Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3735–3743
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Some new fixed point theorems in generalized probabilistic metric spaces

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Communicated by R. Saadati

Abstract

In this paper, we introduced the notion of α-ψ-type contractive mapping in PGM-spaces and established some new fixed point theorems in complete PGM-spaces. Finally, an example is given to support our main results.

Keywords: PGM-space, α-ψ-type contractive mapping, ϕ-function, fixed point.
2010 MSC: 47H10, 47H09.

1. Introduction

In 1942, Menger [9] initiated the study of PM-spaces and then Sehgal and Bharucha-Reid [14] followed Menger's line of research by using the notion of probabilistic q-contraction. They proved a unique fixed point result, which is an extension of the celebrated Banach's contraction principle [1]. For interested reader, a comprehensive study of fixed point theory in the probabilistic metric setting can be found in the book of Hadžić and Pap [6]. Recently, Choudhury and Das [2] gave a generalized unique fixed point theorem by using an altering distance function which was originally introduced by Khan et al. [7]. This extension of altering distance function is called ϕ-function, and has been further used in many related literatures [3, 10].

Dutta et al. [4] defined nonlinear generalized contractive type mapping involving ψ-contractive mapping and proved their theorems for such kind of mappings in the setting of G-complete Menger PM-spaces. Then Kutbi et al. [8] weakened the notion of ψ-contraction mapping and established some fixed point theorems in G-complete Menger PM-spaces. Later Samet et al. [12] introduced α-ψ-type contractive mapping in metric spaces, while Gopal et al. [5] introduced the notion of α-ψ-type contractive mapping and established

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Received 2016-03-16

corresponding fixed point theorems. In 2006, Mustafa and Sims [11] introduced the concept of generalized metric space. After then, many fixed point results have been obtained by many authors. Moreover, Zhou et al. [15] defined the notion of generalized probabilistic metric space as a generalization of PM-spaces and established corresponding fixed point theorems.

In this paper, motivated by the idea of Samet et al. [12] and α-ψ-type contractive mapping in PM-spaces, we weaken the notion of α-ψ-type contractive mapping and establish some fixed point theorems in complete PGM-spaces. Finally, an example is given to support our main results.

2. Preliminaries

In this section, we recall some definitions and theorems which will be needed in the next section.

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, and $\mathbb{N}^+$ be the set of all positive integers.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$.

We shall denote by $\mathscr{D}$ the set of all distribution functions. A special distribution function H of $\mathscr{D}$ is defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 ([13]). A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if the following conditions are satisfied:

(1) $T(a, b) = T(b, a)$ and $T(a, T(b, c)) = T(T(a, b), c)$, for all $a, b, c \in [0, 1]$;

(2) T is continuous;

(3) $T(a, 1) = a$ for all $a \in [0, 1]$;

(4) $T(a, b) \geq T(c, d)$, whenever $a \geq c$ and $b \geq d$, for $a, b, c, d \in [0, 1]$.

From the definition of T, it follows that $T(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. The following are three basic continuous t-norms:

(1) The minimum t-norm, defined by $T_M(a, b) = \min\{a, b\}$;

(2) The product t-norm, defined by $T_P(a, b) = ab$;

(3) The Lukasiewicz t-norm, defined by $T_L(a, b) = \max\{a + b - 1, 0\}$.

These t-norms are related in this way: $T_L \leq T_P \leq T_M$.

Definition 2.2 ([13]). A Menger probabilistic metric space (briefly, Menger PM-space) is a triplet (X, F, T), where X is a nonempty set, T is a continuous t-norm and F is a mapping from $X \times X$ into $\mathscr{D}^+$ ($F_{x,y}$ denote the value of F at the pair (x, y)) satisfying the following conditions:

(PM-1) $F_{x,y}(t) = H(t)$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$;

(PM-3) $F_{x,z}(t + s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 2.3 ([13]). Let (X, F, T) be a PM-space. Then

(1) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, if for every $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer N such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$;

(2) a sequence $\{x_n\}$ in X is called a *Cauchy* sequence if for every $\varepsilon > 0$ and $\lambda > 0$ we can find a positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $m, n \geq N$;

(3) a Menger PM-space is said to be complete if every *Cauchy* sequence is convergent to a point in X;

(4) the sequence $\{x_n\}$ in X is said to be a *G-Cauchy* sequence if $\lim\limits_{n\to\infty} F_{x_n,x_{n+m}}(t) = 1$ for each $m \in \mathbb{N}$ and $t > 0$;

(5) the space (X, F, T) is called G-complete if every *G-Cauchy* sequence in X is convergent.

According to [13], the (ε, λ)-topology in Menger PM-space (X, F, T) is introduced by the family of neighbourhoods N_x of a point $x \in X$ given by $N_x = \{N_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1)\}$ where $N_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$. The (ε, λ)-topology is a Hausdorff topology.

Definition 2.4 ([15])**.** The 3-tuple(X, G^*, T) is called a Menger probabilistic G-metric space (briefly,PGM-space), where X is a nonempty set, T is a continuous t-norm and G^* is a mapping from $X \times X \times X$ into $\mathscr{D}$ ($G^*_{x,y,z}$ denotes the value of G^* at the pair (x, y, z)) satisfying the following conditions:

(PGM-1) $G^*_{x,y,z}(t) = 1$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;

(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x, y, z \in X$ with $z \neq y$ and $t > 0$;

(PGM-3) $G^*_{x,y,z}(t) = G^*_{z,x,y}(t) = G^*_{y,x,z}(t) = \cdots$ (symmetry in all three variables);

(PGM-4) $G^*_{x,y,z}(t + s) \geq T(G^*_{x,a,a}(t), G^*_{a,y,z}(s))$ for all $x, y, z, a \in X$ and $t, s \geq 0$.

Example 2.5 ([15])**.** Let (X, F, T) be a PM-space. Define a function $G^* : X \times X \times X \to \mathbb{R}^+$ by

$$G^*_{x,y,z}(t) = \min\{F_{x,y}(t), F_{y,z}(t), F_{x,z}(t)\}$$

for all $x, y, z \in X$ and $t > 0$. Then (X, G^*, T) is a PGM-space.

For more examples of PGM-space, please refer to Zhou et al. [15].

Definition 2.6 ([15])**.** Let (X, G^*, T) be a PGM-space. Then

(1) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, if for any $\varepsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer $N_{\varepsilon,\lambda}$ such that $G^*_{x,x_n,x_n}(\varepsilon) \geq 1 - \lambda$ whenever $n > N_{\varepsilon,\lambda}$;

(2) a sequence $\{x_n\}$ in X is said to be a *Cauchy* sequence, if for any $\varepsilon > 0$ and $\lambda > 0$ we can find a positive integer $N_{\varepsilon,\lambda}$ such that $G^*_{x_n,x_m,x_l}(\varepsilon) \geq 1 - \lambda$ whenever $m, n, l \geq N_{\varepsilon,\lambda}$;

(3) a PGM-space is said to be complete if every *Cauchy* sequence is convergent to a point in X.

Definition 2.7 ([16])**.** Let (X, G^*, T) be a PGM-space. Then the following statement are equivalent:

(1) the sequence $\{x_n\}$ is a *Cauchy* sequence;

(2) for all $\varepsilon > 0$ and $0 < \lambda < 1$, there exists M such that $G^*_{x_n,x_m,x_m}(\varepsilon) \geq 1 - \lambda$ for all $n, m > M$.

Remark 2.8. Let (X, G^*, T) be a PGM-space. Then a sequence $\{x_n\}$ is said to be a *Cauchy* sequence if $\lim\limits_{n\to\infty} G^*_{x_n,x_{n+m},x_{n+m}}(t) = 1$ for each $m \in \mathbb{N}^+$, and $t > 0$.

Definition 2.9 ([15])**.** Let (X, G^*, T) be a PGM-space and x_0 be any point in X. For every $\varepsilon > 0$ and $0 < \lambda < 1$, an(ε, λ)-neighborhood of x_0 is the set of all points $y \in X$ for which $G^*_{x_0,y,y}(\varepsilon) \geq 1 - \lambda$ and $G^*_{y,x_0,x_0}(\varepsilon) \geq 1 - \lambda$. We write $N_{x_0}(\varepsilon, \lambda) = \{y \in X : G^*_{x_0,y,y}(\varepsilon) \geq 1 - \lambda,\ G^*_{y,x_0,x_0}(\varepsilon) \geq 1 - \lambda\}$.

Theorem 2.10 ([15])**.** *Let (X, G^*, T) be a Menger PGM-space. Then (X,G^*,T) is a Hausdorff space in the topology induced by the family $\{N_{x_0}(\varepsilon, \lambda)\}$ of (ε, λ)-neighborhoods.*

Definition 2.11 ([15])**.** A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a ϕ-function if it satisfies the following conditions:

(1) $\phi(t)$=0 if and only if $t = 0$;

(2) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$;

(3) ϕ is left continuous in (0,∞);

(4) ϕ is continuous at 0.

In the sequel, the class of all ϕ-function will be denoted by Φ.

Definition 2.12 ([16])**.** Let (X, G^*, T) be a PGM-space. If the mapping $f : X \to X$ be a given mapping and $\alpha : X \times X \times X \times (0, \infty) \to \mathbb{R}^+$ be a function, we say that f is generalized α-admissible if

$$\alpha(x, y, z, t) \geq 1 \Rightarrow \alpha(fx, fy, fz, t) \geq 1 \quad \text{for all} \quad t > 0 \quad \text{and} \quad x, y, z \in X.$$

3. Main results

In this section, we denote by Ψ the class of all non-decreasing functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that ψ is continuous at point 0, $\psi(0) = 0$ and $\psi^n(a_n) \to 0$ whenever $a_n \to 0$ as $n \to \infty$.

Theorem 3.1. *Let (X, G^*, T) be a PGM-space, and $f : X \to X$ be a mappings, we say that f is an α-ψ-type contractive mapping, if there exist two functions $\alpha : X \times X \times X \times (0, \infty) \to \mathbb{R}^+$ and $\psi \in \Psi$ satisfying the following inequality:*

$$\alpha(x, y, z, t)\left(\frac{1}{G^*_{fx,fy,fz}(\phi(ct))} - 1\right) \leq \psi\left(\frac{1}{G^*_{x,y,z}(\phi(t))} - 1\right), \tag{3.1}$$

where $x, y, z \in X$, $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi$ and all $t > 0$ such that $G^*_{x,y,z}(\phi(t)) > 0$.

Theorem 3.2. *Let (X, G^*, T) be a complete PGM-space. If the mapping $f : X \to X$ is an α-ψ-type contractive mapping satisfying the following conditions:*

(1) f is a generalized α-admissible mapping;

(2) there exists $x_0 \in X$, such that $\alpha(x_0, fx_0, fx_0, t) \geq 1$, for all $t > 0$;

(3) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, x_{n+1}, t) \geq 1$ for all $t > 0$ and all $n \in \mathbb{N}$, and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, x, t) \geq 1$ for all $t > 0$ and all $n \in \mathbb{N}$.

Then f has a fixed point in X.

Proof. Take an arbitrary point $x_0 \in X$ and $\alpha(x_0, fx_0, fx_0, t) \geq 1$, for all $t > 0$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n$ for $n \in \mathbb{N}^+$. suppose that $x_{n+1} \neq x_n$ for $n \in \mathbb{N}^+$(otherwise f has trivially a fixed point). Then, by using the fact that f is α-admissible, we have

$$\alpha(x_0, fx_0, fx_0, t) = \alpha(x_0, x_1, x_1, t) \geq 1 \Rightarrow \alpha(fx_0, fx_1, fx_1, t) = \alpha(x_1, x_2, x_2, t) \geq 1,$$

By induction, we get

$$\alpha(x_n, x_{n+1}, x_{n+1}, t) \geq 1 \quad \text{for} \quad t > 0 \quad \text{and} \quad n \in \mathbb{N}^+.$$

From the properties of ϕ, there exists $t > 0$ such that $G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t)) > 0$ for $n \in \mathbb{N}^+$. Obviously, $G^*_{x_n,x_{n+1},x_{n+1}}(\phi(\frac{t}{c})) > 0$ and $G^*_{x_n,x_{n+1},x_{n+1}}(\phi(ct)) > 0$ for $c \in (0, 1)$ and $n \in \mathbb{N}^+$. Therefore Theorem 3.1 gives that

$$\begin{aligned}\frac{1}{G^*_{x_1,x_2,x_2}(\phi(t))}-1 =&\frac{1}{G^*_{fx_0,fx_1,fx_1}(\phi(t))}-1\\ \leq&\alpha(x_0,x_1,x_1,t)(\frac{1}{G^*_{fx_0,fx_1,fx_1}(\phi(t))}-1)\\ \leq&\psi(\frac{1}{G^*_{x_0,x_1,x_1}(\phi(\frac{t}{c}))}-1).\end{aligned}$$

Repeating the above procedure successively n times, we obtain

$$\frac{1}{G^*_{x_n,x_{n+1},x_{n+1}}(\phi(t))}-1\leq\psi^n(\frac{1}{G^*_{x_0,x_1,x_1}(\phi(\frac{t}{c^n}))}-1).$$

In general, if we repeat the above step with $r<n$, we get

$$\frac{1}{G^*_{x_n,x_{n+1},x_{n+1}}(\phi(c^rt))}-1\leq\psi^{n-r}(\frac{1}{G^*_{x_r,x_r+1,x_r+1}(\phi(\frac{c^rt}{c^{n-r}}))}-1). \tag{3.2}$$

Since $\psi^n(a_n)\to 0$, whenever $a_n\to 0$, from (3.2), we deduce that

$$\lim_{n\to\infty}G^*_{x_n,x_{n+1},x_{n+1}}(\phi(c^rt))=1 \text{ for all } r>0. \tag{3.3}$$

Now, let $\varepsilon>0$ be given, there exists $r\in\mathbb{N}^+$ such that $\phi(c^rt)\leq\varepsilon$. Therefore, from (3.3), we obtain

$$\lim_{n\to\infty}G^*_{x_n,x_{n+1},x_{n+1}}(\varepsilon)\geq\lim_{n\to\infty}G^*_{x_n,x_{n+1},x_{n+1}}(\phi(c^rt))=1. \tag{3.4}$$

On the other hand, we know that

$$\begin{aligned}G^*_{x_n,x_{n+p},x_{n+p}}(\varepsilon)\geq&T(G^*_{x_n,x_{n+1},x_{n+1}}(\frac{\varepsilon}{p}),G^*_{x_{n+1},x_{n+p},x_{n+p}}(\frac{(p-1)\varepsilon}{p}))\\ \geq&T(G^*_{x_n,x_{n+1},x_{n+1}}(\frac{\varepsilon}{p}),T(G^*_{x_{n+1},x_{n+2},x_{n+2}}(\frac{\varepsilon}{p})\cdots(G^*_{x_{n+p-1},x_{n+p},x_{n+p}}(\frac{\varepsilon}{p}))\cdots).\end{aligned}$$

Letting $n\to\infty$ and making use of (3.4), for any integer p, we get

$$\lim_{n\to\infty}G^*_{x_n,x_{n+p},x_{n+p}}(\varepsilon)=1 \quad\text{for every}\quad \varepsilon>0.$$

It follows that $\{x_n\}$ is a *Cauchy* sequence. Since (X,G^*,T) is a complete PGM-space, there exists $u\in X$ such that $x_n\to u$ as $n\to\infty$. Moreover, we get

$$G^*_{fu,u,u}(\varepsilon)\geq T(G^*_{fu,x_{n+1},x_{n+1}}(\frac{\varepsilon}{2}),G^*_{x_{n+1},u,u}(\frac{\varepsilon}{2})). \tag{3.5}$$

Next, using the properties of ϕ, there exists $s>0$, such that $\phi(s)<\frac{\varepsilon}{2}$. Since $x_n\to u$, as $n\to\infty$, then there exists $n_0\in\mathbb{N}^+$, such that

$$G^*_{u,x_n,x_n}(\phi(s))>0 \text{ for all } n>n_0.$$

Therefore, for $n>n_0$, we obtain

$$\begin{aligned}\frac{1}{G^*_{fu,x_{n+1},x_{n+1}}(\frac{\varepsilon}{2})}-1\leq&\frac{1}{G^*_{fu,fx_n,fx_n}(\phi(s))}-1\\ \leq&\alpha(u,x_n,x_n,\frac{s}{c})\frac{1}{G^*_{fu,fx_n,fx_n}(\phi(s))}-1\\ \leq&\psi(\frac{1}{G^*_{u,x_n,x_n}(\phi(\frac{s}{c}))}-1).\end{aligned}$$

Since ψ is continuous at 0 and $\psi(0) = 0$, we obtain

$$\lim_{n\to\infty} G^*_{fu,x_{n+1},x_{n+1}}(\frac{\varepsilon}{2}) = 1. \tag{3.6}$$

Finally, from (3.5) and (3.6), we get $G^*_{fu,u,u}(\varepsilon) = 1$ for every $\varepsilon > 0$. Thus $fu = u$. This completes the proof. □

Theorem 3.3. *For all $x, y, z \in X$ and for $t > 0$, there exists $z \in X$ such that $\alpha(z, x, x, t) \geq 1$ and $\alpha(z, y, y, t) \geq 1$. Adding this condition to the hypotheses of Theorem 3.1, we obtain the uniqueness of the fixed point.*

Proof. Let $u, v \in X$ be two fixed points of f, that is $u = fu$ and $v = fv$. Now, from the condition, there exists $z \in X$, such that

$$\alpha(z, u, u, t) \geq 1 \quad \text{and} \quad \alpha(z, v, v, t) \geq 1. \tag{3.7}$$

Since f is generalized α-admissible, we get

$$\alpha(f^n z, u, u, t) \geq 1 \quad \text{and} \quad \alpha(f^n z, v, v, t) \geq 1 \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad t > 0. \tag{3.8}$$

Then, using (3.1) and (3.8), we obtain

$$\begin{aligned}\frac{1}{G^*_{f^n z,u,u}(\phi(ct))} - 1 =& \frac{1}{G^*_{f(f^{n-1}z),fu,fu}(\phi(ct))} - 1\\ \leq& \alpha(f^{n-1}z, u, u, t)\frac{1}{G^*_{f(f^{n-1}z),fu,fu}(\phi(ct))} - 1\\ \leq& \psi(\frac{1}{G^*_{f^{n-1}z,u,u}(\phi(t))} - 1).\end{aligned}$$

Repeating the above procedure successively n times, we obtain

$$\frac{1}{G^*_{f^n z,u,u}(\phi(ct))} - 1 \leq \psi^n(\frac{1}{G^*_{z,u,u}(\phi(\frac{t}{c^{n-1}}))} - 1.$$

Finally, letting $n \to \infty$, we obtain $f^n z \to u$. A similar argument shows that $f^n z \to v$ as $n \to \infty$. Thus $u = v$. This completes the proof. □

Taking $\alpha(x, y, z, t) = 1$ in Theorem 3.2, we obtain the following corollary.

Corollary 3.4. *Let (X, G^*, T) be a complete PGM-space and $f : X \to X$ be a mapping satisfying the following inequality:*

$$\frac{1}{G^*_{fx,fy,fz}(\phi(ct))} - 1 \leq \psi(\frac{1}{G^*_{x,y,z}(\phi(t))} - 1), \tag{3.9}$$

where $x, y, z \in X$, $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi$ and $t > 0$ such that $G^*_{x,y,z}(\phi(t)) > 0$. Then f has a unique fixed point in X.

From Example 2.5 and Theorem 3.1 we get the following corollary.

Corollary 3.5 ([4]). *Let (X, F, T) be a G-complete PM-space and $f : X \to X$ be a mapping satisfying the following inequality:*

$$\frac{1}{F_{fx,fy}(\phi(ct))} - 1 \leq \psi(\frac{1}{F_{x,y}(\phi(t))} - 1), \tag{3.10}$$

where $x, y \in X$, $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi$ and $t > 0$ such that $F_{x,y}(\phi(t)) > 0$. Then f has a unique fixed point in X.

Proof. Define $G^*_{x,y,z}(t) = \min\{F_{x,y}(t), F_{y,z}(t), F_{x,z}(t)\}$ for all $x, y, z \in X$ and $t > 0$. Example 2.5 shows that (X, G^*, T) is a PGM-space. Since $G^*_{x_n,x_{n+m},x_{n+m}}(t) = F_{x_n,x_{n+m}}(t)$ and $\lim_{n\to\infty} F_{x_n,x_{n+m}}(t) = 1$ implies $\lim_{n\to\infty} G^*_{x_n,x_{n+m},x_{n+m}}(t) = 1$ for each $m \in \mathbb{N}^+$ and $t > 0$. Thus (X, G^*, T) is a complete PGM-space.

Now we show that f satisfies the inequality of Corollary 3.4.

$$\begin{aligned}
\frac{1}{G^*_{fx,fy,fz}(\phi(ct))} - 1 =& \frac{1}{\min\{F_{fx,fy}(\phi(ct)), F_{fy,fz}(\phi(ct)), F_{fx,fz}(\phi(ct))\}} - 1 \\
=& \max\{\frac{1}{F_{fx,fy}(\phi(ct))} - 1, \frac{1}{F_{fy,fz}(\phi(ct))} - 1, \frac{1}{F_{fx,fz}(\phi(ct))} - 1\} \\
\leq& \max\{\psi(\frac{1}{F_{x,y}(\phi(t))} - 1), \psi(\frac{1}{F_{y,z}(\phi(t))} - 1), \psi(\frac{1}{F_{x,z}(\phi(t))} - 1)\} \\
\leq& \psi(\frac{1}{\min\{F_{x,y}(\phi(t)), F_{y,z}(\phi(t)), F_{x,z}(\phi(t))\}} - 1) \\
=& \psi(\frac{1}{G^*_{x,y,z}(\phi(t))} - 1),
\end{aligned}$$

this implies f satisfies all the hypotheses of Corollary 3.4. Thus f has a unique fixed point. This completes the proof. □

Taking $\psi(t) = t$ in Corollary 3.4, we obtain the following corollary.

Corollary 3.6. *Let (X, G^*, T) be a complete PGM-space, $f : X \to X$ be a given mapping satisfying*

$$G^*_{x,y,z}(\phi(t)) \leq G^*_{fx,fy,fz}(\phi(ct)),$$

where $x, y, z \in X$, $\phi \in \Phi$ and $t > 0$. Then f has a unique fixed point in X.

Taking $\phi(t) = \frac{t}{c}$ in Corollary 3.6, we obtain the following corollary.

Corollary 3.7 ([5])**.** *Let (X, G^*, T) be a complete PGM-space, $f : X \to X$ be a given mapping satisfying*

$$G^*_{x,y,z}(\frac{t}{c}) \leq G^*_{fx,fy,fz}(t),$$

where $x, y, z \in X$, $c \in (0, 1)$, and $t > 0$. Then f has a unique fixed point in X.

Corollary 3.8. *Let (X, G^*, T) be a complete PGM-space and $f : X \to X$ be a mapping satisfying the following inequality:*

$$\frac{1}{G^*_{fx,fy,fz}(\phi(ct))} - 1 \leq \psi(\frac{2}{G^*_{x,y,y}(\phi(t)) + G^*_{y,z,z}(\phi(t))} - 1),$$

where $x, y, z \in X$, $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi$ and all $t > 0$ such that $G^*_{x,y,z}(\phi(t)) > 0$. Then f has a unique fixed point in X.

Proof. Since $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x, y, z \in X$ with $z \neq y$ and $t > 0$, we get

$$G^*_{x,y,z}(t) \leq \min\{G^*_{x,y,y}(t), G^*_{y,z,z}(t)\} \leq \frac{G^*_{x,y,y}(t) + G^*_{y,z,z}(t)}{2}.$$

Then

$$\frac{1}{G^*_{fx,fy,fz}(\phi(ct))} - 1 \leq \psi(\frac{2}{G^*_{x,y,y}(\phi(t)) + G^*_{y,z,z}(\phi(t))} - 1) \leq \psi(\frac{1}{G^*_{x,y,z}(\phi(t))} - 1).$$

This implies that f satisfies all the hypotheses of Corollary 3.4. Thus f has a unique fixed point in X. This completes the proof. □

As an application of Theorems 3.1 and 3.2, we prove the following common fixed point theorem for a finite family of mappings which runs as follows.

Theorem 3.9. *Let (X, G^*, T) be a complete PGM-space, $\{f_i\}_{i=1}^m$ be a finite family of self-mappings defined on X and denote $f = f_1 f_2 f_3 \cdots f_m$. If $f : X \to X$ satisfies all the hypotheses of Theorems 3.1 and 3.2, then the family $\{f_i\}_{i=1}^m$ has a unique common fixed point provided that $f_i f_j = f_j f_i$ whenever $i \neq j$, with $i, j \in \{1, 2, \cdots, m\}$.*

Proof. Notice that all the hypotheses of Theorems 3.1 and 3.2 are satisfied in respect of the mapping f, therefore there exists a unique $x \in X$ such that $fx = x$. Now

$$\begin{aligned} f(f_i x) =& ((f_1 f_2 f_3 \cdots f_m) f_i) x \\ =& (f_1 f_2 f_3 \cdots f_{m-1})((f_m f_i) x) = (f_1 f_2 f_3 \cdots f_{m-1})(f_m f_i x) \\ =& \cdots \\ =& f_1 f_i (f_2 f_3 \cdots f_m x) \\ =& f_i f_1 (f_2 f_3 \cdots f_m x) = f_i(fx) = f_i x, \end{aligned}$$

which shows that $f_i x$ is also a fixed point of f. Since x is the unique fixed point of f, therefore $f_i x = x$ and hence x is also a fixed point of all mappings f_i for $i \in \{1, 2, \cdots, m\}$. □

By setting $f_1 = f_2 = \cdots = f_m = g$ in Theorem 3, we obtain the following fixed point theorem for mth iteration of a mapping g.

Corollary 3.10. *Let (X, G^*, T) be a complete PGM-space and $g : X \to X$ be a mapping such that $\{g^m\}$ satisfies all the hypotheses of Theorems 3.1 and 3.2. Then g has a unique fixed point.*

4. An example

Example 4.1. Let $X = [0, 1]$ and $\mathbb{Q}^+$ be the set of all positive rational numbers. $T(a, b) = min\{a, b\}$ for all $a, b \in X$. Define a function $G^* : X \times X \times X \to \mathscr{D}^+$ as:

$$G^*_{x,y,z}(t) = \frac{t}{t + G(x, y, z)} \tag{4.1}$$

for all $x, y, z \in X$ and $G(x, y, z) = |x-y|+|y-z|+|z-x|$. Obviously, (X, G^*, T) is a complete PGM-space.

Let $f : X \to X$ be a mapping defined by $fx = \frac{1}{3} \sin x$ and $\phi \in \Phi$ by $\phi(x) = \frac{x}{c}$, and $\psi \in \Psi$ by

$$\psi(x) = \begin{cases} x & \text{if } s \in \mathbb{Q}^+; \\ \frac{x}{2} & \textit{otherwise}. \end{cases}$$

Now we show that f satisfies all the hypotheses of Theorem 3.1.

$$\begin{aligned} \frac{1}{G^*_{fx,fy,fz}(\phi(ct))} - 1 =& \frac{G(fx, fy, fz)}{t} \\ =& \frac{|\sin x - \sin y| + |\sin y - \sin z| + |\sin z - \sin x|}{3t} \\ \leq& \frac{|x-y| + |y-z| + |z-x|}{3t} \\ =& \frac{G(x, y, z)}{t} \leq \psi\left(\frac{1}{G^*_{x,y,z}(\phi(t))} - 1\right). \end{aligned}$$

Thus f has a unique fixed point. The fixed point is $u = 0$.

Define the function $\alpha : X \times X \times X \times (0, \infty) \to \mathbb{R}^+$ by

$$\alpha(x,y,z,t) = \begin{cases} 1 & \text{if } x,y,z \in [0,1]; \\ 0 & otherwise. \end{cases} \tag{4.2}$$

Now, let $x, y, z \in X$, and $\alpha(x, y, z, t) \leq 1$ for $t > 0$. This shows that $x, y, z \in [0, 1]$ and by the definitions of f and α, we have

$$fx \in [0,1], fy \in [0,1], fz \in [0,1] \ \text{ and } \ \alpha(fx, fy, fz, t) = 1 \ \text{ for all } \ t > 0,$$

that is, f is generalized α-admissible.

Then, there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, fx_0, t) \leq 1$ for $t > 0$. Indeed for $x_0 = 1$, we have $\alpha(1, f(1), f(1), t) = 1$.

Next, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, x_{n+1}, t) \leq 1$ for $t > 0$ and $n \in \mathbb{N}^+$. Since $x_n \to x$ as $n \to \infty$, this shows that $x_n, x \in [0, 1]$ and $\alpha(x_n, x, x, t) \leq 1$ for $t > 0$ and $n \in \mathbb{N}^+$. Thus, Theorems 3.1 and 3.2 are applicable.

Finally, f has a fixed point in X. The fixed point is 0.

Acknowledgements

Supported by the National Natural Science Foundation of China (11361042,11326099,11071108) and the Provincial Natural Science Foundation of Jiangxi, China (20132BAB201001,2010GZS0147).

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