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Approximate fixed points of set-valued mapping in b-metric space

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Abstract

We establish existence results related to approximate fixed point property of special types of set-valued contraction mappings, in the setting of b-metric spaces. As consequences of the main theorem, we give some fixed point results which generalize and extend various fixed point theorems in the existing literature. A simple example illustrates the new theory. Finally, we apply our results to establishing the existence of solution for some differential and integral problems.

Keywords: b-metric space, η-contraction, fixed point theorem, integral inclusion.
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1. Introduction and Preliminaries

In Mathematical Analysis and related branches of research, the availability of class of functions with useful properties is a key tool for giving explicit or implicit modelization, characterization and solution of real problems. For better understanding and supporting these considerations, it is sufficient to refer to control theory and signal processing.

On the other hand, the notion of fixed point has encountered a great success in mathematics as well as in many areas of applied sciences, in particular in dealing with approximation theory and optimization ([18, 19]). One of the settings, where the theory of fixed points is attractive, is given by metric spaces and

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their abstract extensions. Consequently, we chose to present our results in the generalized framework of b-metric spaces and deduce as particular cases analogous results in metric spaces.

For merging together previously mentioned aspects, we work with set-valued mappings, whose theory lies at the junction of functional analysis, mathematical physics and (general) topology.

Thus, we recall the notion of fixed point of set-valued mapping as follows.

Definition 1.1. Given a nonempty set X and a multi-valued mapping $T : X \to N(X)$, where $N(X)$ denotes the family of all nonempty subsets of X, then:

(i) an element $x \in X$ is called fixed point of T if $x \in Tx$;

(ii) if X is a topological space, an element $x \in X$ is called approximate fixed point of T if $x \in \overline{Tx}$, where $\overline{Tx}$ denotes the closure of Tx.

Finally, the set of all fixed points of T is denoted by $Fix(T)$.

Remark 1.2. In Definition 1.1, the term "approximate fixed point" is not used in its usual sense; the interested reader may refer to the paper by Matoušková and Reich ([13]) for more details.

Thus, the present study is motivated because there is a research direction of combining hypotheses on set-valued mappings and abstract settings to establish sufficient conditions of existence, stability and data dependence of solution of differential and variational problems, via fixed point theory ([5, 12]). In order to clarify this aspect, we mention that Reich ([17]) posed a very interesting question as follows.

Question 1.3. Let (X, d) be a complete metric space and let $T : X \to CB(X)$ satisfy

$$H(Tx, Ty) \leq k(d(x, y))d(x, y), \qquad \text{for all } x, y \in X, \text{ with } x \neq y,$$

where $k :]0, +\infty[\to [0, 1[$ with $\limsup_{r \to t^+} k(r) < 1$ for each $t \in]0, +\infty[$, and $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X. Then, does T have a fixed point?

There is some evidence that this question has affirmative answer. For instance, it is known that T has a fixed point by [15] when k is a constant function, and by [16] when the codomain of T is assumed to be the family of nonempty compact subsets of X, say $K(X)$. Moreover, Mizoguchi and Takahashi ([14, Theorem 5],) gave an answer to Reich's question under the hypothesis of $\limsup_{r \to t^+} k(r) < 1$ for each $t \in [0, +\infty[$.

Based on these considerations, we work with special types of set-valued contraction mappings and establish some new existence theorems related to approximate fixed point property. As consequences of the main theorem, we give some new fixed point results which generalize and extend Du's fixed point theorem, Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem and some well-known results in the literature, to the setting of b-metric spaces. A simple example illustrates the new theory. Finally, we apply our results to establish the existence of solution for some differential and integral problems.

2. Preliminaries

In this section, for convenience of the reader, we collect the hypothesis on set-valued mapping T and some auxiliary notions useful to establishing our results.

Definition 2.1. Let (X, d) be a metric space and let $T : X \to N(X)$ a set-valued mapping. Then T is α-admissible, if there exists a function $\alpha : X \times X \to [0, +\infty[$ such that for all $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, one has $\alpha(y, z) \geq 1$ for all $z \in Ty$.

In 2014, Du and Khojasteh ([11]) introduced a class of manageable functions, which will be used to prove the existence of a special type of Cauchy sequences and approximate fixed point property; see also [9] for other class of auxiliary functions.

Definition 2.2. A function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called manageable if the following conditions hold:

(η_1) $\eta(t, s) < s - t$, for all $t, s > 0$;

(η_2) for any bounded sequence $\{t_n\} \subset]0, +\infty[$ and any nonincreasing sequence $\{s_n\} \subset]0, +\infty[$, we have

$$\limsup_{n\to+\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} < 1. \tag{2.1}$$

For shortness, Du and Khojasteh denoted the set of all manageable functions by $\widehat{Man(\mathbb{R})}$, then established the following result in the setting of metric spaces.

Theorem 2.3. *Let (X, d) be a metric space, $T : X \to N(X)$ be an α-admissible set-valued mapping and $\eta \in \widehat{Man(\mathbb{R})}$. Let*

$$\Omega = \{(\alpha(x, y)d(y, Ty), d(x, y)) \in [0, +\infty[\times[0, +\infty[: x \in X,\ y \in Tx\}.$$

If $\eta(t, s) \geq 0$ for all $(t, s) \in \Omega$ and there exist $x_0 \in X$ and $x_1 \in Tx$ such that $\alpha(x_0, x_1) \geq 1$, then the following statements hold:

1. *There exists a Cauchy sequence $\{w_n\}$ in X such that*
   (i) $w_{n+1} \in Tw_n$ *for all* $n \in \mathbb{N}$;
   (ii) $\alpha(w_n, w_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$;
   (iii) $\lim_{n\to+\infty} d(w_n, w_{n+1}) = \inf_{n\in\mathbb{N}} d(w_n, w_{n+1}) = 0$.
2. $\inf_{x\in X} d(x, Tx) = 0$*, that is, T has the approximate fixed point property on X.*

Moreover, denote by Φ the family of functions $\phi : [0, +\infty[\to [0, 1[$ such that

$$\limsup_{t\to s^+} \phi(t) < 1, \qquad \text{for all } s \in [0, +\infty[.$$

It is evident that if $\phi : [0, +\infty[\to [0, 1[$ is a nondecreasing function or a nonincreasing function, then $\phi \in \Phi$ and hence the family Φ is nonempty.

In this paper, we consider the family Γ of functions $\eta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ satisfying condition (η_1) of Definition 2.2 and the following condition:

(η_3) for any pair of sequences $\{t_n\}, \{s_n\} \subset]0, +\infty[$ such that $t_n \leq s_n$ for all $n \in \mathbb{N}$ and $\{s_n\}$ is a nonincreasing sequence, then (2.1) holds.

Example 2.4. If $\phi \in \Phi$, then the function $\eta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ defined by $\eta(t, s) = s\phi(s) - t$ for all $t, s \geq 0$ belongs to Γ. Trivially, the function η satisfies condition (η_1). Thus, we show that the function η satisfies (η_3).

Let $\{t_n\}$, $\{s_n\} \subset]0, +\infty[$ with $t_n \leq s_n$ for all $n \in \mathbb{N}$ and $\{s_n\}$ be a nonincreasing sequence. If $s_n \to s \geq 0$, then

$$\limsup_{n\to+\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} = \limsup_{n\to+\infty} \phi(s_n) \leq \limsup_{t\to s^+} \phi(t) < 1,$$

and hence $\eta \in \Gamma$.

In establishing our theorem, we need other auxiliary concepts related to the setting of b-metric spaces; see [1–3, 6–8]. We start with definition of the space as follows.

Definition 2.5. Let X be a non-empty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to [0, +\infty[$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

Then, the triplet (X, d, b) is called a b-metric space.

Example 2.6. The triplet $([0, 1], d, 2)$, where $d : X \times X \to [0, +\infty[$ is given by $d(x, y) = |x - y|^2$ for all $x, y \in X$, is a 2-metric space; but it is not a metric space.

Remark 2.7. Each metric space is a b-metric space, with $b = 1$; the converse is not always true as showed by Example 2.6 above.

Let (X, d, b) be a b-metric space. Then, one can deduce some basic notions from their metric counterparts:

(i) a sequence $\{x_n\} \subseteq X$ converges to $x \in X$ if $\lim_{n\to+\infty} d(x_n, x) = 0$;

(ii) a sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence if, for every given $\varepsilon > 0$, there exists a positive integer $n(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n(\varepsilon)$;

(iii) a b-metric space (X, d, b) is said to be complete if and only if each Cauchy sequence converges to some $x \in X$.

Moreover, in dealing with set-valued mappings, it is fundamental to extend the b-metric d to compute distances between sets in a natural way. Then, for $A, B \in N(X)$, define the function $H : N(X) \times N(X) \to [0, +\infty]$ by

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{u\in B} d(u, A)\right\},$$

where

$$d(x, C) = \inf_{y\in C} d(x, y).$$

Note that $H : CB(X) \times CB(X) \to [0, +\infty[$ is called the Pompeiu-Hausdorff b-metric induced by the b-metric d. Also, $H : C(X) \times C(X) \to [0, +\infty]$, where $C(X)$ denote the family of all nonempty closed subsets of X, is a Pompeiu-Hausdorff generalized b-metric.

Without being exhaustive, we collect in few lemmas fundamental properties from the literature (see again [6–8]).

Lemma 2.8. *Let (X, d, b) be a b-metric space. For all $A, B, C \in N(X)$ and all $x, y \in X$, we have:*

(i) $d(x, B) \leq d(x, u)$, *for any* $u \in B$;
(ii) $\sup_{a\in A} d(a, B) \leq H(A, B)$;
(iii) $H(A, A) = 0$;
(iv) $H(A, B) = H(B, A)$;
(v) $H(A, C) \leq b[H(A, B) + H(B, C)]$;
(vi) $d(x, A) \leq b[d(x, y) + d(y, A)]$.

Lemma 2.9. *Let (X, d, b) be a b-metric space and $A, B \in N(X)$. Then, for each $h > 1$ and for each $v \in A$ there exists $u(v) \in B$ such that $d(v, u(v)) \leq h\, H(A, B)$.*

Lemma 2.10. *Let (X, d, b) be a b-metric space. For $A \in N(X)$ and $x \in X$, we have*

$$d(x, A) = 0 \Longleftrightarrow x \in \bar{A}.$$

Definition 2.11. Let (X, d, b) be a b-metric space and let $T : X \to N(X)$. Then:

(i) for each $\alpha \in]1,+\infty[$ and $x \in X$ denote by $I_\alpha^x = \{y \in Tx : d(x,y) < \alpha d(x,Tx)\}$. Clearly, $I_\alpha^x \neq \emptyset$ for all $x \in X$ with $d(x,Tx) > 0$ and $\alpha \in]1,+\infty[$;

(ii) the graph $Gr(T)$ of the set-valued mapping T is the subset $\{(x,y) : x \in X, y \in Tx\}$ of $X \times X$;

(iii) T is a closed set-valued mapping if $Gr(T)$ is a closed subset of $(X \times X, d^*)$, where

$$d^*((x,y),(u,v)) = d(x,u) + d(y,v) \qquad \text{for all } (x,y),(u,v) \in X \times X.$$

As well as, we denote by $\overline{T}$ the set-valued mapping defined by $\overline{T}x = \overline{Tx}$ for all $x \in X$.

In conclusion of section, we have some conditions to be used below, namely:

(a) $d(x,Tx) = 0$ if there exists a sequence $\{x_n\} \subset X$ convergent to $x \in X$ such that $\lim_{n\to+\infty} d(x_n,Tx_n) = 0$;

(b) the function $x \in X \to d(x,Tx)$ is lower semicontinuous;

(c) for each $y \in X$ with $y \notin \overline{Ty}$, we have $\inf_{x\in X}\{d(x,y) + d(x,Tx)\} > 0$;

(d) $\overline{T}$ is a closed set-valued mapping.

3. Main result

We consider a suitable notion of η-contraction for set-valued mapping and establish a result of existence of an approximate fixed point in the framework of complete b-metric spaces.

Definition 3.1. Let (X,d,b) be a b-metric space. A set-valued mapping $T : X \to N(X)$ is called η-contraction if there exists a function $\eta \in \Gamma$ such that for each $\alpha \in]1,+\infty[$ and for all $x \in X$ with $d(x,Tx) > 0$ there exists $y \in I_\alpha^x$ such that $\eta(bd(y,Ty), d(x,y)) \geq 0$.

We state and prove the following theorem.

Theorem 3.2. *Let (X,d,b) be a b-metric space and let $T : X \to N(X)$ be an η-contraction. Then $\inf_{x\in X} d(x,Tx) = 0$, that is, T has the approximate fixed point property on X. Moreover, if (X,d,b) is a complete b-metric space, then T has an approximate fixed point $z \in X$ if one of the conditions (a)–(d) holds.*

Proof. Suppose that T does not have the approximate fixed point property on X. Then $\inf_{x\in X} d(x,Tx) > 0$ and hence $d(x,Tx) > 0$ for all $x \in X$. Fixed $x_0 \in X$ and $\alpha_0 > 1$, since T is an η-contraction, there exists $x_1 \in I_{\alpha_0}^{x_0}$ such that

$$\eta(bd(x_1,Tx_1), d(x_0,x_1)) \geq 0.$$

Clearly, from $d(x,Tx) > 0$ for all $x \in X$, it follows that $d(x_1,Tx_1)$, $d(x_0,x_1) > 0$. By the property (η_1) of function η, one can define a positive real number α_1 given by

$$\alpha_1^2 = \frac{bd(x_1,Tx_1) + \eta(bd(x_1,Tx_1), d(x_0,x_1))}{d(x_0,x_1)} < 1.$$

By hypothesis, there exists $x_2 \in I_{1/\alpha_1}^{x_1}$ such that

$$\eta(bd(x_2,Tx_2), d(x_1,x_2)) \geq 0.$$

Proceeding by induction, one can construct two sequences $\{x_n\} \subset X$ and $\{\alpha_n\} \subset]0,1[$, $n \in \mathbb{N}$, such that

$$d(x_n,Tx_n) > 0,$$

$$\begin{aligned} &d(x_{n-1},x_n)>0,\\ &\eta(bd(x_n,Tx_n),d(x_{n-1},x_n))\geq 0, \end{aligned} \tag{3.1}$$

and $x_{n+1}\in I^{x_n}_{1/\alpha_n}$ for all $n\in\mathbb{N}$, where

$$\alpha_n^2=\frac{bd(x_n,Tx_n)+\eta(bd(x_n,Tx_n),d(x_{n-1},x_n))}{d(x_{n-1},x_n)}. \tag{3.2}$$

From (3.1) and (3.2), we get

$$bd(x_n,Tx_n)\leq bd(x_n,Tx_n)+\eta(bd(x_n,Tx_n),d(x_{n-1},x_n))=\alpha_n^2\, d(x_{n-1},x_n)$$

for all $n\in\mathbb{N}$. Since $x_{n+1}\in, I^{x_n}_{1/\alpha_n}$, from the previous inequality, we obtain

$$bd(x_n,x_{n+1})<\frac{1}{\alpha_n}bd(x_n,Tx_n)\leq \alpha_n\, d(x_{n-1},x_n) \tag{3.3}$$

for all $n\in\mathbb{N}$. This means that the sequence $\{d(x_{n-1},x_n)\}$ is decreasing in $]0,+\infty[$. Thus, there exists a non-negative real number r such that

$$\lim_{n\to+\infty} d(x_{n-1},x_n)=r.$$

Now, we claim that $r=0$. In fact, by condition (η_1) or (3.3), we deduce that

$$bd(x_n,Tx_n)\leq d(x_{n-1},x_n),\quad \text{for all } n\in\mathbb{N}.$$

Then, by condition (η_3), we get

$$\limsup_{n\to+\infty}\alpha_n<1. \tag{3.4}$$

Letting $n\to+\infty$ in (3.3), we obtain $br<r$, a contradiction and so $r=0$. Next, from $\alpha_n<1$ for all $n\in\mathbb{N}$ and (3.4), we deduce that there exists $k\in]0,1[$ such that $\alpha_n<k$ for all $n\in\mathbb{N}$. By (3.3), we get

$$bd(x_n,x_{n+1})\leq k\, d(x_{n-1},x_n),\quad \text{for all } n\in\mathbb{N},$$

and consequently $\{x_n\}$ is a Cauchy sequence. It follows that

$$\inf_{x\in X} d(x,Tx)\leq \inf_{n\in\mathbb{N}} d(x_n,Tx_n)\leq \inf_{n\in\mathbb{N}} d(x_n,x_{n+1})=0.$$

Now, we assume that (X,d,b) is a complete b-metric space. Then, there exists $z\in X$ such that $x_n\to z$ as $n\to+\infty$. We claim that z is an approximate fixed point of T if one of the conditions (a)-(d) holds.

Case 1. Assume that condition (a) holds. From $d(x_n,Tx_n)\leq d(x_n,x_{n+1})\to 0$ as $n\to+\infty$, we deduce that $d(z,Tz)=0$. This implies that $z\in\overline{Tz}$, that is, z is an approximate fixed point of T.

Case 2. Assume that condition (b) holds. From

$$d(z,Tz)\leq \liminf_{n\to+\infty} d(x_n,Tx_n)=0,$$

it follows that condition (a) holds and so T has an approximate fixed point.

Case 3. Assume that condition (c) holds. If $z\notin\overline{Tz}$, it follows

$$\begin{aligned} 0&<\inf_{x\in X}\{d(x,z)+d(x,Tx)\}\\ &\leq \inf_{n\in\mathbb{N}}\{d(x_n,z)+d(x_n,Tx_n)\}=0, \end{aligned}$$

which is a contradiction. Thus $z\in\overline{Tz}$, that is, z is an approximate fixed point of T.

Case 4. Assume that condition (d) holds. From

$$\lim_{n\to+\infty} d^*((x_n,x_{n+1}),(z,z))=\lim_{n\to+\infty}[d(x_n,z)+d(x_{n+1},z)]=0,$$

we obtain that $(z,z)\in Gr(\overline{T})$, that is, $z\in\overline{Tz}$. Hence z is an approximate fixed point of T. □

From Theorem 3.2, one can deduce a result of existence of fixed point for set-valued mappings as follows.

Corollary 3.3. *Let (X, d, b) be a b-metric space and $T : X \to C(X)$ an η-contraction. Then $\inf_{x\in X} d(x, Tx) = 0$; that is, T has the approximate fixed point property on X. Moreover, if (X, d, b) is a complete b-metric space, then T has a fixed point $z \in X$ if one of the conditions (a)–(d) holds.*

4. Consequences and related results

In this section, we discuss some consequences of Theorem 3.2 in the setting of b-metric spaces. First, we give the following corollary, which is a Mizoguchi-Takahashi type result in a b-metric space.

Corollary 4.1. *Let (X, d, b) be a b-metric space and let $T : X \to N(X)$ be a set-valued mapping. Assume that there exists a function $\phi \in \Phi$ such that*

$$bH(Tx, Ty) \leq \phi(d(x, y))d(x, y), \qquad \text{for all } x, y \in X,\ x \neq y. \tag{4.1}$$

Then $\inf_{x\in X} d(x, Tx) = 0$. Moreover, if (X, d, b) is a complete b-metric space, then T has an approximate fixed point.

Proof. From (4.1), we get that, for each $\alpha \in]1, +\infty[$ and for all $x \in X$ with $d(x, Tx) > 0$, there exists $y \in I^x_\alpha$ such that

$$0 \leq \phi(d(x, y))\, d(x, y) - bH(Tx, Ty) \leq \phi(d(x, y))\, d(x, y) - bd(y, Ty),$$

that is, T is an η-contraction with respect to the function $\eta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ defined by $\eta(t, s) = s\phi(s) - t$ for all $t, s \geq 0$. Now, we claim that condition (a) holds. In fact, for all $x, y \in X$, we write

$$d(x, Tx) \leq b\, d(x, y) + b^2 d(y, Ty) + b^2 H(Ty, Tx). \tag{4.2}$$

By using (4.1) and (4.2), we obtain

$$d(x, Tx) \leq b\, d(x, y) + b^2 d(y, Ty) + b\, \phi(d(x, y))d(x, y) \leq 2b\, d(x, y) + b^2 d(y, Ty) \tag{4.3}$$

for all $x, y \in X$ with $x \neq y$. If the sequence $\{x_n\} \subset X$ converges to x and $d(x_n, Tx_n) \to 0$, by (4.3) with $y = x_n$, letting $n \to +\infty$, we get that $d(x, Tx) = 0$.

Thus, the conclusion follows from Theorem 3.2, since all the hypotheses in Theorem 3.2 are satisfied. □

Corollary 4.2. *Let (X, d, b) be a b-metric space and let $T : X \to N(X)$ be a set-valued mapping. Assume that there exist a function $\phi \in \Phi$ and a function $f : X \to [0, +\infty[$ such that*

$$b\, H(Tx, Ty) \leq \phi(d(x, y))d(x, y) + f(y)\, d(y, Tx), \qquad \text{for all } x, y \in X,\ x \neq y. \tag{4.4}$$

Then $\inf_{x\in X} d(x, Tx) = 0$. Moreover, if (X, d, b) is a complete b-metric space, then T has an approximate fixed point.

Proof. From (4.4), we get that, for each $\alpha \in]1, +\infty[$ and for all $x \in X$ with $d(x, Tx) > 0$, there exists $y \in I^x_\alpha$ such that

$$0 \leq \phi(d(x, y))\, d(x, y) - bH(Tx, Ty) \leq \phi(d(x, y))\, d(x, y) - bd(y, Ty),$$

that is, T is an η-contraction with respect to the function $\eta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ defined by $\eta(t, s) = s\phi(s) - t$ for all $t, s \geq 0$.

By using (4.2) and (4.4) (by interchanging the role of x and y in (4.4)), we obtain

$$d(x, Tx) \leq 2b\, d(x, y) + b^2 d(y, Ty) + bf(x)d(x, Ty) \tag{4.5}$$

for all $x, y \in X$ with $x \neq y$. If the sequence $\{x_n\} \subset X$ converges to x and $d(x_n, Tx_n) \to 0$, by (4.5) with $y = x_n$, letting $n \to +\infty$, we get that $d(x, Tx) = 0$.

Thus, the conclusion follows by an application of Theorem 3.2. □

From the previous results, one can obtain some well-known results in the setting of metric spaces. From Corollary 4.1, we obtain the following result that is an extension of a result of Mizoguchi and Takahashi ([14, Theorem 5]).

Corollary 4.3. *Let (X,d) be a complete metric space and let $T : X \to C(X)$ be a set-valued mapping. Assume that there exists a function $\phi \in \Phi$ such that*

$$H(Tx,Ty) \le \phi(d(x,y))d(x,y), \quad \text{for all } x,y \in X,\ x \neq y.$$

Then T has a fixed point.

Proof. By Corollary 4.1, the set-valued mapping T has an approximate fixed point that is a fixed point since Tx is closed for all $x \in X$. □

From Corollary 4.2, we obtain the following corollary.

Corollary 4.4. *Let (X,d) be a complete metric space and let $T : X \to C(X)$ be a set-valued mapping. Assume that there exist a function $\phi \in \Phi$ and a function $f : X \to [0,+\infty[$ such that*

$$H(Tx,Ty) \le \phi(d(x,y))d(x,y) + f(y)\, d(y,Tx), \quad \text{for all } x,y \in X,\ x \neq y.$$

Then T has a fixed point.

Corollary 4.4 generalizes Theorem 2.6 of Du in [10], by extending the range of T from $CB(X)$ to $C(X)$. Corollary 4.4 is also a generalization of Berinde-Berinde's fixed point theorem, see Theorem 4 in [4].

In conclusion of section, an example supports the new theory.

Example 4.5. Let ℓ^∞ be the vector space consisting of all bounded real sequences and let $\{e_n\}$ be the canonical basis of ℓ^∞. Put $x_n = e_n/n$ for each $n \in \mathbb{N}$ and denote by x_0 the null element of ℓ^∞. Consider the set $X = \{x_n : n \in \mathbb{N} \cup \{0\}\}$ endowed with the b-metric $d : X \times X \to [0,+\infty[$ defined by $d(x,y) = \max\{|x_i - y_i|^2 : i \in \mathbb{N}\}$ for all $x = \{x_i\}$, $y = \{y_i\} \in X$. Clearly, $(X,d,2)$ is a complete b-metric space. We have

$$d(x_0,x_n) = \frac{1}{n^2} \text{ for all } n \in \mathbb{N} \quad \text{and} \quad d(x_m,x_n) = \frac{1}{n^2} \text{ for all } n,m \in \mathbb{N},\ m > n.$$

Now, consider the set-valued mapping $T : X \to C(X)$ defined by

$$Tx = \begin{cases} \{x_0,x_1\} & \text{if } x \in \{x_0,x_1\}, \\ X \setminus \{x_1,\ldots,x_{3n-1}\} & \text{if } x = x_n,\ n \ge 2, \end{cases}$$

and the function $\phi : [0,+\infty[\to [0,1[$ defined by

$$\phi(t) = \begin{cases} \frac{n^2}{(n+2)^2} & \text{if } t = \frac{1}{n^2},\ n \in \mathbb{N}, \\ \frac{t}{1+t} & \text{otherwise.} \end{cases}$$

Clearly, $\phi \in \Phi$. We have

$$d(x,Tx) = 0 \text{ if } x \in \{x_0,x_1\} \quad \text{and} \quad d(x,Tx) = \frac{1}{n^2} \text{ if } x = x_n,\ n \ge 2.$$

Now, we choose $y = x_{3n}$ if $x = x_n$ with $n \ge 2$, then $y \in Tx$ for all $x \in X$ with $d(x,Tx) > 0$. Thus, for all $n \in \mathbb{N} \setminus \{1\}$, we have

$$d(x_n,x_{3n}) = \frac{1}{n^2} = d(x_n,Tx_n), \tag{4.6}$$

and

$$2d(x_{3n}, Tx_{3n}) = \frac{2}{9n^2} < \frac{1}{(n+2)^2} = \frac{n^2}{(n+2)^2}\frac{1}{n^2} = \phi(d(x_n, x_{3n}))\, d(x_n, x_{3n}). \tag{4.7}$$

Therefore, from (4.6) and (4.7), we get that, for all $\alpha \in]1, +\infty[$ and for all $x \in X$ with $d(x, Tx) > 0$, there exists $y \in I_\alpha^x$ such that

$$2d(y, Ty) \leq \phi(d(x, y))\, d(x, y),$$

that is, T is an η-contraction with respect to the function $\eta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ defined by $\eta(t, s) = s\phi(s) - t$ for all $t, s \geq 0$. Now, we claim that condition (c) holds. In fact, if $y = x_n$ with $n \geq 2$, then $y \notin \overline{Ty} = Ty$ and we have

$$\inf_{x\in X}\{d(x, x_n) + d(x, Tx)\} \geq \min\{\inf_{x\in X,\, x\neq x_n} d(x, x_n), d(x_n, Tx_n)\} = \frac{1}{n^2}.$$

Thus, since all the hypotheses of Corollary 3.3 hold true, the existence of a fixed point of T follows from Corollary 3.3.

Note that Mizoguchi-Takahashi's fixed point theorem in the setting of b-metric space is not applicable here, since

$$2H(Tx_1, Tx_3) = 2 > \frac{1}{9} = \phi(d(x_1, x_3))\, d(x_1, x_3).$$

5. Differential and integral problems

We establish the existence of solution for some differential and integral problems, by fixed point theory of set-valued operators. Background information may be found in [20].

Let K_{cv} denote the family of all nonempty compact and convex subsets of $\mathbb{R}$, $X = C([0, 1], \mathbb{R})$ be the space of all continuous functions $f : [0, 1] \to \mathbb{R}$. Clearly, $C([0, 1], \mathbb{R})$, endowed with the metric $d : X \times X \to [0, +\infty[$ defined by

$$d(x, y) = \sup_{t\in[0,1]} (x(t) - y(t))^2 = \|(x - y)^2\|_\infty, \text{ for all } x, y \in X,$$

is a complete 2-metric space.

In this setting, we consider the problem of solving the differential inclusion

$$x'(t) \in G(t, s, x(s)), \quad t, s \in [0, 1], \tag{5.1}$$

where $x \in C([0, 1], \mathbb{R})$ and $G : [0, 1]\times[0, 1]\times\mathbb{R} \to K_{cv}(\mathbb{R})$ is a set-valued operator. Let $G_x(t, s) := G(t, s, x(s))$, $t, s \in [0, 1]$. By an application of Corollary 4.1, we establish the existence of solution of (5.1), as follows.

Theorem 5.1. *Suppose that the following conditions hold:*

(i) for each $x \in X$, the set-valued operator $G : [0, 1] \times [0, 1] \times \mathbb{R} \to K_{cv}(\mathbb{R})$ is such that $G(t, s, x(s))$ is lower semicontinuous in $[0, 1] \times [0, 1]$;

(ii) there exists a continuous function $l : [0, 1] \times [0, 1] \to \mathbb{R}$ with

$$\sup_{t\in[0,1]} \int_0^1 l(t, s)ds \leq \sqrt{2^{-1}k}, \quad \textit{for some } k \in]0, 1[,$$

such that for each $g_x : [0, 1] \times [0, 1] \to \mathbb{R}$ with $g_x(t, s) \in G_x(t, s)$, there exists a continuous function $g_y(t, s) \in G_y(t, s)$ satisfying

$$|g_x(t, s) - g_y(t, s)| \leq l(t, s)|x(s) - y(s)|$$

for all $t, s \in [0, 1]$ and for all $x, y \in X$ with $x \neq y$.

Then, the differential inclusion (5.1) *has at least one solution in* X.

Proof. Let $T : X \to C(X)$ be the set-valued operator defined by

$$Tx(t) = \left\{ v \in X \text{ such that } v(t) \in \int_0^1 G(t, s, x(s))ds,\ t \in [0, 1] \right\}$$

for each $x \in C([0, 1], \mathbb{R})$. Clearly, each fixed point of T is a solution of (5.1).

Next, consider the set-valued operator $G_x : [0, 1] \times [0, 1] \to K_{cv}(\mathbb{R})$, defined by $G_x(t, s) = G(t, s, x(s))$. By Michael's selection theorem, we get that there exists a continuous operator $g_x : [0, 1] \times [0, 1] \to \mathbb{R}$ such that $g_x(t, s) \in G_x(t, s)$, for all $t, s \in [0, 1]$. This implies that $\int_0^1 g_x(t, s)ds \in Tx$ and so Tx is a nonempty set. It is an easy matter to show that Tx is closed, and so details are omitted (see also [20]). This implies that Tx is closed in (X, d).

Next, we show that the set-valued operator T satisfies all the conditions of Corollary 4.1. Let $x, y \in X$, with $x \neq y$, be such that $v \in Tx$, then there exists $g_x(t, s) \in G_x(t, s)$ with $t, s \in [0, 1]$ such that $v(t) = \int_0^1 g_x(t, s)ds$, $t \in [0, 1]$. On the other hand, by hypothesis (ii), we have

$$z(t) = \int_0^1 g_y(t, s)ds \in \int_0^1 G(t, s, y(s))ds, \quad t \in [0, 1],$$

such that

$$\begin{aligned}
|v(t) - z(t)|^2 &\leq \left(\int_0^1 |g_x(t, s) - g_y(t, s)|ds \right)^2 \\
&\leq \left(\int_0^1 l(t, s)|x(s) - y(s)|ds \right)^2 \\
&\leq \left(\int_0^1 l(t, s)\sqrt{(x(s) - y(s))^2}ds \right)^2 \\
&\leq \left(\int_0^1 l(t, s)\sqrt{\|(x - y)^2\|_\infty}ds \right)^2 \\
&\leq \|(x - y)^2\|_\infty \left(\int_0^1 l(t, s)ds \right)^2
\end{aligned}$$

for all $t \in [0, 1]$. Thus, $d(v, z) \leq \frac{k}{2} d(x, y)$. Interchanging the roles of x and y, we obtain that

$$2H(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ with $x \neq y$. Thus, all the conditions of Corollary 4.1 are satisfied with $\phi(d(x, y)) = k \in]0, 1[$ for all $x, y \in X$ with $x \neq y$, and hence T has a fixed point, which is a solution of (5.1). □

Obviously, an analogous of Theorem 5.1 holds true in respect of the following general integral inclusion of Fredholm type:

$$x(t) \in h(t) + \int_0^1 G(t, s, x(s))ds, \quad \text{for all } t \in [0, 1], \tag{5.2}$$

where $h \in X$.

Alternatively to the approach described above, we show how it is possible to prove the existence of solution of (5.2) via fixed point theory of single-valued operators. Thus, we consider the integral equation

$$x(t) = h(t) + \int_0^1 F(t, s, x(s))ds, \quad \text{for all } t \in [0, 1], \tag{5.3}$$

where $h \in X$. At this time, we consider the 2^{p-1}-metric $d : X \times X \to [0, +\infty[$ defined by

$$d(x, y) = \sup_{t\in[0,1]} (x(t) - y(t))^p = \|(x - y)^p\|_\infty, \text{ for all } x, y \in X,$$

so that X is a complete 2^{p-1}-metric space.

Theorem 5.2. *Suppose that the following conditions hold:*

(i) *for each $x \in X$, the operator $F : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous;*

(ii) *the function $h : [0, 1] \to \mathbb{R}$ is continuous;*

(iii) *there exists $l(t, \cdot) \in L^1([0, 1])$, for each $t \in [0, 1]$ and*

$$\sup_{t\in[0,1]} \int_0^1 l(t, s)ds \le \frac{1}{\sqrt[p]{2^{p-1}(1 + k)}}$$

for some $k \in]0, +\infty[$, such that

$$0 \le |F(t, s, x(s)) - F(t, s, y(s))| \le l(t, s)|x(s) - y(s)| \tag{5.4}$$

for all $t, s \in [0, 1]$ and for all $x, y \in \mathbb{R}$.

Then, the integral equation (5.3) has at least one solution in X.

Proof. Let $T : X \to X$ be the single-valued operator defined by

$$Tx(t) = h(t) + \int_0^1 F(t, s, x(s))ds, \ t \in [0, 1].$$

By hypotheses (i) and (ii), T is well-defined. Next, by hypothesis (iii), for all $x, y \in X$, we write

$$\begin{aligned}
|Tx(t) - Ty(t)|^p &= \left(\int_0^t |F(t, s, x(s)) - F(t, s, y(s))|ds\right)^p \\
&\le \left(\int_0^1 l(t, s)|x(s) - y(s)|ds\right)^p \\
&\le \left(\int_0^1 l(t, s) \sqrt[p]{(x(s) - y(s))^p}ds\right)^p \\
&\le \left(\int_0^1 l(t, s) \sqrt[p]{\|(x - y)^p\|_\infty}ds\right)^p \\
&\le \|(x - y)^p\|_\infty \left(\int_0^1 l(t, s)ds\right)^p
\end{aligned}$$

for all $t \in [0, 1]$. Thus, $d(Tx, Ty) \le \frac{d(x,y)}{2^{p-1}(1+k)}$, or equivalently

$$2^{p-1}d(Tx, Ty) \le \frac{d(x, y)}{1 + k}, \quad \text{for all } x, y \in X.$$

Thus, all the conditions of Corollary 4.1 are satisfied with $\phi(t) = \frac{1}{1+k}$ for each $t \in [0, +\infty[$ and some $k \in]0, +\infty[$, in respect of a single-valued mapping. Therefore, T has a fixed point and so we have the existence of a solution of (5.3). □

In conclusion, Theorem 5.2 is applicable for solving problem (5.2) every time one has sufficient conditions to guarantee the existence of a continuous selection for G satisfying condition (5.4). Indeed, such a kind of selection satisfies all the conditions of Theorem 5.2.

Example 5.3. Let $X = C([0,1],\mathbb{R})$ and $T : X \to X$ be the single-valued operator defined by

$$Tx(t) = \frac{7}{8}t + \int_0^1 \frac{s}{(t+1)^\alpha}\frac{|x(s)|}{|x(s)|+1}ds, \quad t \in [0,1],\ \alpha \in [1,+\infty[.$$

Clearly, T is well-defined, since the operator $F : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$, given by $F(t,s,x) = \frac{s}{(t+1)^\alpha}\frac{|x|}{|x|+1}$, and the function $h : [0,1] \to \mathbb{R}$, given by $h(t) = \frac{7}{8}t$, are continuous, for all $t,s \in [0,1]$ and $x \in \mathbb{R}$.

Moreover, consider the function $l : [0,1]\times[0,1] \to \mathbb{R}$ defined by

$$l(t,s) = \frac{s}{t+1}, \quad \text{for all } t,s \in [0,1].$$

Clearly, $l(t,\cdot) \in L^1([0,1])$, for each $t \in [0,1]$ and $\sup_{t\in[0,1]}\int_0^1 l(t,s)ds = \frac{1}{2} = \frac{1}{\sqrt[p]{2^{p-1}(1+k)}}$ for $k = 1 \in]0,+\infty[$. Also, we have

$$\begin{aligned} 0 &\le |F(t,s,x(s)) - F(t,s,y(s))| \\ &= \frac{s}{(t+1)^\alpha}\left|\frac{|x(s)|}{|x(s)|+1} - \frac{|y(s)|}{|y(s)|+1}\right| \\ &= \frac{s}{(t+1)^\alpha}\frac{||x(s)|-|y(s)||}{(|x(s)|+1)(|y(s)|+1)} \\ &\le \frac{s}{t+1}|x(s)-y(s)| = l(t,s)|x(s)-y(s)| \end{aligned}$$

for all $t,s \in [0,1]$ and for all $x,y \in X$.

Thus, since all the hypotheses of Theorem 5.2 hold true, then the integral equation

$$x(t) = \frac{7}{8}t + \int_0^1 \frac{s}{(t+1)^\alpha}\frac{|x(s)|}{|x(s)|+1}ds, \quad \text{for all } t \in [0,1],\ \alpha \in [1,+\infty[$$

has at least one solution in X.

Example 5.4. Let $X = C([0,1],\mathbb{R})$ and $T : X \to X$ be the single-valued operator defined by

$$Tx(t) = \sqrt{t} + 1 - t\sin t + \int_0^1 \frac{\sin s}{(\sqrt{t}+1)^2}x(s)ds, \quad t \in [0,1].$$

Clearly, T is well-defined, since the operator $F : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$, given by $F(t,s,x) = \frac{\sin s}{(\sqrt{t}+1)^2}x$, and the function $h : [0,1] \to \mathbb{R}$, given by $h(t) = \sqrt{t} + 1 - t\sin t$, are continuous, for all $t,s \in [0,1]$ and $x \in \mathbb{R}$.

Moreover, consider the function $l : [0,1]\times[0,1] \to \mathbb{R}$ defined in Example 5.3. Thus, we have

$$\begin{aligned} 0 &\le |F(t,s,x(s)) - F(t,s,y(s))| \\ &= \frac{\sin s}{(\sqrt{t}+1)^2}|x(s)-y(s)| \\ &\le \frac{s}{t+1}|x(s)-y(s)| \\ &= l(t,s)|x(s)-y(s)| \end{aligned}$$

for all $t,s \in [0,1]$ and for all $x,y \in X$.

Thus, since all the hypotheses of Theorem 5.2 hold true, then the integral equation

$$x(t) = \sqrt{t} + 1 - t\sin t + \int_0^1 \frac{\sin s}{(\sqrt{t}+1)^2}x(s)ds, \quad \text{for all } t \in [0,1]$$

has at least one solution in X.

Conclusions

Fixed point and set-valued mappings theories are actual branches of research. The key success factor is the possibility of applications in many fields of mathematics and applied sciences. Thus, we work with approximate fixed point property and prove an existence theorem for set-valued mapping, by using a notion of manageable function. This approach is useful to generalize various results in the existing literature. A discussion on the solution of differential and integral problems completes the paper.

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