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Coincidence type alternatives for Φ–essential maps

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Abstract

In this paper we present some criteria for Φ-essential maps and as a consequence these generate a number of new Leray-Schauder type alternatives.

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1. Introduction

Essential maps were introduced by Granas in [3] and extended in a variety of settings in [1, 2, 4]. Recently a new notion of Φ–essential maps was discussed in [5]. In [5] the author presented some coincidence alternatives in a very general setting. He showed (see Theorem 2.1 below) that if G is Φ–essential and $G \cong F$ (in a particular setting) then Φ and F have a coincidence point. This paper puts criteria on a map G to guarantees that G is Φ–essential so this together with our above result will guarantee that Φ and F have a coincidence point.

Let E be a completely regular topological space and U an open subset of E.

We will consider classes $\mathbf{A}$ and $\mathbf{B}$ of maps.

Definition 1.1. We say $F \in A(\overline{U}, E)$ if $F \in \mathbf{A}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map; here $\overline{U}$ denotes the closure of U in E and $K(E)$ denotes the family of nonempty compact subsets of E.

Definition 1.2. We say $F \in B(\overline{U}, E)$ if $F \in \mathbf{B}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map.

In this paper we fix a $\Phi \in B(\overline{U}, E)$ as indicated in our results.

Definition 1.3. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

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Definition 1.4. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists an upper semicontinuous map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Psi_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $\Psi_1 = F$, $\Psi_0 = G$ and $\{x \in \overline{U} : \ \Phi(x) \cap \Psi(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\}$ is relatively compact (here $\Psi_t(x) = \Psi(x,t)$).

Remark 1.5. If E is a normal topological space the condition

$$\{x \in \overline{U} : \ \Phi(x) \cap \Psi(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\}$$

is relatively compact can be removed in Definition 1.4.

Definition 1.6. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to K(E)$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $J\,(x) \cap \Phi\,(x) \neq \emptyset$.

2. Leray-Schauder nonlinear alternatives.

The following result was established in [5].

Theorem 2.1. *Let E be a normal topological space, U an open subset of E and let $G \in A_{\partial U}(\overline{U}, E)$ be Φ–essential in $A_{\partial U}(\overline{U}, E)$. Suppose there exists an upper semicontinuous map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap \Psi_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1]$ and $\Psi_0 = G$. Then there exists $x \in U$ with $\Phi(x) \cap \Psi_1(x) \neq \emptyset$.*

Remark 2.2. We can replace in Theorem 2.1 the assumption that E is normal with E being completely regular provided in addition we assume $\{x \in \overline{U} : \ \Phi(x) \cap \Psi(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\}$ is relatively compact in the statement of Theorem 2.1.

We now rewrite Theorem 2.1 as a nonlinear alternative of Leray-Schauder type.

Theorem 2.3. *Let E be a normal topological space and U an open subset of E. Suppose $G \in A_{\partial U}(\overline{U}, E)$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$ and $F \in A(\overline{U}, E)$. Also assume there exists an upper semicontinuous map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, and with $\Psi_0 = G$, $\Psi_1 = F$. Then either*

(A1). there exists $x \in \overline{U}$ with $F(x) \cap \Phi(x) \neq \emptyset$,

or

(A2). there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $\Psi_\lambda(x) \cap \Phi(x) \neq \emptyset$.

Proof. Suppose $(A2)$ does not hold and $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$ (otherwise $(A1)$ is true). Then

$$\Psi_\lambda(x) \cap \Phi(x) = \emptyset \quad \text{for} \quad x \in \partial U \quad \text{and} \quad \lambda \in (0,1].$$

Then Theorem 2.1 implies there exists a $x \in U$ with $F(x) \cap \Phi(x) \neq \emptyset$. □

Let $L : E \to E$ be a continuous single valued map (a particular example is when $L = i$, the identity map). We now consider a special case of Theorem 2.3 when $\Phi = L$.

Theorem 2.4. *Let E be a normal topological space and U an open subset of E. Suppose $L : E \to E$ is a continuous map with*

$$L \in \mathbf{B}(\overline{U}, E). \tag{2.1}$$

Assume G, $F \in A(\overline{U}, E)$ with $L(x) \notin G(x)$ for $x \in \partial U$ and suppose G is L–essential in $A_{\partial U}(\overline{U}, E)$. Also suppose there exists an upper semicontinuous map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, and with $\Psi_0 = G$, $\Psi_1 = F$. Then either

(A1). there exists $x \in \overline{U}$ with $L(x) \in F(x)$,

or

(A2). there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $L(x) \in \Psi_\lambda(x)$.

Proof. Note $L \in B(\overline{U}, E)$ and $G \in A_{\partial U}(\overline{U}, E)$ since $G(x) \cap L(x) = \emptyset$ for $x \in \partial U$. The result follows from Theorem 2.3. □

We next discuss L–essential maps (which could be used in Theorem 2.4).

Theorem 2.5. *Let E be a normal topological vector space and U an open subset of E. Suppose $L : E \to E$ is a continuous map with $L(y) \neq 0$ for $y \in E \setminus U$. Let $G \in A(\overline{U}, E)$ and $L \in \mathbf{B}(\overline{U}, E)$. Assume the following conditions hold:*

$$\text{there exists } \ x \in U \ \text{ with } \ L(x) = 0, \tag{2.2}$$

$$L(x) \notin \lambda\, G(x) \ \text{ for } \ x \in \partial U \ \text{ and } \ \lambda \in (0, 1], \tag{2.3}$$

$$\text{for any map } \ Q \in A(E, E) \ \text{ there exists } \ x \in E \ \text{ with } \ L(x) \in Q(x), \tag{2.4}$$

$$\text{there exists a retraction (continuous) } \ r : E \to \overline{U} \tag{2.5}$$

and

$$\begin{cases} \text{for any continuous map } \ \mu : E \to [0, 1] \ \text{ with } \mu(E \setminus U) = 0 \ \text{ and} \\ J \in A_{\partial U}(\overline{U}, E) \ \text{ with } \ J|_{\partial U} = G|_{\partial U} \ \text{ and } \ J \cong G \ \text{ in } \ A_{\partial U}(\overline{U}, E) \\ \text{the map } \ H \in A(E, E) \ \text{ where } \ H(x) = \mu(x)\, J(r(x)). \end{cases} \tag{2.6}$$

Then G is L–essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Now

$$L(x) \notin \lambda\, G(x) \ \text{ for } \ x \in \partial U \ \text{ and } \ \lambda \in [0, 1], \tag{2.7}$$

(note (2.3) and $L(y) \neq 0$ for $y \in E \setminus U$). Now $G \in A_{\partial U}(\overline{U}, E)$ and to show G is L–essential in $A_{\partial U}(\overline{U}, E)$ let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = G|_{\partial U}$ and $J \cong G$ in $A_{\partial U}(\overline{U}, E)$. We must show there exists $x \in U$ with $L(x) \in J(x)$ (note $\Phi = L$). Let

$$D = \left\{ x \in \overline{U} : \ L(x) \in \lambda\, J(x) \ \text{ for some } \ \lambda \in [0, 1] \right\}.$$

Note $D \neq \emptyset$ (see (2.2)), D is closed (note J is upper semicontinuous) and $D \subseteq \overline{U}$. We claim $D \subseteq U$. To see this let $x \in D$ and $x \in \partial U$. Then since $J|_{\partial U} = G|_{\partial U}$ we have

$$L(x) \in \lambda\, J(x) = \lambda\, G(x),$$

which contradicts (2.7). Thus $D \subseteq U$. Now Urysohn's Lemma guarantees there exists a continuous map $\mu : E \to [0, 1]$ with $\mu(E \setminus U) = 0$ and $\mu(D) = 1$. Let $r : E \to \overline{U}$ be as in (2.5) and consider the map H given by $H(x) = \mu(x)\, J(r(x))$. Now (2.4), (2.6) guarantee there exists $x \in E$ with $L(x) \in H(x) = \mu(x)\, J(r(x))$. If $x \in E \setminus U$ then $\mu(x) = 0$, which yields a contradiction since $L(y) \neq 0$ for $y \in E \setminus U$. Thus $x \in U$ so $L(x) \in \mu(x)\, J(x)$. Hence $x \in D$ so $\mu(x) = 1$. Thus $L(x) \in J(x)$ with $x \in U$. □

Remark 2.6. We can remove the assumption that E is normal in the statement of Theorem 2.5 provided we have that (so we need to put conditions on the maps) the set D (see the proof of Theorem 2.5) is relatively compact (note the existence of μ in Theorem 2.5 is then guaranteed since topological vector spaces are completely regular).

A special case of Theorem 2.5 is when $L = i$.

Theorem 2.7. *Let E be a normal topological vector space and U an open subset of E with $0 \in U$. Suppose $G \in A(\overline{U}, E)$ and $i \in \mathbf{B}(\overline{U}, E)$. Assume*

$$x \notin \lambda\, G(x) \ \text{ for } \ x \in \partial U \ \text{ and } \ \lambda \in (0, 1] \tag{2.8}$$

$$\text{any map } \ Q \in A(E, E) \ \text{ has a fixed point} \tag{2.9}$$

and (2.5), (2.6) hold. Then G is i–essential in $A_{\partial U}(\overline{U}, E)$.

Remark 2.8. If U is convex and $G(\partial U) \subseteq U$ then (2.8) holds. To see this note if there exists $x \in \partial U$ and $\lambda \in (0,1]$ with $x \in \lambda\, G(x)$ then since $G(\partial U) \subseteq U$, U convex and $0 \in U$, we have $\lambda\, G(x) \subseteq U$, a contradiction.

The argument in Theorem 2.5 can be extended to a multivalued Φ as can be seen in our next result (here $\Phi \in B(\overline{U}, E)$ is fixed).

Theorem 2.9. *Let E be a normal topological vector space and U an open subset of E. Suppose $\Phi : E \to 2^E$ with $0 \notin \Phi(E \setminus U)$. Let $G \in A(\overline{U}, E)$, $\Phi \in B(\overline{U}, E)$ and assume the following conditions hold:*

$$0 \in \Phi(U), \tag{2.10}$$

$$\Phi(x) \cap \lambda\, G(x) = \emptyset \ \ \textit{for} \ \ x \in \partial U \ \ \textit{and} \ \ \lambda \in (0,1] \tag{2.11}$$

and

$$\textit{for any map} \ \ Q \in A(E,E) \ \ \textit{there exists} \ \ x \in E \ \ \textit{with} \ \ \Phi(x) \cap Q(x) \neq \emptyset. \tag{2.12}$$

Also suppose (2.5) *and* (2.6) *hold. Then G is Φ–essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. Note (2.11) and $0 \notin \Phi(E \setminus U)$ implies

$$\Phi(x) \cap \lambda\, G(x) = \emptyset \ \ \text{for} \ \ x \in \partial U \ \ \text{and} \ \ \lambda \in [0,1]. \tag{2.13}$$

Now $G \in A_{\partial U}(\overline{U}, E)$ and to show G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = G|_{\partial U}$ and $J \cong G$ in $A_{\partial U}(\overline{U}, E)$. Let

$$D = \left\{ x \in \overline{U} : \ \Phi(x) \cap \lambda\, J(x) \neq \emptyset \ \ \text{for some} \ \ \lambda \in [0,1] \right\}.$$

Note $D \neq \emptyset$ (see (2.10)). Also a standard argument (see [5]) guarantees that D is closed. Note $D \subseteq \overline{U}$ and we claim $D \subseteq U$. To see this let $x \in D$ and $x \in \partial U$. Then since $J|_{\partial U} = G|_{\partial U}$ we have $\Phi(x) \cap \lambda\, G(x) \neq \emptyset$, and this contradicts (2.13). Thus $D \subseteq U$. Now Urysohn's Lemma guarantees there exists a continuous map $\mu : E \to [0,1]$ with $\mu(E \setminus U) = 0$ and $\mu(D) = 1$. Let $r : E \to \overline{U}$ be as in (2.5) and consider the map H given by $H(x) = \mu(x)\, J(r(x))$. Now (2.6) and (2.12) guarantee there exists $x \in E$ with $\Phi(x) \cap H(x) \neq \emptyset$ i.e. $\Phi(x) \cap \mu(x)\, J(r(x)) \neq \emptyset$. If $x \in E \setminus U$ then $\mu(x) = 0$, which yields a contradiction since $0 \notin \Phi(E \setminus U)$. Thus $x \in U$ so $\Phi(x) \cap \mu(x)\, J(x) \neq \emptyset$. Hence $x \in D$ so $\mu(x) = 1$, and consequently $\Phi(x) \cap J(x) \neq \emptyset$. □

Remark 2.10. We can remove the assumption that E is normal in the statement of Theorem 2.9 provided we have that the set D (see the proof of Theorem 2.9) is relatively compact.

In our next two results we assume $\Phi : \overline{U} \to 2^E$ (we do not assume $\Phi : E \to 2^E$). Here $\Phi \in B(\overline{U}, E)$ is fixed.

Theorem 2.11. *Let E be a normal topological vector space and U an open subset of E with $0 \in U$. Let $G \in A(\overline{U}, E)$, $\Phi \in B(\overline{U}, E)$ and assume the following condition holds:*

$$G(x) \cap \Phi(x) = \emptyset \ \ \textit{for} \ \ x \in \partial U. \tag{2.14}$$

Suppose (2.5), (2.9) *and the following holds:*

$$\begin{cases} \textit{for any continuous map} \ \ \mu : E \to [0,1] \ \ \textit{with} \ \ \mu(E \setminus U) = 0 \ \ \textit{and} \\ J \in A_{\partial U}(\overline{U}, E) \ \ \textit{with} \ \ J|_{\partial U} = G|_{\partial U} \ \ \textit{and} \ \ J \cong G \ \ \textit{in} \ \ A_{\partial U}(\overline{U}, E) \\ \textit{the map} \ \ H \in A(E,E) \ \ \textit{where} \ \ H(x) = \mu(x)\, [\, J(r(x)) \cap \Phi(r(x))\,]. \end{cases} \tag{2.15}$$

Then G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ [in fact there exists a $x \in U$ with $x \in J(x) \cap \Phi(x)$ where J is described in (2.15)*].*

Proof. Note $G \in A_{\partial U}(\overline{U}, E)$ (see (2.14)). To show G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = G|_{\partial U}$ and $J \cong G$ in $A_{\partial U}(\overline{U}, E)$. We must show there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$. Let

$$D = \left\{ x \in \overline{U} : \ x \in \lambda \left[J(x) \cap \Phi(x) \right] \ \text{ for some } \ \lambda \in [0,1] \right\}.$$

Note $0 \in D$, D is closed and $D \subseteq U$ since if $x \in \partial U$ then $J(x) \cap \Phi(x) = G(x) \cap \Phi(x) = \emptyset$. Now Urysohn's Lemma guarantees there exists a continuous map $\mu : E \to [0,1]$ with $\mu(E \setminus U) = 0$ and $\mu(D) = 1$. Let $r : E \to \overline{U}$ be as in (2.5) and consider the map H given by

$$H(x) = \mu(x) \left[J(r(x)) \cap \Phi(r(x)) \right].$$

Now (2.9) and (2.15) guarantee there exists $x \in E$ with $x \in \mu(x) \left[J(r(x)) \cap \Phi(r(x)) \right]$. If $x \in E \setminus U$ then $\mu(x) = 0$ so $x = 0$, a contradiction since $0 \in U$. Thus $x \in U$ so $x \in \mu(x) \left[J(x) \cap \Phi(x) \right]$. Hence $x \in D$ so $\mu(x) = 1$. Thus $x \in U$ and $x \in J(x) \cap \Phi(x)$. □

Remark 2.12. A special case of Theorem 2.11 is when $G = i$ (the identity map) or $G = 0$ (the zero map). Theorem 2.11 was motivated in part by our result in [4, Theorem 2.9] (note in [4, Theorem 2.9] the assumption $0 \in \psi(\partial U)$ in not needed and also there are some typos in the proof there).

A more general version of Theorem 2.11 is the following result.

Theorem 2.13. *Let E be a normal topological vector space and U an open subset of E. Let $G \in A(\overline{U}, E)$, $\Phi \in B(\overline{U}, E)$, $\Psi : E \to 2^E$ with $\Psi : \overline{U} \to K(E)$ an upper semicontinuous map and $0 \notin \Psi(E \setminus U)$. Also assume the following conditions hold:*

$$0 \in \Psi(U) \tag{2.16}$$

and

$$\textit{for any map } \ Q \in A(E, E) \ \textit{ there exists } \ x \in E \ \textit{ with } \ \Psi(x) \cap Q(x) \neq \emptyset. \tag{2.17}$$

Suppose (2.5), (2.14) *and* (2.15) *hold. Then G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ [in fact there exists a $x \in U$ with $\Psi(x) \cap [\, J(x) \cap \Phi(x)\,] \neq \emptyset$ where J is described in* (2.15)*].*

Proof. To show G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ let $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = G|_{\partial U}$ and $J \cong G$ in $A_{\partial U}(\overline{U}, E)$. We must show there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$. Let

$$D = \left\{ x \in \overline{U} : \ \Psi(x) \cap \lambda \left[J(x) \cap \Phi(x) \right] \neq \emptyset \ \text{ for some } \ \lambda \in [0,1] \right\}.$$

Note $D \neq \emptyset$ (see (2.16)), D is closed and $D \subseteq U$ since if $x \in \partial U$ then $J(x) \cap \Phi(x) = G(x) \cap \Phi(x) = \emptyset$. Now Urysohn's Lemma guarantees there exists a continuous map $\mu : E \to [0,1]$ with $\mu(E \setminus U) = 0$ and $\mu(D) = 1$. Let $r : E \to \overline{U}$ be as in (2.5) and consider the map H given by

$$H(x) = \mu(x) \left[J(r(x)) \cap \Phi(r(x)) \right].$$

Now (2.15), (2.17) guarantee there exists $x \in E$ with $\Psi(x) \cap \mu(x) \left[J(r(x)) \cap \Phi(r(x)) \right] \neq \emptyset$. If $x \in E \setminus U$ then $\mu(x) = 0$, which yields a contradiction since $0 \notin \Psi(E \setminus U)$. Thus $x \in U$ so $\Psi(x) \cap \mu(x) \left[J(x) \cap \Phi(x) \right] \neq \emptyset$. Hence $x \in D$ so $\mu(x) = 1$ and $\Psi(x) \cap [\, J(x) \cap \Phi(x)\,] \neq \emptyset$. □

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