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Approximation of a common minimum-norm fixed point of a finite family of σ-asymptotically quasi-nonexpansive mappings with applications

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Abstract

In this paper, we use the iterative method proposed by Zegeye and Shahzad [H. Zegeye, N. Shahzed, Fixed Point Theory Appl., **2013** (2013), 12 pages] which converges strongly to the common minimum-norm fixed point of a finite family of σ-asymptotically quasi-nonexpansive mappings. As consequence, convergence results to a common minimum-norm fixed point of a finite family of asymptotically nonexpansive mappings is proved. Our result generalize and improve a recent result of Zegeye and Shahzad [H. Zegeye, N. Shahzed, Fixed Point Theory Appl., **2013** (2013), 12 pages]. In the sequel, we apply our main result to find solution of minimizer of a continuously Frechet-differentiable convex functional which has the minimum norm in Hilbert spaces.

Keywords: Asymptotically quasi-nonexpansive mappings, asymptotically nonexpansive mappings, nonexpansive mappings, minimum-norm fixed point, strong convergence.
2010 MSC: 47H09, 54H25, 47J25, 65J15.

1. Introduction

Unless otherwise mentioned, throughout this paper, let H denote a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let K be a nonempty closed convex subset of H, $T : K \to K$ be a mapping

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and let $F(T)$ denote the set of fixed points of T, i.e., $F(T) = \{u \in K : Tu = u\}$. T is said to be:

(1) *nonexpansive* [11] if $\|Tu - Tv\| \leq \|u - v\|$ for all $u, v \in K$;

(2) *quasi-nonexpansive* [24] if $\|Tu - p\| \leq \|u - p\|$ for all $u \in K$ and $p \in F(T)$;

(3) *asymptotically nonexpansive* [13] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that
$$\|T^n u - T^n v\| \leq k_n \|u - v\|$$
for all $u, v \in K$ and $n \geq 1$;

(4) *asymptotically quasi-nonexpansive* [20] if there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that
$$\|T^n u - p\| \leq k_n \|u - p\|$$
for all $u \in K$ and $p \in F(T)$;

(5) *generalized quasi-nonexpansive* [21] with respect to $\{s_n\}$ if there exists a sequence $\{s_n\} \subset [0, 1)$ with $s_n \to 0$ as $n \to \infty$ such that
$$\|T^n u - p\| \leq \|u - p\| + s_n \|u - T^n u\|$$
for all $u \in K$ and $p \in F(T)$ and $n \geq 1$;

(6) *generalized asymptotically quasi-nonexpansive* [22] if there exist two sequences $\{k_n\}, \{c_n\}$ of real numbers with $\lim_{n\to\infty} k_n = 0 = \lim_{n\to\infty} c_n$ such that
$$\|T^n u - p\| \leq (1 + k_n)\|u - p\| + c_n$$
for all $u \in K$ and $p \in F(T)$, $\quad n \geq 1$.

In 1916, Tricomi [24] introduced quasi-nonexpansive for real functions and later studied by Diaz and Metcalf [10] for mappings in Banach spaces. In 1972, the class of asymptotically nonexpansive mappings was introduced as a generalization of the class of nonexpansive mappings by Goebel and Kirk [13]. In 2001, the class of asymptotically quasi-nonexpansive mapping was introduced as a generalization of the class of asymptotically nonexpansive mappings by Qihou [20]. Furthermore, it is easy to observe that, if $F(T) \neq \emptyset$, then a nonexpansive mapping must be quasi-nonexpansive and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mapping. But the converse implications need not be true.

In 1973, Petryshan and Williamson [19] proved a sufficient and necessary condition for Mann iterative sequences to convergence to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [12] extended the results of [19] and gave a sufficient and necessary condition for Ishikawa iterative sequences to converge to fixed points for quasi-nonexpansive mappings. Using these, they have also obtained some sufficient conditions for Ishikawa iterative sequences converge to fixed points for nonexpansive mappings.

The foregoing discussion arose a natural question:

Is it possible to extend the result of Ghosh and Debnath to the class of asymptotically quasi-nonexpansive mappings ?

In 2001, Qihou [20] answered this question affirmatively by proving some sufficiency and necessary conditions for Ishikawa iterative sequences of asymptotically quasi-nonexpansive mappings to converge to fixed points.

From the above definitions, it is clear that:

(1) a nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping,

(2) a quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping,

(3) an asymptotically nonexpansive mapping is generalized asymptotically quasi-nonexpansive mapping,

(4) a generalized asymptotically quasi-nonexpansive mapping is not asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive because it is not Lipschitz (see [22]).

Let K and D be nonempty closed convex subset of real Hilbert space H_1 and H_2, respectively. The *split feasibility problem* is formulated as follows:

Find a point $\overline{u}$ such that

$$\overline{u} \in K \quad \text{and} \quad A\overline{u} \in D, \tag{1.1}$$

where A is bounded linear operator from H_1 *to* H_2. A split feasibility problem in finite dimensional Hilbert spaces was introduced by Censor and Elfving [6] for modeling inverse problems which arise in medical image reconstruction, image restoration and radiation therapy treatment planing (see, for example, [3, 5, 6]). It is clear that $\overline{u}$ is a solution to the split feasibility problem (1.1) if and only if $\overline{u} \in K$ and $A\overline{u} - P_D A\overline{u} = 0$, where P_D is the metric (nearest point) projection from H_2 onto D. Set

$$\min_{u \in K} \psi(u) := \min_{u \in K} \frac{1}{2}\|Au - P_D Au\|^2. \tag{1.2}$$

Then $\overline{u}$ is a solution of the split feasibility problem (1.1) if and only if $\overline{u}$ solves the minimum problem (1.2) with a minimum equal to zero.

Recall that a point $\overline{u} \in K$ is said to be a *fixed point* of T if $T\overline{u} = \overline{u}$. We denote the set of fixed points of T by $F(T) := \{\overline{u} \in K : T\overline{u} = \overline{u}\}$. Therefore, finding a solution to the split feasibility problem (1.1) is equivalent to finding the minimum-norm problem fixed point of the mapping $u \mapsto P_K(u - \gamma A^*(I - P_D)Au)$, where A^* is the adjoint of A and $\gamma > 0$ is any positive scalar.

Motivated by the above split feasibility problem, we study the general case of finding the minimum-norm fixed point of a generalized asymptotically quasi-nonexpansive mapping $T : K \to K$, that is, we find a minimum-norm fixed point of (T) which satisfies

$$\overline{u} \in F(T) \quad \text{such} \quad \text{that} \quad \|\overline{u}\| = \min\{\|u\| : u \in F(T)\}. \tag{1.3}$$

That is, $\overline{u}$ is the minimum-norm fixed point of T. In other words, $\overline{u}$ is the metric projection of the origin into $F(T)$, i.e., $\overline{u} = P_{F(T)}0$.

Next, we briefly review two historic approaches which relate to the minimum-norm fixed point problem (1.3). In 1967, Browder [1] introduced an implicit scheme as follows:

Let $u \in K$ and $t \in (0, 1)$, u_t be the unique fixed point in K of the contraction $T_t : K \to K$ by

$$T_t x = tu + (1 - t)Tx, \tag{1.4}$$

for all $x \in K$. Also, he proved that $s - \lim_{t \downarrow 0^+} x_t = P_{F(T)}u$, that is, the strong limit of $\{x_t\}$ as $t \to 0^+$ is the fixed point of T which is nearest from F(T) to u.

Besides, in 1967, Halpern [14] introduced an explicit scheme. Let $x_0 \in K$ and define a sequence $\{x_n\}$ by

$$x_{n+1} = t_n u + (1 - t_n)Tx_n, \tag{1.5}$$

for all $n \geq 0$, where $\{t_n\} \subset (0, 1)$. It is known that the sequence $\{x_n\}$ generated by (1.5) converges in norm to the same limit $P_{F(T)}x$ as Browder's implicit scheme (1.4) if the sequence $\{t_n\}$ satisfies the conditions:

(A1) $\lim_{n \to \infty} t_n = 0$;

(A2) $\sum_{n=1}^{\infty} t_n = \infty$;

(A3) either $\sum_{n=1}^{\infty} |t_{n+1} - t_n| = \infty$ or $\lim_{n\to\infty} \left(t_n/t_{n+1}\right) = 1$.

Some more recent progress on the investigation of the implicit and explicit schemes (1.4) and (1.5) can be found in [2, 8, 9, 15, 17, 25, 26].

We notice that the above two methods find the minimum-norm fixed point $\overline{x}$ of T if $0 \in K$. However, if $0 \notin K$, then neither Browder's nor Halpern's methods work to find the minimum-norm element $\overline{x}$. The reason is simple: if $0 \notin K$, then we cannot take $u = 0$ either in (1.4) or (1.5) since the contraction $x \mapsto (1-t)Tx$ is no longer a self-mapping of K or $(1 - t_n)Tx_n$ may not belong to K and, consequently, x_{n+1} may be undefined.

For Browder's method, we consider a contraction $x \mapsto P_K((1-t)Tx)$. Since this contraction clearly maps K into K, it has a unique fixed point which is still denoted by x_t, that is,

$$x_t = P_K((1-t)Tx_t) \tag{1.6}$$

is well-defined. For Halpern's method, we consider the following iterative algorithm:

$$x_{n+1} = P_K((1-t_n)Tx_n), \tag{1.7}$$

for each $n \geq 0$. It is easily seen that the sequence $\{x_n\}$ is well-defined (i.e., $x_n \in K$ for all $n \geq 1$). Note that, if $0 \in K$, then (1.6) and (1.7) are reduced to (1.4) and (1.5) with $u = 0$, respectively.

In 2011, Yao and Xu [28] proved that both implicit and explicit methods (1.6) and (1.7) converge strongly to the minimum-norm fixed point $\overline{x}$ of the nonexpansive mapping T as $t \to 0^+$ and $n \to \infty$, respectively, (for (1.7)) provided that $\{t_n\}$ satisfies the conditions (A_1), (A_2) and (A_3).

In connection with the iterative approximation of the minimum-norm fixed point of a nonexpansive self-mapping T, in 2011, Yang et al. [27] introduced an explicit scheme given by

$$x_{n+1} = \beta Tx_n + (1-\beta)P_K[(1-\alpha_n)x_n],$$

for each $n \geq 1$. They proved that, under certain conditions on $\{\alpha_n\}$ and β, the sequence $\{x_n\}$ converges strongly to the minimum-norm fixed point of T in real Hilbert spaces. More recently, in 2012, Cai et al.[4] have also shown that the implicit and explicit methods for $\lambda \in (0, 1)$, respectively,

$$x_t = (1-t)(\lambda Tx_y + (1-\lambda)x_t), \tag{1.8}$$

$$x_{n+1} = (1-\alpha_n)(\lambda Tx_n + (1-\lambda)x_n), \tag{1.9}$$

for each $n \geq 0$, where $\{\alpha_n\} \subset (0, 1)$. They proved that the sequence $\{x_n\}$ generated by (1.8) and (1.9) converge strongly to the element of minimum-norm fixed point of nonexpansive mappings.

The aim of this paper is to introduce a new class of σ-asymptotically quasi-nonexpansive mappings and prove some strong convergence theorems for a common minimum-norm fixed point of a finite family of σ-asymptotically quasi-nonexpansive mappings which extends some known results on strong convergences for the class of generalized asymptotically quasi-nonexpansive mappings using iterative process propounded by Zegeye and Shahzad [30]. In the sequel, we apply our main result to find a solution of minimizer of a continuously Fréchet-differentiable convex functional which has the minimum norm in Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Recall that the *nearest point* (or *metric projection*) $P_K x$ of x onto a nonempty closed convex subset K is defined as follows:

$$P_K x = \min_{y \in K} \|x - y\|.$$

Now, we make use of the following lemmas for our main results:

Lemma 2.1. *Let H be a real Hilbert space. Then, for any $x, y \in H$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Lemma 2.2 ([18])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n,$$

for each $n \geq n_0$, where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([23])**.** *Let K be a closed and convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_K x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0,$$

for all $z \in K$.

Lemma 2.4 ([29])**.** *Let E be a real Hilbert space and $B_R(0)$ be a closed ball of H. Then, for any subset $\{x_0, x_1, x_2, \cdots, x_N\} \subset B_r(0)$ and for any positive numbers $\alpha_0, \alpha_1, \cdots, \alpha_N$ with $\sum_{i=0}^{N} \alpha_i = 1$, we have*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N\|^2 = \sum_{i=0}^{N} \alpha_i \|x_i\|^2 - \sum_{0\leq i,j\leq N} \alpha_i\alpha_j \|x_i - x_j\|^2.$$

Lemma 2.5 ([16])**.** *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.6 ([7])**.** *Let H be a real Hilbert space, K be a closed convex subset of H and $T : K \to K$ be an asymptotically nonexpansive mapping. Then $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in K such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, as $n \to \infty$, $x = T(x)$.*

Definition 2.7. Let E be a real normed linear space and K be a nonempty subset of E. A mapping $T : K \to K$ is said to be *σ-asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exist two sequences of real numbers $\{k_n\}$, $\{c_n\}$ with $\lim_{n\to\infty} k_n = 0$ and $\sum c_n < \infty$ such that the following inequality holds:

$$\|T^n u - p\| \leq (1 + k_n)\|u - p\| + c_n,$$

for all $u \in K$, $p \in F(T)$ and $n \geq 1$.

Since $\sum c_n < \infty$ implies $\lim_{n\to\infty} c_n = 0$, it follows that every σ-asymptotically quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping. However, the converse is not true. The following Example 2.8 below shows that the class of σ-asymptotically quasi-nonexpansive mappings contains the class of generalized asymptotically quasi-nonexpansive mappings.

Example 2.8. Let $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and define $Tx = \frac{x}{2}\cos(\frac{2}{x})$, if $x \neq 0$ and $Tx = 0$ if $x = 0$. Then $T^n x \to 0$. Clearly, $F(T) = \{0\}$. For each fixed $n \geq 1$, define

$$f_n(x) = \|T^n x\| - \|x\|,$$

for all $x \in K$. Set

$$k_n = \frac{1}{n^2+1}, \quad c_n = \max\left\{\sup_{x\in K} f_n(x),\ \frac{1}{n}\right\} = \max\left\{\sup_{x\in K}\big(\|T^n x\| - \|x\|\big),\ \frac{1}{n}\right\},$$

for all $n \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty}\frac{1}{n^2+1} = 0, \quad \lim_{n\to\infty} c_n = \lim_{n\to\infty}\frac{1}{n} = 0$$

and

$$\begin{aligned}\|T^n x\| - \|x\| &= f_n(x) \le \sup f_n(x)\\ &\le \max\left\{\sup f_n(x), \frac{1}{n}\right\}\\ &= c_n\\ &\le k_n\|x\| + c_n.\end{aligned}$$

Thus, for all $n \ge 1$, the above inequality yields

$$\|T^n x\| \le (1+k_n)\|x\| + c_n.$$

Therefore, T is a generalized asymptotically quasi-nonexpansive mapping with $k_n = \frac{1}{n^2+1}$ and $c_n = \frac{1}{n}$ for all $n \ge 1$. However, we notice that T is not a σ-asymptotically quasi-nonexpansive mapping because $\sum c_n = \infty$.

Proposition 2.9. *Let H be a real Hilbert space, K be a closed convex subset of H and T be a σ-asymptotically quasi-nonexpansive mappings from K into itself. Then $F(T)$ is closed and convex.*

Proof. Clearly, the continuity of T implies that $F(T)$ is closed. Now, we show that $F(T)$ is convex. For any $x, y \in F(T)$ and $t \in (0,1)$, put $z = tx + (1-t)y$. Now, we show that $z = T(z)$. In fact, we have

$$\begin{aligned}\|z - T^n z\|^2 &= \|z\|^2 - 2\langle z, T^n z\rangle + \|T^n z\|^2\\ &= \|z\|^2 - 2\langle tx + (1-t)y, T^n z\rangle + \|T^n z\|^2\\ &= \|z\|^2 - 2t\langle x, T^n z\rangle - 2(1-t)\langle y, T^n z\rangle + \|T^n z\|^2\\ &= \|z\|^2 + t\|x - T^n z\|^2 + (1-t)\|y - T^n z\|^2 - t\|x\|^2 - (1-t)\|y\|^2\\ &\le \|z\|^2 + t[(1+k_n)\|x - z\| + c_n]^2 + (1-t)[(1+k_n)\|y - z\| + c_n]^2\\ &\quad - t\|x\|^2 - (1-t)\|y\|^2\\ &\le \|z\|^2 + t(1+k_n)^2\langle x - z, x - z\rangle + (1-t)(1+k_n)^2\langle y - z, y - z\rangle\\ &\quad - t\|x\|^2 - (1-t)\|y\|^2 + 2t(1+k_n)c_n\|x - z\|\\ &\quad + 2(1-t)(1+k_n)c_n\|y - z\| + c_n^2\\ &\le \big[(1+k_n)^2 - 1\big]\big[t\|x\|^2 + (1-t)\|y\|^2\big] + [1 + (1+k_n)^2]\|z\|^2\\ &\quad - 2(1+k_n)^2[t\langle x, z\rangle + (1-t)\langle y, z\rangle] + 2(1+k_n)c_n[t\|x - z\|\\ &\quad + (1-t)\|y - z\|] + c_n^2\\ &\le \big[(1+k_n)^2 - 1\big]\big[t\|x\|^2 + (1-t)\|y\|^2\big] - [(1+k_n)^2 - 1]\|z\|^2\\ &\quad + 2(1+k_n)c_n[t\|x - z\| + (1-t)\|y - z\|] + c_n^2\\ &\le k_n(k_n+2)\big[t\|x\|^2 + (1-t)\|y\|^2 - \|z\|^2\big] + 2(1+k_n)c_n[t\|x - z\|\\ &\quad + (1-t)\|y - z\|] + c_n^2\ ,\end{aligned}$$

and hence, since $k_n \to 0$ and $c_n \to 0$ as $n \to \infty$, it follows that $\lim_{n\to\infty} \|z - T^n z\|^2 = 0$, which implies that $\lim_{n\to\infty} T^n z = z$. Now, by the continuity of T, we obtain that

$$z = \lim_{n\to\infty} = \lim_{n\to\infty} T(T^{n-1}z) = T(\lim_{n\to\infty} T^{n-1}z) = T(z).$$

Hence $z \in F(T)$ and that $F(T)$ is convex. □

3. Main results

In this section, we establish some strong convergence theorems for finding a common element of the set of solutions for common minimum-norm fixed point and the set of fixed points of a σ-asymptotically quasi-nonexpansive mappings in a Hilbert space.

Theorem 3.1. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T_i : K \to K$ be a σ-asymptotically quasi-nonexpansive mappings with sequences of real numbers $\{k_{n,i}\}$ and $\{c_{n,i}\}$ for each $i = 1, 2, \cdots, N$. Assume that $F := \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_1 \in K, \quad \text{chosen arbitrarily}, \\ v_n = P_K[(1-\alpha_n)u_n], \\ u_{n+1} = \beta_{n,0}u_n + \sum_{i=1}^N \beta_{n,i}T_i^n v_n, \end{cases} \tag{3.1}$$

for each $n \geq 1$, where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$ for each $i = 0, 1, 2, \cdots, N$ satisfying $\beta_{n,0} + \beta_{n,1} + \beta_{n,2} + \cdots + \beta_{n,N} = 1$ for each $n \geq 1$. Then the sequence $\{u_n\}$ converges strongly to a common minimum-norm fixed point of F.

Proof. Since $F(T)$ is closed and convex for any operator $T : K \to K$, $P_{F(T)}0$ is unique. Let $u^* = P_F 0$. Then, from (3.1) and σ-asymptotically quasi-nonexpansive mappings of T_i for each $i \in \{1, 2, \cdots, N\}$, we have

$$\begin{aligned} \|v_n - u^*\| &= \|P_k(1-\alpha_n)u_n - P_k u^*\| \\ &\leq \|(1-\alpha_n)u_n - u^*\| \\ &= \|\alpha_n(0 - u^*) + (1-\alpha_n)(u_n - u^*)\| \\ &\leq \alpha_n\|u^*\| + (1-\alpha_n)\|u_n - u^*\|, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|u_{n+1} - u^*\| &= \|\beta_{n,0}u_n + \sum_{i=1}^N \beta_{n,i}T_i^n v_n - u^*\| \\ &\leq \beta_{n,0}\|u_n - u^*\| + \sum_{i=1}^N \beta_{n,i}\|T_i^n v_n - u^*\| \\ &\leq \beta_{n,0}\|u_n - u^*\| + (1-\beta_{n,0})\big[(1+k_n)\|v_n - u^*\| + c_n\|\big] \\ &\leq \beta_{n,0}\|u_n - u^*\| + (1-\beta_{n,0})(1+k_n)[\alpha_n\|u^*\| \\ &\quad + (1-\alpha_n)\|u_n - u^*\|] + (1-\beta_{n,0})c_n \\ &\leq \beta_{n,0}\|u_n - u^*\| + (1-\beta_{n,0})(1+k_n)(1-\alpha_n)\|u_n - u^*\| \\ &\quad + (1-\beta_{n,0})(1+k_n)\alpha_n\|u^*\| + (1-\beta_{n,0})c_n \\ &\leq [\beta_{n,0} + (1-\beta_{n,0})(1+k_n)(1-\alpha_n)]\|u_n - u^*\| \\ &\quad + (1-\beta_{n,0})(1+k_n)\alpha_n\|u^*\| + (1-\beta_{n,0})c_n \end{aligned}$$

$$
\begin{aligned}
&\leq [1 + k_n(1-\beta_{n,0}) - \alpha_n(1-\beta_{n,0}) - k_n\alpha_n(1-\beta_{n,0})]\|u_n - u^*\| \\
&\quad + (1-\beta_{n,0})(1+k_n)\alpha_n\|u^*\| + (1-\beta_{n,0})c_n \\
&\leq [1 - (1-\beta_{n,0})(-k_n + \alpha_n + k_n\alpha_n)]\|u_n - u^*\| \\
&\quad + (1-\beta_{n,0})(1+k_n)\alpha_n\|u^*\| + (1-\beta_{n,0})c_n \\
&\leq [1 - (1-\beta_{n,0})(\alpha_n(1+k_n) - k_n)]\|u_n - u^*\| \\
&\quad + (1-\beta_{n,0})(1+k_n)\alpha_n\|u^*\| + (1-\beta_{n,0})c_n \\
&\leq \Big(\prod_{i=1}^{n}\beta_{i,0}\Big)\|u_n - u^*\| + (1-\beta_{n-1,0})\|u^*\| + \sum_{j=1}^{n} c_j \\
&\leq b_1\|u_n - u^*\| + (1-b_{n-1})\|u^*\| + \sum_{j=1}^{n} c_j \ ,
\end{aligned}
$$

where $b_1 = \big(\prod_{i=1}^{n}\beta_{i,0}\big)$, $b_{n-1} = \beta_{n-1,0}\beta_{n-2,0}\cdots\beta_{1,0}$ and $\sum_{j=1}^{n} c_j = c_1 + c_2 + \cdots + c_{n-1} + c_n$. Moreover, from (3.2) and Lemma 2.1, it follows that

$$
\begin{aligned}
\|v_n - u^*\|^2 &= \|P_k[(1-\alpha_n)u_n] - P_k u^*\|^2 \\
&\leq \|\alpha_n(0 - u^*) + (1-\alpha_n)(u_n - u^*)\|^2 \\
&\leq (1-\alpha_n)\|u_n - u^*\|^2 - 2\alpha_n\langle u^*, v_n - u^*\rangle.
\end{aligned} \tag{3.3}
$$

Furthermore, from (3.1), Lemma 2.4 and σ-asymptotically quasi-nonexpansive mappings of T_i for each $i = 1, 2, \cdots, N$, it follows that

$$
\begin{aligned}
\|u_{n+1} - u^*\|^2 &= \|\beta_{n,0}u_n + \sum_{i=1}^{N}\beta_{n,i}T_i^n v_n - u^*\|^2 \\
&\leq \beta_{n,0}\|u_n - u^*\|^2 + \sum_{i=1}^{N}\beta_{n,i}\|T_i^n v_n - u^*\|^2 - \sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n - T_i^n v_n\|^2 \\
&\leq \beta_{n,0}\|u_n - u^*\|^2 + (1-\beta_{n,0})[(1+k_n)\|v_n - u^*\| + c_n]^2 \\
&\quad - \sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n - T_i^n v_n\|^2 \\
&\leq \beta_{n,0}\|u_n - u^*\|^2 + (1-\beta_{n,0})[(1+k_n)^2\|v_n - u^*\|^2 + c_n^2 \\
&\quad + 2(1+k_n)c_n\|v_n - u^*\|] - \sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n - T_i^n v_n\|^2 \\
&\leq \beta_{n,0}\|u_n - u^*\|^2 + (1-\beta_{n,0})(1+k_n)^2\|v_n - u^*\|^2 + (1-\beta_{n,0})c_n^2 \\
&\quad + 2(1-\beta_{n,0})(1+k_n)c_n\|v_n - u^*\| - \sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n - T_i^n v_n\|^2,
\end{aligned}
$$

which implies, using (3.2) and (3.3), that

$$
\begin{aligned}
\|u_{n+1} - u^*\|^2 \leq\ & \beta_{n,0}\|u_n - u^*\|^2 + (1-\beta_{n,0})(1+k_n)^2 \\
& [(1-\alpha_n)\|u_n - u^*\|^2 - 2\alpha_n\langle u^*, v_n - u^*\rangle] + (1-\beta_{n,0})c_n^2 \\
& + 2(1-\beta_{n,0})(1+k_n)c_n[\alpha_n\|u^*\| + (1-\alpha_n)\|u_n - u^*\|] \\
& - \sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n - T_i^n v_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big(1-\frac{\theta_n}{\alpha_n}\Big)\|u_n-u^*\|^2+\frac{\theta_n}{\alpha_n}(1+k_n)^2(1-\alpha_n)\|u_n-u^*\|^2\\
&\quad -2\theta_n(1+k_n)^2\langle u^*,v_n-u^*\rangle+\frac{\theta_n}{\alpha_n}c_n^2+2\frac{\theta_n}{\alpha_n}(1+k_n)c_n\alpha_n\|u^*\|\\
&\quad +2\frac{\theta_n}{\alpha_n}(1+k_n)c_n(1-\alpha_n)\|u_n-u^*\|-\sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n-T_i^nv_n\|^2\\
&\leq \Big[1-\frac{\theta_n}{\alpha_n}+\frac{\theta_n}{\alpha_n}(1+k_n)^2(1-\alpha_n)\Big]\|u_n-u^*\|^2\\
&\quad -2\theta_n(1+k_n)^2\langle u^*,v_n-u^*\rangle\\
&\quad +\frac{\theta_n}{\alpha_n}\big[c_n^2+2(1+k_n)c_n\alpha_n\|u^*\|+2(1+k_n)c_n(1-\alpha_n)\|u_n-u^*\|\big]\\
&\quad -\sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n-T_i^nv_n\|^2\\
&\leq \Big[1-\frac{\theta_n}{\alpha_n}+\frac{\theta_n}{\alpha_n}(1+k_n)^2-\frac{\theta_n}{\alpha_n}(1+k_n)^2\alpha_n)\Big]\|u_n-u^*\|^2\\
&\quad -2\theta_n(1+k_n)^2\langle u^*,v_n-u^*\rangle\\
&\quad +\frac{\theta_n}{\alpha_n}\big[c_n^2+2(1+k_n)c_n\alpha_n\|u^*\|+2(1+k_n)c_n(1-\alpha_n)\|u_n-u^*\|\big]\\
&\quad -\sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n-T_i^nv_n\|^2\\
&\leq \Big[1-\theta_n(1+k_n)^2+\frac{\theta_n}{\alpha_n}[(1+k_n)^2-1]\Big]\|u_n-u^*\|^2\\
&\quad -2\theta_n(1+k_n)^2\langle u^*,v_n-u^*\rangle\\
&\quad +\frac{\theta_n}{\alpha_n}\big[c_n^2+2(1+k_n)c_n\alpha_n\|u^*\|+2(1+k_n)c_n(1-\alpha_n)\|u_n-u^*\|\big]\\
&\quad -\sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n-T_i^nv_n\|^2\\
&\leq [1-\theta_n(1+k_n)^2]\|u_n-u^*\|^2+\frac{\theta_n}{\alpha_n}[(1+k_n)^2-1]\|u_n-u^*\|^2\\
&\quad -2\theta_n(1+k_n)^2\langle u^*,v_n-u^*\rangle\\
&\quad +\frac{\theta_n}{\alpha_n}\big[c_n^2+2(1+k_n)c_n\alpha_n\|u^*\|+2(1+k_n)c_n(1-\alpha_n)\|u_n-u^*\|\big]\\
&\quad -\sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n-T_i^nv_n\|^2\\
&\leq (1-\theta_n)\|u_n-u^*\|^2-2\theta_n\langle u^*,v_n-u^*\rangle+[(1+k_n)^2-1]M\\
&\quad -\sum_{i=1}^{N}\beta_{n,0}\beta_{n,i}\|u_n-T_i^nv_n\|^2\\
&\quad +\frac{\theta_n}{\alpha_n}\big[c_n^2+2(1+k_n)c_n\alpha_n\|u^*\|+2(1+k_n)c_n(1-\alpha_n)\|u_n-u^*\|\big] \qquad (3.4)\\
&\leq (1-\theta_n)\|u_n-u^*\|^2-2\theta_n\langle u^*,v_n-u^*\rangle+[(1+k_n)^2-1]M\\
&\quad +\frac{\theta_n}{\alpha_n}\big[c_n^2+2(1+k_n)c_n\alpha_n\|u^*\|+2(1+k_n)c_n(1-\alpha_n)\|u_n-u^*\|\big], \qquad (3.5)
\end{aligned}
$$

for some $M > 0$, where $\theta_n := \alpha_n(1 - \beta_{n,0})$ for all $n \geq 1$.

Now, we consider the following two cases:

Case 1. Suppose that there exists $n \in \mathbb{N}$ such that $\{\|u_n - u^*\|\}$ is non-increasing for all $n \geq N$. In this situation, $\{\|u_n - u^*\|\}$ is convergent. Then it follows from (3.4) that

$$\sum_{i=1}^{N} \beta_{n,0}\beta_{n,i}\|u_n - T_i^n v_n\|^2 \to 0,$$

which implies that

$$u_n - T_i^n v_n \to 0, \tag{3.6}$$

as $n \to \infty$ for each $i \in \{1, 2, \cdots, N\}$. Moreover, from (3.1) and (3.6) and the fact that $\alpha_n \to 0$, we have

$$\begin{aligned} \|u_{n+1} - u_n\| &= \left\|\beta_{n,0}u_n + \sum_{i=1}^{N} \beta_{n,i}T_i^n v_n - u_n\right\| \\ &= \sum_{i=1}^{N} \beta_{n,i}\|T_i^n v_n - u_n\| \\ &= \beta_{n,1}\|T_1^n v_n - u_n\| + \cdots + \beta_{n,N}\|T_i^n v_n - u_n\| \to 0, \end{aligned} \tag{3.7}$$

and

$$\|v_n - u_n\| = \|P_k[(1 - \alpha_n)u_n] - P_k u_n\| \leq \| - \alpha_n u_n\| \to 0, \tag{3.8}$$

as $n \to \infty$ and hence, from (3.7) and (3.8), we have

$$\|v_{n+1} - v_n\| \leq \|v_{n+1} - u_{n+1}\| + \|u_{n+1} - u_n\| + \|u_n - v_n\| \to 0, \tag{3.9}$$

as $n \to \infty$. Furthermore, from (3.6) and (3.8), it follows that

$$\|v_n - T_i^n v_n\| \leq \|v_n - u_n\| + \|u_n - T_i^n v_n\| \to 0, \tag{3.10}$$

as $n \to \infty$. Therefore, since

$$\begin{aligned} \|v_n - T_i v_n\| \leq\ & \|v_n - v_{n+1}\| + \|v_{n+1} - T_i^{n+1}v_{n+1}\| \\ & + \|T_i^{n+1}v_{n+1} - T_i^{n+1}v_n\| + \|T_i^{n+1}v_n - T_i v_n\| \\ \leq\ & \|v_n - v_{n+1}\| + \|v_{n+1} - T_i^{n+1}v_{n+1}\| \\ & + [(1 + k_{n+1})\|v_{n+1} - v_n\| + c_n] + \|T_i^{n+1}v_n - T_i v_n\|, \end{aligned} \tag{3.11}$$

it follows from (3.9), (3.10), (3.11) and the uniform continuity of T_i that

$$\|v_n - T_i v_n\| \to 0, \tag{3.12}$$

as $n \to \infty$ for each $i = 1, 2, \cdots, N$. Let $\{v_{n_k}\}$ be subsequence of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle u^*, v_n - u^*\rangle = \lim_{k\to\infty}\langle u^*, v_{n_k} - u^*\rangle,$$

and $v_{n_k} \rightharpoonup z$. Then, from (3.8), we have $u_{n_k} \rightharpoonup z$. Therefore, by Lemma 2.3, we obtain

$$\limsup_{n\to\infty}\langle u^*, v_n - u^*\rangle = \lim_{k\to\infty}\langle u^*, v_{n_k} - u^*\rangle = \langle u^*, z - u^*\rangle \geq 0. \tag{3.13}$$

Now, we show that $u_{n+1} \to u^*$ as $n \to \infty$. But, from (3.12) and Lemma 2.6, it follows that $z \in F(T_i)$ for each $i \in \{1, 2, \cdots, N\}$ and $z \in \bigcap_i^N F(T_i)$. Then, from (3.5), we have

$$\begin{aligned}\|u_{n+1} - u^*\|^2 \leq (1 - \theta_n)\|u_n - u^*\|^2 - 2\theta_n \langle u^*, v_n - u^*\rangle + [(1 + k_n)^2 - 1]M \\ + \frac{\theta_n}{\alpha_n}\big[c_n^2 + 2(1 + k_n)c_n\alpha_n\|u^*\| + 2(1 + k_n)c_n(1 - \alpha_n)\|u_n - u^*\|\big],\end{aligned} \tag{3.14}$$

for some $M > 0$. We also notice that

$$\limsup_{n\to\infty} \theta_n = \limsup_{n\to\infty} \alpha_n(1 - \beta_{n,0}) \leq \limsup_{n\to\infty} \alpha_n \cdot (1 - \liminf_{n\to\infty} \beta_{n,0}) = 0 \cdot (1 - a) = 0,$$

and

$$\sum_{n=1}^{\infty} \theta_n = \sum_{n=1}^{\infty} \alpha_n(1 - \beta_{n,0}) \geq \sum_{n=1}^{\infty} \alpha_n \cdot (1 - \limsup_{n\to\infty} \beta_{n,0}) = (1 - b)\sum_{n=1}^{\infty} \alpha_n = \infty.$$

Thus, $\lim_{n\to\infty} \theta_n = 0$ and $\sum_{n=1}^{\infty} \theta_n = \infty$. Now it follows from (3.14) and Lemma 2.2 that $\|u_n - u^*\| \to 0$ as $n \to \infty$. Consequently, $u_n \to u^*$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - u^*\| \leq \|u_{n_i+1} - u^*\|,$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.5, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\|u_{m_k} - u^*\| \leq \|u_{m_k+1} - u^*\|, \quad \|u_k - u^*\| \leq \|u_{n_i+1} - u^*\|,$$

for all $k \in \mathbb{N}$. Then, from (3.4) and the fact that $\theta_n \to 0$, we have

$$\begin{aligned}\sum_{i=1}^{N} \beta_{m_k,0}\beta_{m_k,i}\|u_{m_k} - T_i^{m_k} v_{m_k}\|^2 \leq \|u_{m_k} - u^*\|^2 - \|u_{m_k+1} - u^*\|^2 - \theta_{m_k}\|u_{m_k} - u^*\|^2 \\ - 2\theta_{m_k}\langle u^*, v_{m_k} - u^*\rangle \\ + [(1 + k_{m_k})^2 - 1]M + \frac{\theta_{m_k}}{\alpha_{m_k}}\big[c_{m_k}^2 + 2(1 + k_{m_k})c_{m_k}\alpha_{m_k}\|u^*\| \\ + 2(1 + k_{m_k})c_{m_k}(1 - \alpha_{m_k})\|u_{m_k} - u^*\|\big] \to 0,\end{aligned}$$

as $k \to \infty$. This implies that $u_{m_k} - T_i^{m_k} v_{m_k} \to 0$ as $k \to \infty$. Thus, following the method of Case 1, we obtain that $u_{m_k} - v_{m_k} \to 0$ and $v_{m_k} - T_i v_{m_k} \to 0$ as $k \to \infty$ for each $i = 1, 2, \cdots, N$ and hence there exists $z_1 \in F$ such that

$$\limsup_{n\to\infty} \langle u^*, v_{m_k} - u^*\rangle = \lim_{k\to\infty} \langle u^*, v_{m_k} - u^*\rangle = \langle u^*, z_1 - u^*\rangle \geq 0. \tag{3.15}$$

Then it follows form (3.5) that

$$\begin{aligned}\|u_{m_k+1} - u^*\|^2 \leq (1 - \theta_{m_k})\|u_{m_k} - u^*\|^2 - 2\theta_{m_k}\langle u^*, v_{m_k} - u^*\rangle \\ + [(1 + k_{m_k})^2 - 1]M + \frac{\theta_{m_k}}{\alpha_{m_k}}\big[c_{m_k}^2 + 2(1 + k_{m_k})c_{m_k}\alpha_{m_k}\|u^*\| \\ + 2(1 + k_{m_k})c_{m_k}(1 - \alpha_{m_k})\|u_{m_k} - u^*\|\big].\end{aligned} \tag{3.16}$$

Since $\|u_{m_k} - u^*\| \leq \|u_{m_k+1} - u^*\|$, (3.16) implies that

$$\begin{aligned}\theta_{m_k}\|u_{m_k} - u^*\|^2 \leq \|u_{m_k} - u^*\|^2 - \|u_{m_k+1} - u^*\|^2 - 2\theta_{m_k}\langle u^*, v_{m_k} - u^*\rangle \\ + [(1 + k_{m_k})^2 - 1]M + \frac{\theta_{m_k}}{\alpha_{m_k}}\big[c_{m_k}^2 + 2(1 + k_{m_k})c_{m_k}\alpha_{m_k}\|u^*\|\end{aligned}$$

$$
\begin{aligned}
&+2(1+k_{m_k})c_{m_k}(1-\alpha_{m_k})\|u_{m_k}-u^*\|\big] \\
\le\; & -2\theta_{m_k}\langle u^*, v_{m_k}-u^*\rangle + [(1+k_{m_k})^2-1]M \\
&+\frac{\theta_{m_k}}{\alpha_{m_k}}\big[c_{m_k}^2 + 2(1+k_{m_k})c_{m_k}\alpha_{m_k}\|u^*\| \\
&+2(1+k_{m_k})c_{m_k}(1-\alpha_{m_k})\|u_{m_k}-u^*\|\big].
\end{aligned}
$$

In particular, since $\theta_{m_k} > 0$, we have

$$
\begin{aligned}
\|u_{m_k}-u^*\|^2 \le -2\langle u^*, v_{m_k}-u^*\rangle &+ \frac{[(1+k_{m_k})^2-1]M}{\theta_{m_k}} \\
&+\frac{1}{\alpha_{m_k}}\big[c_{m_k}^2 + 2(1+k_{m_k})c_{m_k}\alpha_{m_k}\|u^*\| \\
&+2(1+k_{m_k})c_{m_k}(1-\alpha_{m_k})\|u_{m_k}-u^*\|\big],
\end{aligned}
$$

and so $\|u_{m_k}-u^*\| \to 0$ as $k\to\infty$, which, together with (3.16), gives $\|u_{m_k+1}-u^*\| \to 0$ as $k\to\infty$. But $\|u_k-u^*\| \le \|u_{m_k+1}-u^*\|$ for all $k\in\mathbb{N}$ and so we obtain that $u_k \to u^*$. Therefore, from the above two Cases, we can conclude that the sequence $\{u_n\}$ converges strongly to a point u^* of F which is the common minimum-norm fixed point of the family $\{T_i : i = 1, 2, \cdots, N\}$. This completes the proof. □

If, in Theorem 3.1, we assume that $N = 1$, then we get the following results:

Corollary 3.2. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be a σ-asymptotically quasi-nonexpansive mapping with two sequences of real numbers $\{k_n\}$ and $\{c_n\}$. Assume that $F(T)$ is nonempty. Let $\{u_n\}$ be a sequence generated by*

$$
\begin{cases}
u_1 \in K, \quad \textit{chosen arbitrarily}, \\
v_n = P_K[(1-\alpha_n)u_n], \\
u_{n+1} = \beta_n u_n + (1-\beta_n)T^n v_n,
\end{cases}
\tag{3.17}
$$

for each $n \ge 1$, where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\{\beta_n\} \subset [a,b] \subset (0,1)$ for each $n \ge 1$. Then the sequence $\{u_n\}$ converges strongly to a minimum-norm point of $F(T)$.

If, in Theorem 3.1, we assume that each T_i is an asymptotically nonexpansive mapping and a nonexpansive mapping for $i = 1, 2, \cdots, N$, then the method of proof of Theorem 3.1 provides the following results:

Corollary 3.3 ([30])**.** *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $i \in \{1, 2, \cdots, N\}$, let $T_i : K \to K$ be an asymptotically nonexpansive mapping with sequence of real number $\{k_n\}$. Assume that $F := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{u_n\}$ be a sequence generated by*

$$
\begin{cases}
u_1 \in K, \quad \textit{chosen arbitrarily}, \\
v_n = P_K[(1-\alpha_n)u_n], \\
u_{n+1} = \beta_{n,0} u_n + \sum_{i=1}^{N}\beta_{n,i}T_i^n v_n,
\end{cases}
\tag{3.18}
$$

for each $n \ge 1$, where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\{\beta_{n,i}\} \subset [a,b] \subset (0,1)$ for $i = 1, 2, \cdots, N$ satisfying $\beta_{n,0} + \beta_{n,1} + \beta_{n,2} + \cdots + \beta_{n,N} = 1$ for each $n \ge 1$. Then the sequence $\{u_n\}$ converges strongly to a common minimum-norm point of $F(T_i)$.

Corollary 3.4 ([30]). *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T_i : K \to K$ be a nonexpansive mapping. Assume that $F := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_1 \in K, \quad \textit{chosen arbitrarily}, \\ v_n = P_K[(1-\alpha_n)u_n], \\ u_{n+1} = \beta_{n,0}u_n + \sum_{i=1}^{N} \beta_{n,i}T_i v_n, \end{cases} \tag{3.19}$$

for each $n \geq 1$, where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\beta_{n,i}\} \subset [a,b] \subset (0,1)$ for $i = 1,2,\cdots,N$ satisfying $\beta_{n,0} + \beta_{n,1} + \beta_{n,2} + \cdots + \beta_{n,N} = 1$ for each $n \geq 1$. Then the sequence $\{u_n\}$ converges strongly to a minimum-norm point of $F(T)$.

If, in Corollaries 3.3 and 3.4 we assume that $N = 1$, then we have the following results:

Corollary 3.5 ([30]). *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers. Assume that $F(T)$ is nonempty. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_1 \in K, \quad \textit{chosen arbitrarily}, \\ v_n = P_K[(1-\alpha_n)u_n], \\ u_{n+1} = \beta_n u_n + (1-\beta_n)T^n v_n, \end{cases} \tag{3.20}$$

for each $n \geq 1$, where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\beta_n\} \subset [a,b] \subset (0,1)$ for each $n \geq 1$. Then the sequence $\{u_n\}$ converges strongly to a minimum-norm point of $F(T)$.

Corollary 3.6 ([30]). *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be a nonexpansive mappings with $F(T)$ nonempty. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_1 \in K, \quad \textit{chosen arbitrarily}, \\ v_n = P_K[(1-\alpha_n)u_n], \\ u_{n+1} = \beta_n u_n + (1-\beta_n)T v_n, \end{cases} \tag{3.21}$$

for each $n \geq 1$, where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\beta_n\} \subset [a,b] \subset (0,1)$ for each $n \geq 1$. Then the sequence $\{u_n\}$ converges strongly to a minimum-norm point of $F(T)$.

4. Applications

In this section, we study the problem of finding a minimizer of a continuously Fréchet-differentiable convex functional which has the minimum norm in Hilbert spaces.

We consider the following minimization problem

$$\min_{x \in K} \psi(x), \tag{4.1}$$

where K is a closed convex subset of a real Hilbert space H and $\psi : K \to \mathbb{R}$ is a continuously Fréchet-differentiable convex function. Denote by S the solution set of the minimization problem (4.1), that is,

$$S = \{z \in K : \psi(z) = \min_{x \in K} \psi(x)\}. \tag{4.2}$$

Assume $S \neq \emptyset$. It is known that a point $z \in K$ is a solution of the minimization problem (4.1) if and only if the following optimality condition holds:

$$z \in K, \quad \langle \nabla\psi(z), x - z \rangle \geq 0, \tag{4.3}$$

for all $x \in K$, where $\nabla\psi(x)$ is denotes the gradient of ψ at $x \in K$. It is also known that the optimality condition (4.3) is equivalent to the following fixed point problem

$$z = T_\mu z, \quad T_\mu = P_K(I - \mu\nabla\psi), \tag{4.4}$$

where P_K is the metric projection onto K and $\mu > 0$ is any positive number.

We assume that each T_μ is nonexpansive mappings for some $\mu > 0$, then Corollary 3.6 deduce following result:

Corollary 4.1. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $\psi : K \to \mathbb{R}$ is a continuously Fréchet-differentiable convex function such that $T_\mu := P_K(I - \mu\nabla\psi)$ be a nonexpansive mapping for some $\mu > 0$. Assume that the solution of the minimization problem* (4.1) *is nonempty. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_1 \in K, \quad \textit{chosen arbitrarily}, \\ v_n = P_K[(1-\alpha_n)u_n], \\ u_{n+1} = \beta_n u_n + (1-\beta_n)[P_K(I - \mu\nabla\psi)]v_n, \end{cases} \tag{4.5}$$

for each $n \geq 1$, where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\beta_n\} \subset [a,b] \subset (0,1)$ for each $n \geq 1$. Then the sequence $\{u_n\}$ converges strongly to a common minimum-norm solution of the minimization problem (4.1).

5. Conclusion

In this paper, we use the iterative algorithm proposed by Zegeye and Shahzad [30] which converges strongly to a common minimum-norm fixed point of a finite family of σ-asymptotically quasi-nonexpansive mappings. We also study the convergence analysis of this process, besides proving convexity of this algorithm for the set of common fixed points of a finite family of σ-asymptotically quasi-nonexpansive mappings and boundedness of the sequence of this algorithm. Our main result generalize and improve the recent results of Zegeye and Shahzad [30]. Our result also extend and improve the known results of Yang et al. [27] (Theorems 3.1, 3.2), Yao et al. [28] (Theorems 3.1, 3.2) and Cai et al. [4] (Theorems 3.1, 3.2) by using the above iterative algorithm for finding a minimum-norm fixed point of a nonexpansive mapping in lies of the implicit and explicit methods. Finally, we furnish an application of our main result to find solution of a minimizer of continuously Fréchet-differentiable convex functional which has the minimization problem.

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