinear operator defined on a set equipped with some topological or order structure.

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One of the basic and the most widely applied fixed point theorem in all of analysis is "Banach (or Banach- Cassioppoli) contraction principle" due to Banach [2]. It states that if (X, d) is a complete metric space and $T : X \to X$ satisfies

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ with $k \in (0, 1)$, then T has a unique fixed point. The basic idea of this principle rest in the use of successive approximations to establish the existence and uniqueness of solution of an operator equation $T(x) = x$, particularly it can be employed to prove the existence of solution of differential or integral equations.

Different iterative procedures have been used to approximate the solution of fixed point problems. For instance, the sequence of Picard [17] iterates $\{T^n x\}$ where x is an initial guess, converges to the fixed point of a mapping T satisfying certain contractive condition. This procedure may fail to converge for some important classes of nonlinear mappings such as nonexpansive mappings, so was the justification of Mann iteration [13] procedure (see pp.8, Example 1.8 in [4]). Chidume and Mutangadura [5] showed that the Mann iterative sequence [13] fails to converge for Lipschitzian pseudocontractive mappings whereas the Ishikawa iterative procedure [9] works. It is worth mentioning that in many cases, Mann as well as Ishikawa iterative procedures converge to some fixed point of the mapping T . But, this does not hold in general. Recently, Rhoades and Soltuz ([18–20]) proved that Mann and Ishikawa iteration procedures are equivalent for several classes of mappings.

It is an important subject for research to determine whether an iteration procedure converges to the fixed point of a mapping. For the results dealing with the convergence of various iteration methods, we refer to ([4, 9, 12–14, 23]).

The aim of this paper is to introduce a new iteration procedure and to prove some convergence results for contractive-like operators. We also show the equivalence among convergence of iteration methods.

2. Preliminaries

In the sequel the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real numbers and the set of all positive integers. Let E be a nonempty closed convex subset of a Banach space X, and $T : E \longrightarrow E$ a mapping. We denote the set of all fixed points of T by $F(T) := \{p \in E : p = Tp\}$.

Let us first recall the following definitions and iterative procedures:

Suppose that there exist real numbers a, b, c satisfying $0 < a < 1, 0 < b, c < 1/2$ such that, for each pair $x, y \in E$, at least one of the following is true:

$$\begin{cases} (z_1) & \|Tx - Ty\| \leq a \|x - y\|, \\ (z_2) & \|Tx - Ty\| \leq b (\|x - Tx\| + \|y - Ty\|), \\ (z_3) & \|Tx - Ty\| \leq c (\|x - Ty\| + \|y - Tx\|). \end{cases} \tag{2.1}$$

Such a mapping is called a Zamfirescu mapping. Zamfirescu [26] obtained an important generalization of Banach fixed point theorem using Zamfirescu mapping.

It was shown in [3], the contractive condition (2.1) gives

$$\begin{cases} (b_1) & \|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \text{ if one uses } (z_2), \text{ and} \\ (b_2) & \|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Ty\| \text{ if one uses } (z_3) \end{cases} \tag{2.2}$$

for all $x, y \in E$, where $\delta := \max \left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\}, \delta \in [0, 1)$.

A mapping satisfying condition (b_1) or (b_2) is called a quasi-contractive mapping. This class of mappings is more general than the class of Zamfirescu mappings.

Extending the above class of mappings, Osilike and Udomene [15] considered a mapping T satisfying the following contractive condition:

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|x - Tx\| \tag{2.3}$$

for all $x, y \in E$, where $L \geq 0$ and $\delta \in [0,1)$.

A treatment for a similar kind of operators can be found in [1] and [11].

Further in this direction, Imoru and Olantiwo [8] gave the following definition:

Definition 2.1. A self mapping T on E is called a contractive-like mapping if there exists a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that the following holds

$$\|Tx - Ty\| \leq \delta \|x - y\| + \varphi(\|x - Tx\|) \tag{2.4}$$

for each $x, y \in E$.

Remark 2.2 ([4]). It is known that condition (2.4) alone does not ensures that T has a fixed point. But if T satisfying (2.4) has a fixed point, it is certainly unique.

Rhoades and Şoltuz [20] introduced a multistep iterative algorithm as follows:

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1, \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \\ y_n^{r-1} = (1-\beta_n^{r-1})x_n + \beta_n^{r-1} T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\beta_n^i\}, i = 1, 2, ...r-2, r \geq 2$ are real sequences in $[0,1)$ satisfying certain conditions.

Recently, Yildirim and Ozdemir [25] proved some convergence result by using the following multistep-Mann iteration process: For an arbitrary fixed order $r \geq 2$,

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}, \\ y_{n+r-2} = (1-\alpha_{2n})\, y_{n+r-3} + \alpha_{2n} T y_{n+r-3}, \\ \vdots \\ y_{n+1} = \left(1-\alpha_{(r-1)n}\right) y_n + \alpha_{(r-1)n} T y_n, \\ y_n = (1-\alpha_{rn})\, x_n + \alpha_{rn} T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.6}$$

or, in short,

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}, \\ y_{n+r-i} = (1-\alpha_{in})\, y_{n+r-(i+1)} + \alpha_{in} T y_{n+r-(i+1)}, \\ y_n = (1-\alpha_{rn})\, x_n + \alpha_{rn} T x_n, \ n \in \mathbb{N}, \end{cases} \tag{2.7}$$

where $\{\alpha_{1n}\}$ and $\{\alpha_{in}\}$, $i = 2, ...r$, are real sequence in $[0,1)$.

Remark 2.3. If we take $r = 2$ and $r = 3$ in (2.7), respectively, we obtain the two-step iteration procedure given in [23] and SP iteration method in [16].

Şoltuz and Grosan [22] proved that the Ishikawa iteration [9] converges to the fixed point of T, where $T : E \to E$ is a mapping satisfying condition (2.4). In 2007, Şoltuz [21] proved that the Mann [13], Ishikawa [9], Noor [14] and multistep (2.5) iterations are equivalent for quasi-contractive mappings in a normed space. In 2011, Chugh and Kumar [6] showed that the Picard [17], Mann [13], Ishikawa [9], new two step [23], Noor [14] and SP [16] iterations are equivalent for quasi-contractive mappings in a Banach space. In 2013, Karakaya et al. [10] proved the data dependence results for the multistep (2.5) and CR [7] iteration processes for the class of contractive-like operators satisfying (2.4).

In 2013, Khan [12] introduced Picard-Mann hybrid iterative process as follows:

$$\begin{cases} x_0 \in E, \\ x_{n+1} = T y_n, \\ y_n = (1-\alpha_n)\, x_n + \alpha_n T x_n, n \in \mathbb{N}. \end{cases}$$

Inspired by the work of Khan [12], we introduce the following new iterative algorithm. For an arbitrary fixed order $r \geq 2$,

$$\begin{cases} x_0 \in E, \\ x_{n+1} = T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}\right], \\ y_{n+r-2} = T\left[(1-\alpha_{2n})\, y_{n+r-3} + \alpha_{2n} T y_{n+r-3}\right], \\ \vdots \\ y_{n+1} = T\left[\left(1-\alpha_{(r-1)n}\right) y_n + \alpha_{(r-1)n} T y_n\right], \\ y_n = T\left[(1-\alpha_{rn})\, x_n + \alpha_{rn} T x_n\right], \quad n \in \mathbb{N}, \end{cases} \tag{2.8}$$

or, in short,

$$\begin{cases} x_0 \in E, \\ x_{n+1} = T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}\right], \\ y_{n+r-i} = T\left[(1-\alpha_{in})\, y_{n+r-(i+1)} + \alpha_{in} T y_{n+r-(i+1)}\right], \\ y_n = T\left[(1-\alpha_{rn})\, x_n + \alpha_{rn} T x_n\right], \ n \in \mathbb{N}, \end{cases} \tag{2.9}$$

where $\{\alpha_{1n}\} \subset [0,1)\,, \sum_{n=0}^{\infty} \alpha_{1n} = \infty$ and $\{\alpha_{in}\} \subset [0,1)$, $i = 2, ...r$.

Following definitions and lemmas will be needed in proving our main results.

Definition 2.4 ([4])**.** Let T, $S : X \to X$ be two operators. We say that S is an approximate operator for T if, for some $\varepsilon > 0$, and any $x \in X$, the following hold:

$$\|Tx - Sx\| \leq \varepsilon.$$

Lemma 2.5 ([24])**.** *Let $\{a_n\}$ and $\{\rho_n\}$ be nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1-\mu_n)\, a_n + \rho_n,$$

where $\mu_n \in (0,1)$ for all $n \geq n_0$, $\sum_{n=0}^{\infty} \mu_n = \infty$, and $\rho_n = o\,(\mu_n)$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([22])**.** *Let $\{a_n\}$ be a nonnegative real sequence and there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ satisfying the following condition:*

$$a_{n+1} \leq (1-\mu_n)\, a_n + \mu_n \eta_n,$$

where $\mu_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \mu_n = \infty$, and $\eta_n \geq 0$ $\forall n \in \mathbb{N}$. Then the following inequality holds:

$$0 \leq \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup a_n \eta_n.$$

3. Main Results

Theorem 3.1. *Let $T : E \to E$ be a mapping satisfying (2.4) with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (2.8), where $\{\alpha_{1n}\}$ and $\{\alpha_{in}\} \subset [0,1)$ for all $i = 2, ...r$ satisfy the condition $\sum_{k=0}^{n} \alpha_{1k} = \infty$. Then the iterative sequence $\{x_n\}$ converges to a unique fixed point of T.*

Proof. Let $p \in F(T)$. We now show that $x_n \to p$ as $n \to \infty$. From (2.4) and (2.7), we have

$$\begin{aligned} \|y_n - p\| &= \|T\left[(1-\alpha_{rn})\, x_n + \alpha_{rn} T x_n\right] - p\| \\ &\leq \delta \left\|(1-\alpha_{rn})\, x_n + \alpha_{rn} T x_n - p\right\| \\ &\leq \delta \left[(1-\alpha_{rn}) \|x_n - p\| + \alpha_{rn} \|T x_n - p\|\right] \\ &\leq \delta \left[(1-\alpha_{rn}) \|x_n - p\| + \alpha_{rn} \delta \|x_n - p\| + \alpha_{rn} \varphi\left(\|p - Tp\|\right)\right] \\ &= \delta \left[1 - \alpha_{rn}(1-\delta)\right] \|x_n - p\| + \delta \alpha_{rn} \varphi\left(\|p - Tp\|\right) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\|y_{n+1}-p\| &= \left\|T\left[\left(1-\alpha_{(r-1)n}\right)y_n+\alpha_{(r-1)n}Ty_n\right]-p\right\| \\ &\le\delta\left\|\left(1-\alpha_{(r-1)n}\right)y_n+\alpha_{(r-1)n}Ty_n-p\right\| \\ &\le\delta\left[\left(1-\alpha_{(r-1)n}\right)\|y_n-p\|+\alpha_{(r-1)n}\|Ty_n-p\|\right] \\ &\le\delta\left[\left(1-\alpha_{(r-1)n}\right)\|y_n-p\|+\alpha_{(r-1)n}\delta\|Ty_n-p\|+\alpha_{(r-1)n}\varphi\left(\|p-Tp\|\right)\right] \\ &=\delta\left[1-\alpha_{(r-1)n}(1-\delta)\right]\|y_n-p\|+\delta\alpha_{(r-1)n}\varphi\left(\|p-Tp\|\right).\end{aligned} \tag{3.2}$$

Similarly, we have

$$\begin{aligned}\|y_{n+r-2}-p\| &= \|T\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]-p\| \\ &\le\delta\|\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}-p\| \\ &\le\delta\left[\left(1-\alpha_{2n}\right)\|y_{n+r-3}-p\|+\alpha_{2n}\|Ty_{n+r-3}-p\|\right] \\ &\le\delta\left[\left(1-\alpha_{2n}\right)\|y_{n+r-3}-p\|+\alpha_{2n}\delta\|y_{n+r-3}-p\|+\alpha_{2n}\varphi\left(\|p-Tp\|\right)\right] \\ &=\delta\left[1-\alpha_{2n}(1-\delta)\right]\|y_{n+r-3}-p\|+\delta\alpha_{2n}\varphi\left(\|p-Tp\|\right)\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\|x_{n+1}-p\| &= \|T\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]-p\| \\ &\le\delta\|\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}-p\| \\ &\le\delta\left[\left(1-\alpha_{1n}\right)\|y_{n+r-2}-p\|+\alpha_{1n}\|Ty_{n+r-2}-p\|\right] \\ &\le\delta\left[\left(1-\alpha_{1n}\right)\|y_{n+r-2}-p\|+\alpha_{1n}\delta\|x_n-p\|+\alpha_{1n}\varphi\left(\|p-Tp\|\right)\right] \\ &=\delta\left[1-\alpha_{1n}(1-\delta)\right]\|y_{n+r-2}-p\|+\delta\alpha_{1n}\varphi\left(\|p-Tp\|\right).\end{aligned} \tag{3.4}$$

By (3.1), (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}\|x_{n+1}-p\| &\le\delta\left[1-\alpha_{1n}(1-\delta)\right]\|y_{n+r-2}-p\|+\delta\alpha_{1n}\varphi\left(\|p-Tp\|\right) \\ &\le\delta\left[1-\alpha_{1n}(1-\delta)\right]\left[\delta\left[1-\alpha_{2n}(1-\delta)\right]\|y_{n+r-3}-p\|+\delta\alpha_{2n}\varphi\left(\|p-Tp\|\right)\right] \\ &\quad+\delta\alpha_{1n}\varphi\left(\|p-Tp\|\right) \\ &=\delta^2\left[1-\alpha_{1n}(1-\delta)\right]\left[1-\alpha_{2n}(1-\delta)\right]\|y_{n+r-3}-p\| \\ &\quad+\delta^2\left[1-\alpha_{1n}(1-\delta)\right]\alpha_{2n}\varphi\left(\|p-Tp\|\right)+\delta\alpha_{1n}\varphi\left(\|p-Tp\|\right) \\ &\le\delta^2\left[1-\alpha_{1n}(1-\delta)\right]\left[1-\alpha_{2n}(1-\delta)\right] \\ &\quad\left[\delta\left[1-\alpha_{3n}(1-\delta)\right]\|y_{n+r-4}-p\|+\delta\alpha_{3n}\varphi\left(\|p-Tp\|\right)\right] \\ &\quad+\delta^2\left[1-\alpha_{1n}(1-\delta)\right]\alpha_{2n}\varphi\left(\|p-Tp\|\right)+\delta\alpha_{1n}\varphi\left(\|p-Tp\|\right) \\ &\quad\vdots \\ &\le\delta^r\left[1-\alpha_{1n}(1-\delta)\right]\left[1-\alpha_{2n}(1-\delta)\right]\cdots\left[1-\alpha_{rn}(1-\delta)\right]\|x_n-p\| \\ &\quad+\delta^r\left[1-\alpha_{1n}(1-\delta)\right]\left[1-\alpha_{2n}(1-\delta)\right]\cdots\left[1-\alpha_{(r-1)n}(1-\delta)\right] \\ &\quad\alpha_{rn}\varphi\left(\|p-Tp\|\right)+\cdots+\delta^2\left[1-\alpha_{1n}(1-\delta)\right]\alpha_{2n}\varphi\left(\|p-Tp\|\right) \\ &\quad+\delta\alpha_{1n}\varphi\left(\|p-Tp\|\right).\end{aligned} \tag{3.5}$$

As $\varphi\left(\|p-Tp\|\right)=0$, so (3.5) becomes

$$\|x_{n+1}-p\|\le\delta^r\left[1-\alpha_{1n}(1-\delta)\right]\left[1-\alpha_{2n}(1-\delta)\right]\cdots\left[1-\alpha_{rn}(1-\delta)\right]\|x_n-p\|. \tag{3.6}$$

From $\{\alpha_{in}\}\subset[0,1)$ for $i=1,2,...r$, we have

$$\left[1-\alpha_{1n}(1-\delta)\right]\left[1-\alpha_{2n}(1-\delta)\right]\cdots\left[1-\alpha_{rn}(1-\delta)\right]\le\left[1-\alpha_{1n}(1-\delta)\right]. \tag{3.7}$$

From inequalities (3.6), (3.7) and the fact that $\delta \in (0,1)$, we have

$$\|x_{n+1} - p\| \leq [1 - \alpha_{1n}(1-\delta)] \|x_n - p\| . \tag{3.8}$$

Continuing this process in (3.8), we obtain the following inequalities:

$$\begin{aligned} \|x_n - p\| &\leq \left[1 - \alpha_{1(n-1)}(1-\delta)\right] \|x_{n-1} - p\| \\ \|x_{n-1} - p\| &\leq \left[1 - \alpha_{1(n-2)}(1-\delta)\right] \|x_{n-2} - p\| \\ &\vdots \\ \|x_1 - p\| &\leq [1 - \alpha_{10}(1-\delta)] \|x_0 - p\| . \end{aligned}$$

Therefore we have

$$\|x_{n+1} - p\| \leq \prod_{k=0}^{n} [1 - \alpha_{1k}(1-\delta)] \|x_0 - p\| . \tag{3.9}$$

Using the fact $1 - x \leq e^{-x}$ for all $x \in [0,1]$, inequality (3.9) gives

$$\begin{aligned} \|x_{n+1} - p\| &\leq \prod_{k=0}^{n} e^{-\alpha_{1k}(1-\delta)} \|x_0 - p\| \\ &= \|x_0 - p\|^{n+1} \left[e^{(1-\delta)\sum\limits_{k=0}^{n} \alpha_{1k}} \right]^{-1} . \end{aligned} \tag{3.10}$$

The result follows on taking the limit as $n \to \infty$ on both sides of inequality (3.10). $\square$

Theorem 3.2. *Let $T : E \to E$ be a mapping satisfying* (2.4) *with $F(T) \neq \emptyset$. If $u_0 = x_0 \in E$ and $\alpha_{1n} \geq A > 0$ $\forall n \in \mathbb{N}$, then the followings are equivalent:*

(i) The Mann iteration [13] *converges to $p \in F(T)$,*

(ii) The multistep Picard-Mann iteration (2.8) *converges to $p \in F(T)$.*

Proof. We will show that $(i) \Rightarrow (ii)$. Suppose that the Mann iteration [13] converges to p. Using the Mann iteration [13] and multistep Picard-Mann iteration (2.8), we have

$$\begin{aligned} \|u_{n+1} - x_{n+1}\| &= \|(1-\alpha_{1n})\, u_n + \alpha_{1n} T u_n - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}\right]\| \\ &\leq (1-\alpha_{1n}) \|u_n - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}\right]\| \\ &\quad + \alpha_{1n} \|T u_n - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}\right]\| \\ &\leq (1-\alpha_{1n}) \left\{\|u_n - T u_n\| + \|T u_n - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}\right]\|\right\} \\ &\quad + \alpha_{1n} \|T u_n - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}\right]\| \\ &= (1-\alpha_{1n}) \|u_n - T u_n\| + \|T u_n - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}\right]\| \\ &\leq (1-\alpha_{1n}) \|u_n - T u_n\| + \delta \|u_n - (1-\alpha_{1n})\, y_{n+r-2} - \alpha_{1n} T y_{n+r-2}\| \\ &\quad + \varphi\left(\|u_n - T u_n\|\right) \\ &\leq (1-\alpha_{1n}) \|u_n - T u_n\| + \delta(1-\alpha_{1n}) \|u_n - y_{n+r-2}\| \\ &\quad + \delta\alpha_{1n} \|u_n - T y_{n+r-2}\| + \varphi\left(\|u_n - T u_n\|\right) \\ &\leq (1-\alpha_{1n}) \|u_n - T u_n\| + \delta(1-\alpha_{1n}) \|u_n - y_{n+r-2}\| \\ &\quad + \delta\alpha_{1n} \|u_n - T u_n\| + \delta\alpha_{1n} \|T u_n - T y_{n+r-2}\| + \varphi\left(\|u_n - T u_n\|\right) \\ &\leq \left[1 - \alpha_{1n}(1-\delta)\right] \|u_n - T u_n\| + \delta(1-\alpha_{1n}) \|u_n - y_{n+r-2}\| \\ &\quad + \delta^2\alpha_{1n} \|u_n - y_{n+r-2}\| + \delta\alpha_{1n}\varphi\left(\|u_n - T u_n\|\right) + \varphi\left(\|u_n - T u_n\|\right) \\ &= \left[1 - \alpha_{1n}(1-\delta)\right] \|u_n - T u_n\| + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right] \|u_n - y_{n+r-2}\| \\ &\quad + (1+\delta\alpha_{1n})\varphi\left(\|u_n - T u_n\|\right) , \end{aligned} \tag{3.11}$$

$$\begin{aligned}
\|u_n - y_{n+1}\| &= \left\|u_n - T\left[\left(1-\alpha_{(r-1)n}\right) y_n + \alpha_{(r-1)n} T y_n\right]\right\| \\
&\leq \|u_n - Tu_n\| + \left\|Tu_n - T\left[\left(1-\alpha_{(r-1)n}\right) y_n + \alpha_{(r-1)n} T y_n\right]\right\| \\
&\leq \|u_n - Tu_n\| + \delta\left\|u_n - \left(1-\alpha_{(r-1)n}\right) y_n - \alpha_{(r-1)n} T y_n\right\| \\
&\quad + \varphi\left(\|u_n - Tu_n\|\right) \\
&\leq \|u_n - Tu_n\| + \delta\left(1-\alpha_{(r-1)n}\right)\|u_n - y_n\| + \delta\alpha_{(r-1)n}\|u_n - Ty_n\| \\
&\quad + \varphi\left(\|u_n - Tu_n\|\right) \\
&\leq \|u_n - Tu_n\| + \delta\left(1-\alpha_{(r-1)n}\right)\|u_n - y_n\| + \delta\alpha_{(r-1)n}\|u_n - Tu_n\| \\
&\quad + \delta^2\alpha_{(r-1)n}\|u_n - y_n\| + \delta\alpha_{(r-1)n}\varphi\left(\|u_n - Tu_n\|\right) + \varphi\left(\|u_n - Tu_n\|\right) \\
&= (1+\delta\alpha_{(r-1)n})\|u_n - Tu_n\| + \left[\delta\left(1-\alpha_{(r-1)n}\right) + \delta^2\alpha_{(r-1)n}\right]\|u_n - y_n\| \\
&\quad + \left(1+\delta\alpha_{(r-1)n}\right)\varphi\left(\|u_n - Tu_n\|\right),
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\|u_n - y_n\| &\leq \|u_n - T\left[(1-\alpha_{rn}) x_n + \alpha_{rn} T x_n\right]\| \\
&\leq \|u_n - Tu_n\| + \|Tu_n - T\left[(1-\alpha_{rn}) x_n + \alpha_{rn} T x_n\right]\| \\
&\leq (1+\delta\alpha_{rn})\|u_n - Tu_n\| + \left[\delta(1-\alpha_{rn}) + \delta^2\alpha_{rn}\right]\|u_n - x_n\| \\
&\quad + (1+\delta\alpha_{rn})\varphi\left(\|u_n - Tu_n\|\right).
\end{aligned} \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), we obtain that

$$\begin{aligned}
\|u_{n+1} - x_{n+1}\| &\leq \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\left[\delta(1-\alpha_{2n}) + \delta^2\alpha_{2n}\right] \ldots \left[\delta(1-\alpha_{rn}) + \delta^2\alpha_{rn}\right]\|u_n - x_n\| \\
&\quad + \left\{\left[1-\alpha_{1n}(1-\delta)\right] + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right](1+\delta\alpha_{2n}) + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\right. \\
&\quad \left[\delta(1-\alpha_{2n}) + \delta^2\alpha_{2n}\right](1+\delta\alpha_{3n}) + \cdots + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\left[\delta(1-\alpha_{2n}) + \delta^2\alpha_{2n}\right] \\
&\quad \left.\cdots\left[\delta\left(1-\alpha_{(r-1)n}\right) + \delta^{r-1}\alpha_{(r-1)n}\right](1+\delta\alpha_{rn})\right\}\left[\|u_n - Tu_n\| + \varphi\left(\|u_n - Tu_n\|\right)\right] \\
&\quad + \alpha_{1n}\varphi\left(\|u_n - Tu_n\|\right).
\end{aligned} \tag{3.14}$$

Since $\delta \in (0,1)$ and $\{\alpha_{in}\} \subset [0,1)$ for $i = 1, 2, \ldots r$, we have

$$\begin{aligned}
&\left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\left[\delta(1-\alpha_{2n}) + \delta^2\alpha_{2n}\right] \ldots \left[\delta(1-\alpha_{rn}) + \delta^2\alpha_{rn}\right] \\
&\qquad = \delta\left[1-\alpha_{1n}(1-\delta)\right]\delta\left[1-\alpha_{2n}(1-\delta)\right] \ldots \delta\left[1-\alpha_{rn}(1-\delta)\right] \\
&\qquad \leq \left[1-\alpha_{1n}(1-\delta)\right].
\end{aligned} \tag{3.15}$$

From inequality (3.15) and the assumption that $\alpha_{1n} \geq A > 0 \ \forall n \in \mathbb{N}$ in (3.14), we obtain

$$\begin{aligned}
\|u_{n+1} - x_{n+1}\| &\leq \left[1 - A(1-\delta)\right]\|u_n - x_n\| + \left\{\left[1-A(1-\delta)\right] + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right](1+\delta\alpha_{2n})\right. \\
&\quad + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\left[\delta(1-\alpha_{2n}) + \delta^2\alpha_{2n}\right](1+\delta\alpha_{3n}) + \cdots \\
&\quad + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\left[\delta(1-\alpha_{2n}) + \delta^2\alpha_{2n}\right]\cdots\left[\delta\left(1-\alpha_{(r-1)n}\right) + \delta^{r-1}\alpha_{(r-1)n}\right] \\
&\quad \left.(1+\delta\alpha_{rn})\right\}\left[\|u_n - Tu_n\| + \varphi\left(\|u_n - Tu_n\|\right)\right] + \alpha_{1n}\varphi\left(\|u_n - Tu_n\|\right).
\end{aligned}$$

Set:

$$\begin{aligned}
a_n &:= \|u_n - x_n\|, \\
\mu_n &:= A(1-\delta) \in (0,1), \\
\rho_n &:= \left\{\left[1-A(1-\delta)\right] + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right](1+\delta\alpha_{2n}) + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\right. \\
&\quad \left[\delta(1-\alpha_{2n}) + \delta^2\alpha_{2n}\right](1+\delta\alpha_{3n}) + \cdots + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\left[\delta(1-\alpha_{2n}) + \delta^2\alpha_{2n}\right] \\
&\quad \left.\cdots\left[\delta\left(1-\alpha_{(r-1)n}\right) + \delta^{r-1}\alpha_{(r-1)n}\right](1+\delta\alpha_{rn})\right\}\left[\|u_n - Tu_n\| + \varphi\left(\|u_n - Tu_n\|\right)\right] + \alpha_{1n}\varphi\left(\|u_n - Tu_n\|\right).
\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n - p\| = 0$ and $Tp = p \in F(T)$, it follows from (2.4) that

$$\begin{aligned}0 &\le \|u_n - Tu_n\|\\ &\le \|u_n - p\| + \|Tp - Tu_n\|\\ &\le \|u_n - p\| + \delta\|p - u_n\| + \varphi(\|p - Tp\|)\\ &= (1+\delta)\|u_n - p\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence $\lim_{n\to\infty}\|u_n - Tu_n\| = 0$, that is $\rho_n = o(\mu_n)$. It follows from Lemma 2.5 that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Now

$$\|x_n - p\| \le \|x_n - u_n\| + \|u_n - p\|$$

implies that $\lim_{n\to\infty} x_n = p$.

Now, we will show that $(ii) \Rightarrow (i)$. Suppose that multistep Picard-Mann iteration $\{x_n\}$ converges to p. Thus,

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &= \|x_{n+1} - p\| + \|p - u_{n+1}\|\\ &= \|Tp - T[(1-\alpha_{1n})y_{n+r-2} + \alpha_{1n}Ty_{n+r-2}]\| + \|(1-\alpha_{1n})u_n + \alpha_{1n}Tu_n - p\|\\ &\le \delta\|p - [(1-\alpha_{1n})y_{n+r-2} + \alpha_{1n}Ty_{n+r-2}]\| + \varphi(\|p - Tp\|)\\ &\quad + (1-\alpha_{1n})\|u_n - p\| + \alpha_{1n}\|Tu_n - Tp\|\\ &\le \delta(1-\alpha_{1n})\|p - y_{n+r-2}\| + \delta\alpha_{1n}\|Tp - Ty_{n+r-2}\|\\ &\quad + (1-\alpha_{1n})\|p - u_n\| + \alpha_{1n}\|Tp - Tu_n\|\\ &\le \delta(1-\alpha_{1n})\|p - y_{n+r-2}\| + \delta^2\alpha_{1n}\|p - y_{n+r-2}\| + \delta\alpha_{1n}\varphi(\|p - Tp\|)\\ &\quad + (1-\alpha_{1n})\|p - u_n\| + \delta\alpha_{1n}\|p - u_n\| + \alpha_{1n}\varphi(\|p - Tp\|)\\ &\le \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\|y_{n+r-2} - p\| + [1 - \alpha_{1n}(1-\delta)]\|u_n - p\|,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\|y_{n+r-2} - p\| &= \|T[(1-\alpha_{2n})y_{n+r-3} + \alpha_{2n}Ty_{n+r-3}] - p\|\\ &\le [\delta\|(1-\alpha_{2n})y_{n+r-3} + \alpha_{2n}Ty_{n+r-3} - p\| + \varphi(\|p - Tp\|)]\\ &\le [\delta(1-\alpha_{2n})\|y_{n+r-3} - p\| + \delta\alpha_{2n}\|Ty_{n+r-3} - p\|]\\ &\le \left[\delta(1-\alpha_{2n})\|y_{n+r-3} - p\| + \delta^2\alpha_{2n}\|y_{n+r-3} - p\| + \delta\alpha_{2n}\varphi(\|p - Tp\|)\right]\\ &= \delta[1 - \alpha_{2n}(1-\delta)]\|y_{n+r-3} - p\| + [1 - \alpha_{1n}(1-\delta)]\|u_n - p\|\\ &= \delta[1 - \alpha_{2n}(1-\delta)]\|T[(1-\alpha_{3n})y_{n+r-4} + \alpha_{3n}Ty_{n+r-4}] - p\|\\ &\le \cdots\\ &\le \delta[1 - \alpha_{2n}(1-\delta)]\cdots\delta\left[1 - \alpha_{(r-1)n}(1-\delta)\right]\|x_n - p\|.\end{aligned} \tag{3.17}$$

Using (3.17) in (3.16), we have

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\le [1 - \alpha_{1n}(1-\delta)]\|u_n - p\|\\ &\quad + \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\delta[1 - \alpha_{2n}(1-\delta)]\cdots\delta\left[1 - \alpha_{(r-1)n}(1-\delta)\right]\|x_n - p\|.\end{aligned}$$

Let

$$\begin{aligned}\mu_n &= \alpha_{1n}(1-\delta)\\ a_n &= \|u_n - p\|\\ \rho_n &= \left[\delta(1-\alpha_{1n}) + \delta^2\alpha_{1n}\right]\delta[1 - \alpha_{2n}(1-\delta)]\cdots\delta\left[1 - \alpha_{(r-1)n}(1-\delta)\right]\|x_n - p\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - p\| = 0$, we obtain $\rho_n = o(\mu_n)$. It follows from Lemma 2.5 that $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\|u_n - p\| = 0$. □

As a sequence of Theorem 3.2, we obtain the following corollary:

Corollary 3.3. *Let $T : E \to E$ be a mapping satisfying (2.4) with $F(T) \neq \emptyset$. The followings are equivalent:*
(i) The Picard iteration [17] converges to $p \in F(T)$,
(ii) The Mann iteration [13] converges to $p \in F(T)$,
(iii) The SP iteration [16] converges to $p \in F(T)$,
(iv) The multistep iteration (2.5) converges to $p \in F(T)$,
(v) The multistep Picard-Mann iteration (2.8) converges to $p \in F(T)$.

provided that the initial guess is same for all iterations.

As a consequence of Theorem 3.1, we obtain the following result which gives a relationship between the fixed point of approximate operators.

Theorem 3.4. *Let $T : E \to E$ be a mapping satisfying (2.4) with $F(T) \neq \emptyset$, and S an approximate operator of T. Let $\{x_n\}$ and $\{u_n\}$ be two iterative sequences defined by (2.8) associated to T and S, respectively with $\{\alpha_{1n}\}$, $\{\alpha_{in}\} \subset [0,1)$ for all $i = 2, ...r$ satisfying*
(i) $0 \leq \alpha_{in} < \alpha_{1n} \leq 1$ for all $i = 2, ...r$, and $\frac{1}{2} \leq \alpha_{1n}$ for all $n \in \mathbb{N}$.
(ii) $\sum_{k=0}^{n} \alpha_{1k} = \infty$.
If $p = Tp$ and $q = Sq$, then we have

$$\|p - q\| \leq \frac{3r\varepsilon}{1-\delta}.$$

Proof. For $u_0 \in E$, consider the following multistep Picard-Mann iteration procedure for S:

$$\begin{cases} u_0 \in E, \\ u_{n+1} = S\left[(1-\alpha_{1n})\, v_{n+r-2} + \alpha_{1n} S v_{n+r-2}\right], \\ v_{n+r-2} = S\left[(1-\alpha_{2n})\, v_{n+r-3} + \alpha_{2n} S v_{n+r-3}\right], \\ \vdots \\ v_{n+1} = S\left[\left(1-\alpha_{(r-1)n}\right) v_n + \alpha_{(r-1)n} S v_n\right], \\ v_n = S\left[(1-\alpha_{rn})\, u_n + \alpha_{rn} S u_n\right], \quad n \in \mathbb{N}. \end{cases} \tag{3.18}$$

From (2.8), (2.4) and (3.18), we have

$$\begin{aligned} &\|x_{n+1} - u_{n+1}\| \\ &\quad = \|T[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}] - S[(1-\alpha_{1n})\, v_{n+r-2} + \alpha_{1n} S v_{n+r-2}]\| \\ &\quad \leq \|T[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}] - T[(1-\alpha_{1n})\, v_{n+r-2} + \alpha_{1n} S v_{n+r-2}]\| \\ &\qquad + \|T[(1-\alpha_{1n})\, v_{n+r-2} + \alpha_{1n} S v_{n+r-2}] - S[(1-\alpha_{1n})\, v_{n+r-2} + \alpha_{1n} S v_{n+r-2}]\| \\ &\quad \leq \delta \|(1-\alpha_{1n})(y_{n+r-2} - v_{n+r-2}) + \alpha_{1n}(T y_{n+r-2} - S v_{n+r-2})\| \\ &\qquad + \varphi(\|[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}] - T[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}]\|) + \varepsilon \\ &\quad \leq \delta(1-\alpha_{1n}) \|y_{n+r-2} - v_{n+r-2}\| + \delta\alpha_{1n} \|T y_{n+r-2} - T v_{n+r-2}\| \\ &\qquad + \delta\alpha_{1n} \|T v_{n+r-2} - S v_{n+r-2}\| \\ &\qquad + \varphi(\|[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}] - T[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}]\|) + \varepsilon \\ &\quad \leq \delta(1-\alpha_{1n}) \|y_{n+r-2} - v_{n+r-2}\| + \delta^2\alpha_{1n} \|y_{n+r-2} - v_{n+r-2}\| \\ &\qquad + \delta\alpha_{1n}\varphi(\|y_{n+r-2} - T y_{n+r-2}\|) + \delta\alpha_{1n}\varepsilon \\ &\qquad + \varphi(\|[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}] - T[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}]\|) + \varepsilon \\ &\quad \leq \delta(1-\alpha_{1n}) \|y_{n+r-2} - v_{n+r-2}\| + \delta^2\alpha_{1n} \|y_{n+r-2} - v_{n+r-2}\| \\ &\qquad + \delta\alpha_{1n}\varphi(\|y_{n+r-2} - T y_{n+r-2}\|) + \delta\alpha_{1n}\varepsilon \\ &\qquad + \varphi(\|[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}] - T[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T y_{n+r-2}]\|) + \varepsilon \end{aligned} \tag{3.19}$$

$$\begin{aligned}
=&\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\left\|y_{n+r-2}-v_{n+r-2}\right\|+\delta\alpha_{1n}\varphi\left(\left\|y_{n+r-2}-Ty_{n+r-2}\right\|\right)\\
&+\varphi\left(\left\|\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]-T\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\right\|\right)\\
&+\left(1+\delta\alpha_{1n}\right)\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\left\|y_{n+r-2}-v_{n+r-2}\right\|=&\left\|T\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]-S\left[\left(1-\alpha_{2n}\right)v_{n+r-3}+\alpha_{2n}Sv_{n+r-3}\right]\right\|\\
\leq&\left\|T\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]-T\left[\left(1-\alpha_{2n}\right)v_{n+r-3}+\alpha_{2n}Sv_{n+r-3}\right]\right\|\\
&+\left\|T\left[\left(1-\alpha_{2n}\right)v_{n+r-3}+\alpha_{2n}Sv_{n+r-3}\right]-S\left[\left(1-\alpha_{2n}\right)v_{n+r-3}+\alpha_{2n}Sv_{n+r-3}\right]\right\|\\
\leq&\delta\left\|\left(1-\alpha_{2n}\right)\left(y_{n+r-3}-v_{n+r-3}\right)+\alpha_{2n}\left(Ty_{n+r-3}-Sv_{n+r-3}\right)\right\|\\
&+\varphi(\left\|\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]-T\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]\right\|)\\
&+\varepsilon\\
\leq&\delta(1-\alpha_{2n})\left\|y_{n+r-3}-v_{n+r-3}\right\|+\delta\alpha_{2n}\left\|Ty_{n+r-3}-Tv_{n+r-3}\right\|\\
&+\delta\alpha_{2n}\left\|Tv_{n+r-3}-Sv_{n+r-3}\right\|\\
&+\varphi\left(\left\|\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]-T\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]\right\|\right)\\
&+\varepsilon\\
\leq&\delta\left[1-\alpha_{2n}\left(1-\delta\right)\right]\left\|y_{n+r-3}-v_{n+r-3}\right\|+\delta\alpha_{2n}\varphi\left(\left\|y_{n+r-3}-Ty_{n+r-3}\right\|\right)\\
&+\varphi\left(\left\|\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]-T\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]\right\|\right)\\
&+\left(1+\delta\alpha_{2n}\right)\varepsilon.
\end{aligned}$$

By this induction, it follows that

$$\begin{aligned}
\left\|y_n-v_n\right\|=&\left\|T\left[\left(1-\alpha_{rn}\right)x_n+\alpha_{rn}Tx_n\right]-S\left[\left(1-\alpha_{rn}\right)u_n+\alpha_{rn}Su_n\right]\right\|\\
\leq&\left\|T\left[\left(1-\alpha_{rn}\right)x_n+\alpha_{rn}Tx_n\right]-T\left[\left(1-\alpha_{rn}\right)u_n+\alpha_{rn}Su_n\right]\right\|\\
&+\left\|T\left[\left(1-\alpha_{rn}\right)u_n+\alpha rSu_n\right]-S\left[\left(1-\alpha_{rn}\right)u_n+\alpha_{rn}Su_n\right]\right\|\\
\leq&\delta\left\|\left(1-\alpha_{rn}\right)\left(x_n-u_n\right)+\alpha_{rn}\left(Tx_n-Su_n\right)\right\|\\
&+\varphi\left(\left\|\left[\left(1-\alpha_{rn}\right)x_n+\alpha_{rn}Tx_n\right]-T\left[\left(1-\alpha_{rn}\right)x_n+\alpha_{rn}Tx_n\right]\right\|\right)\\
&+\varepsilon\\
\leq&\delta(1-\alpha_{rn})\left\|x_n-u_n\right\|+\delta\alpha_{rn}\left\|Tx_n-Tu_n\right\|+\delta\alpha_{rn}\left\|Tu_n-Su_n\right\|\\
&+\varphi\left(\left\|\left[\left(1-\alpha_{rn}\right)x_n+\alpha_{rn}Tx_n\right]-T\left[\left(1-\alpha_{rn}\right)x_n+\alpha_{rn}Tx_n\right]\right\|\right)\\
&+\varepsilon\\
\leq&\delta\left[1-\alpha_{rn}\left(1-\delta\right)\right]\left\|x_n-u_n\right\|+\delta\alpha_{rn}\varphi\left(\left\|x_n-Tx_n\right\|\right)\\
&+\varphi\left(\left\|\left[\left(1-\alpha_{rn}\right)x_n+\alpha_{rn}Tx_n\right]-T\left[\left(1-\alpha_{rn}\right)x_n+\alpha_{rn}Tx_n\right]\right\|\right)\\
&+\left(1+\delta\alpha_{rn}\right)\varepsilon.
\end{aligned}$$

Thus, using this inequality in (3.19), we get

$$\begin{aligned}
&\left\|x_{n+1}-u_{n+1}\right\|\\
&\leq\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\left\{\delta\left[1-\alpha_{2n}\left(1-\delta\right)\right]\left\|y_{n+r-3}-v_{n+r-3}\right\|\right.\\
&\quad+\delta\alpha_{2n}\varphi\left(\left\|y_{n+r-3}-Ty_{n+r-3}\right\|\right)\\
&\quad+\varphi\left(\left\|\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]-T\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]\right\|\right)\\
&\quad\left.+\left(1+\delta\alpha_{2n}\right)\varepsilon\right\}+\delta\alpha_{1n}\varphi\left(\left\|y_{n+r-2}-Ty_{n+r-2}\right\|\right)\\
&\quad+\varphi\left(\left\|\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]-T\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\right\|\right)\\
&\quad+\left(1+\delta\alpha_{1n}\right)\varepsilon\\
&=\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\delta\left[1-\alpha_{2n}\left(1-\delta\right)\right]\left\|y_{n+r-3}-v_{n+r-3}\right\|
\end{aligned}$$

$$
\begin{aligned}
&+\delta\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)+\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\delta\alpha_{2n}\varphi\left(\|y_{n+r-3}-Ty_{n+r-3}\|\right)\\
&+\varphi\left(\|\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]-T\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\|\right)\\
&+\left(1+\delta\alpha_{1n}\right)\varepsilon+\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\left(1+\delta\alpha_{2n}\right)\varepsilon\\
\leq&\cdots\\
\leq&\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\delta\left[1-\alpha_{2n}\left(1-\delta\right)\right]\cdots\delta\left[1-\alpha_{rn}\left(1-\delta\right)\right]\|x_n-u_n\|\\
&+\delta\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)+\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\delta\alpha_{2n}\varphi\left(\|y_{n+r-3}-Ty_{n+r-3}\|\right)\\
&+\cdots+\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\cdots\delta\left[1-\alpha_{(r-1)n}\left(1-\delta\right)\right]\delta\alpha_{rn}\varphi\left(\|x_n-Tx_n\|\right)\\
&+\varphi\left(\|\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]-T\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\|\right)\\
&+\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\\
&\varphi\left(\|\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]-T\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]\|\right)\\
&+\cdots+\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\cdots\delta\left[1-\alpha_{(r-1)n}\left(1-\delta\right)\right]\\
&\varphi\left(\|\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]-T\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]\|\right)\\
&+\left(1+\delta\alpha_{1n}\right)\varepsilon+\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\left(1+\delta\alpha_{2n}\right)\varepsilon\\
&+\cdots+\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\cdots\delta\left[1-\alpha_{(r-1)n}\left(1-\delta\right)\right]\left(1+\delta\alpha_{rn}\right)\varepsilon.
\end{aligned}
\tag{3.20}
$$

Since $\delta\in[0,1)$ and $\{\alpha_{1n}\}$, $\{\alpha_{in}\}\subset[0,1)$ for all $i=2,...r$, we have

$$
\begin{aligned}
&\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]\delta\left[1-\alpha_{2n}\left(1-\delta\right)\right]\cdots\delta\left[1-\alpha_{rn}\left(1-\delta\right)\right]\leq\left[1-\alpha_{1n}\left(1-\delta\right)\right],\\
&1-\alpha_{1n}\leq\alpha_{1n},\\
&\delta\left[1-\alpha_{1n}\left(1-\delta\right)\right]<1.
\end{aligned}
$$

Using the above inequality and the assumption *(i)* in (3.20), we get

$$
\begin{aligned}
\|x_{n+1}-u_{n+1}\|\leq&\left[1-\alpha_{1n}\left(1-\delta\right)\right]\|x_n-u_n\|\\
&+\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)+\alpha_{1n}\varphi\left(\|y_{n+r-3}-Ty_{n+r-3}\|\right)\\
&+\cdots+\alpha_{1n}\varphi\left(\|x_n-Tx_n\|\right)\\
&+2\alpha_{1n}\varphi(\|\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\\
&-T\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\|)\\
&+2\alpha_{1n}\varphi(\|\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]\\
&-T\left[\left(1-\alpha_{2n}\right)y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]\|)\\
&+\cdots+2\alpha_{1n}\varphi\left(\|\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]-T\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]\|\right)\\
&+r\left[\left(1+\delta\alpha_{1n}\right)\varepsilon\right]\\
\leq&\left[1-\alpha_{1n}\left(1-\delta\right)\right]\|x_n-u_n\|\\
&\alpha_{1n}\left(1-\delta\right)\{\frac{\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)+\cdots+\varphi\left(\|x_n-Tx_n\|\right)}{\left(1-\delta\right)}\\
&\frac{2\varphi\left(\|\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]-T\left[\left(1-\alpha_{1n}\right)y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\|\right)}{\left(1-\delta\right)}\\
&+\cdots+\frac{2\varphi\left(\|\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]-T\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]\|\right)}{\left(1-\delta\right)}\\
&+\frac{3r\varepsilon}{\left(1-\delta\right)}\}.
\end{aligned}
\tag{3.21}
$$

Let

$$\begin{aligned}
a_n =& \left\| x_n - u_n \right\|, \\
\mu_n =& \alpha_{1n} \left(1-\delta\right), \\
\eta_n =& \frac{\varphi\left(\left\| y_{n+r-2} - Ty_{n+r-2} \right\|\right) + \cdots + \varphi\left(\left\| x_n - Tx_n \right\|\right)}{(1-\delta)} \\
&+ \frac{2\varphi\left(\left\| \left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} Ty_{n+r-2}\right] - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} Ty_{n+r-2}\right] \right\|\right)}{(1-\delta)} \\
&+ \cdots + \frac{2\varphi\left(\left\| \left[(1-\alpha_{rn})x_n + \alpha_{rn} Tx_n\right] - T\left[(1-\alpha_{rn})x_n + \alpha_{rn} Tx_n\right] \right\|\right)}{(1-\delta)} \\
&+ \frac{3r\varepsilon}{(1-\delta)}.
\end{aligned}$$

From Theorem 3.1 it follows that $\lim_{n\to\infty} \left\| x_n - p \right\| = 0$. Since T satisfies condition (2.4) and $Tp = p \in F(T)$. Therefore

$$\begin{aligned}
0 \leq& \left\| x_n - Tx_n \right\| \\
\leq& \left\| x_n - p \right\| + \left\| Tp - Tx_n \right\| \\
\leq& \left\| x_n - p \right\| + \delta \left\| p - x_n \right\| + \varphi\left(\left\| p - Tp \right\|\right) \\
=& (1+\delta) \left\| x_n - p \right\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

By (2.4) and (2.8) and the fact that $\alpha_{1n}, \alpha_{in} \in [0,1)$ $\forall n \in \mathbb{N}$, $i = 2, ..., r$, we have

$$\begin{aligned}
0 \leq& \left\| y_{n+r-2} - Ty_{n+r-2} \right\| \\
\leq& \left\| y_{n+r-2} - p \right\| + \left\| Tp - Ty_{n+r-2} \right\| \\
\leq& \left\| y_{n+r-2} - p \right\| + \delta \left\| p - y_{n+r-2} \right\| + \varphi\left(\left\| p - Tp \right\|\right) \\
=& (1+\delta) \left\| y_{n+r-2} - p \right\| \\
=& (1+\delta) \left\| T\left[(1-\alpha_{2n})\, y_{n+r-3} + \alpha_{2n} Ty_{n+r-3}\right] - p \right\| \\
\leq& (1+\delta)\delta \left\| (1-\alpha_{2n})\, y_{n+r-3} + \alpha_{2n} Ty_{n+r-3} - p \right\| \\
\leq& (1+\delta)\delta \left[(1-\alpha_{2n}) \left\| y_{n+r-3} - p \right\| + \alpha_{2n} \left\| Ty_{n+r-3} - p \right\|\right] \\
\leq& (1+\delta)\delta \left[(1-\alpha_{2n}) \left\| y_{n+r-3} - p \right\| + \alpha_{2n}\delta \left\| y_{n+r-3} - p \right\| + \alpha_{2n}\varphi\left(\left\| p - Tp \right\|\right)\right] \\
=& (1+\delta)\delta \left[1 - \alpha_{2n}(1-\delta)\right] \left\| y_{n+r-3} - p \right\| \\
=& (1+\delta)\delta \left[1 - \alpha_{2n}(1-\delta)\right] \left\| T\left[(1-\alpha_{3n})\, y_{n+r-4} + \alpha_{3n} Ty_{n+r-4}\right] - p \right\| \\
\leq& \cdots \\
\leq& (1+\delta)\delta \left[1 - \alpha_{2n}(1-\delta)\right] \cdots \delta \left[1 - \alpha_{(r-1)n}(1-\delta)\right] \left\| x_n - p \right\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Above also holds for $\left\| y_{n+r-3} - Ty_{n+r-3} \right\|, \cdots, \left\| y_n - Ty_n \right\|$.

As φ is continuous, so we obtain

$$\lim_{n\to\infty} \varphi\left(\left\| x_n - Tx_n \right\|\right) = \lim_{n\to\infty} \varphi\left(\left\| y_{n+r-2} - Ty_{n+r-2} \right\|\right) = \cdots = \lim_{n\to\infty} \varphi\left(\left\| y_n - Ty_n \right\|\right) = 0.$$

From (2.4) and (3.2), we have

$$\begin{aligned}
0 \leq& \left\| \left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} Ty_{n+r-2}\right] - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} Ty_{n+r-2}\right] \right\| \\
\leq& \left\| (1-\alpha_{1n}) \left(y_{n+r-2} - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} Ty_{n+r-2}\right]\right) \right\| \\
&+ \left\| \alpha_{1n} \left(Ty_{n+r-2} - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} Ty_{n+r-2}\right]\right) \right\| \\
\leq& (1-\alpha_{1n}) \left\| y_{n+r-2} - T\left[(1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} Ty_{n+r-2}\right] \right\|
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_{1n}\delta\left\|y_{n+r-2}-\left[(1-\alpha_{1n})\,y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\right\|\\
&+\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)\\
=&(1-\alpha_{1n})\left\|y_{n+r-2}-T\left[(1-\alpha_{1n})\,y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\right\|\\
&+\alpha_{1n}^2\delta\|y_{n+r-2}-Ty_{n+r-2}\|+\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)\\
\leq&(1-\alpha_{1n})\left[\|y_{n+r-2}-p\|+\left\|Tp-T\left[(1-\alpha_{1n})\,y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\right\|\right]\\
&+\alpha_{1n}^2\delta\|y_{n+r-2}-Ty_{n+r-2}\|+\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)\\
\leq&(1-\alpha_{1n})\left[\|y_{n+r-2}-p\|+\delta\left\|p-\left[(1-\alpha_{1n})\,y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\right\|+\varphi\left(\|p-Tp\|\right)\right]\\
&+\alpha_{1n}^2\delta\|y_{n+r-2}-Ty_{n+r-2}\|+\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)\\
\leq&(1-\alpha_{1n})\left[\|y_{n+r-2}-p\|+\delta(1-\alpha_{1n})\|y_{n+r-2}-p\|+\delta\alpha_{1n}\|Tp-Ty_{n+r-2}\|\right]\\
&+\alpha_{1n}^2\delta\|y_{n+r-2}-Ty_{n+r-2}\|+\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)\\
\leq&(1-\alpha_{1n})\left[(1+\delta(1-\alpha_{1n}))\|y_{n+r-2}-p\|+\delta^2\alpha_{1n}\|p-y_{n+r-2}\|+\delta\alpha_{1n}\varphi\left(\|p-Tp\|\right)\right]\\
&+\alpha_{1n}^2\delta\|y_{n+r-2}-Ty_{n+r-2}\|+\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right)\\
=&(1-\alpha_{1n})\left(\delta^2\alpha_{1n}-\delta\alpha_{1n}+\delta+1\right)\|y_{n+r-2}-p\|\\
&+\alpha_{1n}^2\delta\|y_{n+r-2}-Ty_{n+r-2}\|+\alpha_{1n}\varphi\left(\|y_{n+r-2}-Ty_{n+r-2}\|\right).
\end{aligned}
$$

By this induction,

$$
\begin{aligned}
0\leq&\left\|\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]-T\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]\right\|\\
\leq&(1-\alpha_{rn})\left\|x_n-T\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]\right\|\\
&+\alpha_{rn}^2\delta\|x_n-Tx_n\|+\alpha_{rn}\varphi\left(\|x_n-Tx_n\|\right)\\
\leq&(1-\alpha_{rn})\left[\|x_n-p\|+\left\|Tp-T\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]\right\|\right]\\
&+\alpha_{rn}^2\delta\|x_n-Tx_n\|+\alpha_{rn}\varphi\left(\|x_n-Tx_n\|\right)\\
\leq&(1-\alpha_{rn})\left[\|x_n-p\|+\delta\left\|p-\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]\right\|+\varphi\left(\|p-Tp\|\right)\right]\\
&+\alpha_{rn}^2\delta\|x_n-Tx_n\|+\alpha_{rn}\varphi\left(\|x_n-Tx_n\|\right)\\
\leq&(1-\alpha_{rn})\left[\|x_n-p\|+\delta(1-\alpha_{rn})\|x_n-p\|+\delta\alpha_{rn}\|Tp-Tx_n\|\right]\\
&+\alpha_{rn}^2\delta\|x_n-Tx_n\|+\alpha_{rn}\varphi\left(\|x_n-Tx_n\|\right)\\
\leq&(1-\alpha_{rn})\left[(1+\delta(1-\alpha_{rn}))\|x_n-p\|+\delta^2\alpha_{rn}\|p-x_n\|+\delta\alpha_{rn}\varphi\left(\|p-Tp\|\right)\right]\\
&+\alpha_{rn}^2\delta\|x_n-Tx_n\|+\alpha_{rn}\varphi\left(\|x_n-Tx_n\|\right)\\
=&(1-\alpha_{rn})\left(\delta^2\alpha_{rn}-\delta\alpha_{rn}+\delta+1\right)\|x_n-p\|\\
&+\alpha_{rn}^2\delta\|x_n-Tx_n\|+\alpha_{rn}\varphi\left(\|x_n-Tx_n\|\right).
\end{aligned}
$$

Since

$$\lim_{n\to\infty}\|y_{n+r-2}-p\|=\lim_{n\to\infty}\|y_{n+r-3}-p\|=\cdots=\lim_{n\to\infty}\|x_n-p\|=0$$

and

$$\lim_{n\to\infty}\|y_{n+r-2}-Ty_{n+r-2}\|=\lim_{n\to\infty}\|y_{n+r-3}-Ty_{n+r-3}\|=\cdots=\lim_{n\to\infty}\|x_n-Tx_n\|=0,$$

we get

$$
\begin{aligned}
&\lim_{n\to\infty}\varphi\left(\left\|\left[(1-\alpha_{1n})\,y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]-T\left[(1-\alpha_{1n})\,y_{n+r-2}+\alpha_{1n}Ty_{n+r-2}\right]\right\|\right)=0,\\
&\lim_{n\to\infty}\varphi\left(\left\|\left[(1-\alpha_{2n})\,y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]-T\left[(1-\alpha_{2n})\,y_{n+r-3}+\alpha_{2n}Ty_{n+r-3}\right]\right\|\right)=0,\\
&\qquad\vdots\\
&\lim_{n\to\infty}\varphi\left(\left\|\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]-T\left[(1-\alpha_{rn})x_n+\alpha_{rn}Tx_n\right]\right\|\right)=0.
\end{aligned}
$$

Hence, from Lemma 2.6, we obtain that

$$0 \leq \lim_{n\to\infty} \sup \|x_n - u_n\| \leq \lim_{n\to\infty} \sup \eta_n = \frac{3r\varepsilon}{1-\delta}.$$

Applying Lemma 2.6 to (3.21) implies that

$$\|p - q\| \leq \frac{3r\varepsilon}{1-\delta}.$$

□

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