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New fixed point theorem for φ-contractions in KM-fuzzy metric spaces

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Abstract

In this paper, by modifying the class of gauge functions Φ_w and Φ_{w^*}, a new fixed point theorem for φ-contractions in KM-fuzzy metric spaces with a t-norm of H-type is established. We also give an example to illustrate the validity of our main results.

Keywords: Fixed point, Menger probabilistic metric space, H-type, φ-contraction.
2010 MSC: 47H10, 47H09.

1. Introduction

Fuzzy metric spaces were introduced by Kramosil and Mickálek [7], which could be considered as modifications of the concept of probabilistic metric space given by Menger. Then the existence of fixed points or solutions of nonlinear equations under various conditions in Menger spaces were studied by some authors. But many results were obtained under the assumption that φ satisfies $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for $t > 0$ and some other conditions. Ćirić has pointed out that the condition $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for $t > 0$ is very strong and difficult for testing in practice. Therefore, a natural question arises, that is, whether this condition can be omitted or improved.

Ćirić firstly tried to answer this question by using the condition (CBW), and gave a fixed point theorem in Menger PM spaces [2]. Jachymiski [6] gave a counterexample to show that there exists a probabilistic φ-contraction satisfying the condition (CBW), but it has no fixed point. Moreover, Jachymiski provided a corrected version. After the paper was accepted for publication, Ćirić has found the error and pointed out that the result in [2] would be correct, if we replace condition (CBW) by the Boyd-Wong condition [1].

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Recently, Fang [3] improved and generalized those theorems and introduced a new fixed point theorem for probabilistic φ-contractions in KM-fuzzy metric spaces, where the gauge function φ only needs to satisfy $\varphi \in \Phi_w$. Then Hua et al. [5] improved the results of Fang and introduced a new fixed point theorem for probabilistic φ-contractions in Menger spaces, where the gauge function φ only needs to satisfy $\varphi \in \Phi_{w^*}$.

In this paper, under φ satisfying that for $t > 0$, there exist $r > t$ and $N \in \mathbb{N}^+$ such that $\varphi^n(r) < t$ for $n \geq N$, we prove a fixed point theorem for φ-contractions in fuzzy metric spaces and give an example to illustrate our result.

2. Preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{N}^+$ be the set of all positive integer numbers, and $\mathbb{N} = \mathbb{N}^+ \cup \{0\}$.

A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm, if the following conditions are satisfied:

- $\Delta(a, 1) = a$;
- $\Delta(a, b) = \Delta(b, a)$;
- $a \geq b, c \geq d \Longrightarrow \Delta(a, c) \geq \Delta(b, d)$;
- $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Definition 2.1 ([4])**.** A t-norm Δ is said to be of H-type, if the family of functions $\{\Delta^m(t)\}_{m\in\mathbb{N}}$ is equicontinuous at $t = 1$, where $\Delta^0(t) = t$ and $\Delta^m(t) = \Delta(t, \Delta^{m-1}(t))$, $m \in \mathbb{N}^+$, $t \in [0, 1]$.

Example 2.2 ([9])**.** Let $\sigma \in [0, 1]$ be a real number and let Δ be a t-norm. Define $\Delta_\sigma(x, y) = \Delta(x, y)$, if $\max\{x, y\} \leq 1 - \sigma$, and $\Delta_\sigma(x, y) = \min\{x, y\}$, if $\max\{x, y\} > 1 - \sigma$. Then Δ_σ is a t-norm of H-type.

Definition 2.3 ([7])**.** A fuzzy metric space in the sense of Kramosil and Michalek is a triple (X, M, Δ) where X is a nonempty set, Δ is a t-norm and M is fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions for $x, y, z \in X$ and $s, t > 0$:

(FM-1) $M(x, y, 0) = 0$;

(FM-2) $M(x, y, t) = 1$ for $t > 0$, if and only if $x = y$;

(FM-3) $M(x, y, t) = M(y, x, t)$;

(FM-4) $M(x, z, t + s) = \Delta(M(x, y, t), \Delta(y, z, s))$;

(FM-5) $M(x, y, \cdot) : R^+ \to [0, 1]$ is left continuous.

From (FM-2) and (FM-5), we deduce that if $t > s > 0$, then

$$M(x, y, t) = M(x, y, t - s + s) \geq \Delta(M(x, y, s), M(y, y, t - s)) = M(x, y, s).$$

Hence, if $(X, M, *)$ is a KM-fuzzy metric space, then $M(x, y, \cdot)$ is a nondecreasing in $\mathbb{R}^+$ for $x, y \in X$.

Definition 2.4 ([8])**.** Let (X, M, Δ) be a KM-fuzzy metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to $x \in X$, if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for $t > 0$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be a Cauchy sequence, if for $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for $m, n \geq n_0$. A fuzzy metric space is called complete, if every Cauchy sequence is convergent in X.

3. Main results

Denote by Φ_w the family of all functions $\varphi : (0, \infty) \to (0, \infty)$ such that the following property is fulfilled [3].

$$\text{For } t > 0, \text{ there exists } r \geq t \text{ such that } \lim_{n\to\infty} \varphi^n(r) = 0. \tag{3.1}$$

Let Φ_{w^*} denote the class of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition [5].

$$\text{For } t_1, t_2 > 0, \text{ there exists } r \geq \max\{t_1, t_2\} \text{ ,such that } \varphi^n(r) < \min\{t_1, t_2\} \text{ for } n \geq N(r). \tag{3.2}$$

Let Φ_k denote the set of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following property is fulfilled.

$$\text{For } t > 0, \text{ there exists } r > t \text{ and } N(r) \in \mathbb{N}^+, \text{ such that } 0 < \varphi^n(r) < t \text{ for } n \geq N(r). \tag{3.3}$$

Then we give an example to explain that (3.3) is different from (3.1) and (3.2).

Example 3.1. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\varphi(t) = \begin{cases} \frac{t}{1+t}, & 0 \leq t < 1, \\ \frac{4-t}{3}, & 1 \leq t < 2, \\ [\frac{t}{2}], & 2 \leq t < \infty. \end{cases} \tag{3.4}$$

It is obvious that $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for $t \in (0, 1) \cup (1, 2)$ and $\lim\limits_{n\to\infty} \varphi^n(t) = 1$ for $t \in [2, +\infty) \cup \{1\}$, which implies $\varphi \notin \Phi_w$. For $t_1 = 0.5$, $t_2 = 3$ and $r \geq \max\{t_1, t_2\}$, we have $\lim\limits_{n\to\infty} \varphi^n(r) = 1 > t_1$, which implies $\varphi \notin \Phi_{w^*}$. For $t \in (0, 1)$, there exists $r \in (0, 1) \cup (1, 2)$ such that $\lim\limits_{n\to\infty} \varphi^n(r) = 0$. If $t = 1$, there exists $r \in (1, 2)$, such that $\lim\limits_{n\to\infty} \varphi^n(r) = 0$. For $t \geq 2$, there exists $r > t$ such that $\lim\limits_{n\to\infty} \varphi^n(r) = 1$. So we obtain $\varphi \in \Phi_k$.

Suppose that $\varphi \in \Phi_k$. Then for each $t > 0$, we have $L_t = \{r > t |\ 0 < \varphi^n(r) < t \text{ for } n \geq N(r)\} \neq \varnothing$. Let $\varphi \in \Phi_k$ be a function. For any $t_1 > 0$, there exists $L_{t_1} = \{r > t_1 | 0 < \varphi^n(r) < t_1, \text{ for } n \geq N(r)\}$ such that $L_{t_1} \neq \emptyset$. Then for $t_0^{(1)} \in L_{t_1}$, there exists $N(t_0^{(1)}) \in \mathbb{N}$ such that $0 < \varphi^n(t_0^{(1)}) < t_1$ for $n \geq N(t_0^{(1)})$. Denote $t_2 = \varphi^{N(t_0^{(1)})}(t_0^{(1)}) < t_1$. Then by the same way, there exists $t_0^{(m)} \in L_{t_m}$, such that $0 < t_{m+1} = \varphi^{N(t_0^{(m)})}(t_0^{(m)}) < t_m$. Then we obtain $\{t_m\}$ is a decreasing sequence. Denote $T = \{\{t_m\} | t_{m+1} = \varphi^{N(t_0^{(m)})}(t_0^{(m)}),\ t_0^{(m)} \in L_{t_m},\ m \in \mathbb{N}^+\}$. Then we obtain the following results.

Lemma 3.2. Let $\varphi \in \Phi_k$ be given. Then for each $t_1 > 0$, we have $\inf\limits_{\{t_m\}\in T} \{t_0 | \lim\limits_{m\to\infty} t_m = t_0\} = 0$.

Proof. It is obvious that $\inf\limits_{\{t_m\}\in T} \{t_0 | \lim\limits_{m\to\infty} t_m = t_0\} \geq 0$. Suppose that $\inf\limits_{\{t_m\}\in T} \{t_0 | \lim\limits_{m\to\infty} t_m = t_0\} = t' > 0$. Since $\{t_m\}$ is a decreasing sequence for $\{t_m\} \in T$, we have $t_m > t'$ for $m \in \mathbb{N}^+$, i.e.,

$$\varphi^{N(x)}(x) > t' \text{ for } \ m \in \mathbb{N}^+. \tag{3.5}$$

Since $\varphi \in \Phi_k$, there exists $t'_0 > t'$, such that

$$0 \leq \varphi^{N(t'_0)}(t'_0) < t'. \tag{3.6}$$

For $\varepsilon_0 = \frac{t'_0 - t'}{2}$, by the definition of infimum, there exists $\{y_m\} \in T$ such that

$$t' \leq \lim_{m\to\infty} y_m < t' + \varepsilon_0 = \frac{t'_0 + t'}{2} < t'_0.$$

Hence there exists $N \in \mathbb{N}^+$, such that

$$y_m < \frac{t'_0 + t'}{2} < t'_0 \text{ for } m \geq N,$$

which implies $t'_0 \in L_{y_m}$. By (3.5), we have $\lim\limits_{n\to\infty} \varphi^n(t'_0) > t'$, which contradicts (3.6). Thus, the conclusion follows. $\square$

Lemma 3.3. Let $\varphi \in \Phi_k$ be a function. For any $t > 0$ and each $r \in L_t$, there exists $N'(r) \in \mathbb{N}^+$ such that $\varphi^{N'(r)}(r) < t \leq \varphi^{N'(r)-1}(r)$ and $\varphi^n(r) < t$ for $n \geq N'(r)$.

Proof. By $\varphi \in \Phi_k$, for each $r \in L_t$, there exists $N(r) \in \mathbb{N}^+$ such that $\varphi^n(r) < t$ for $n \geq N(r)$. It is easy to see that $N(r) \geq 1$, because of $\varphi^0(r) = r > t$. Thus, the set $\{N(r) \in \mathbb{N}^+ | \varphi^n(r) < t, \text{ for } n \geq N(r)\}$ has a lower bound, i.e., there exists $N'(r) = \inf\{N(r) \in \mathbb{N}^+ | \varphi^{N(r)}(r) < t \text{ for } n \geq N(r)\}$ such that $\varphi^{N'(r)}(r) < t \leq \varphi^{N'(r)-1}(r)$. □

Lemma 3.4. Let (X, M, Δ) be a KM-fuzzy metric space with a t-norm Δ of H-type satisfying $\lim_{t\to\infty} M(x, y, t) = 1$ for $x, y \in X$. Let $\{x_n\}$ be a sequence in (X, M, Δ). If there exists a function $\varphi \in \Phi_k$, such that

(1) $\varphi(t) > 0$ for $t > 0$;

(2) $M(x_n, x_m, \varphi(t)) \geq M(x_{n-1}, x_{m-1}, t)$ for $n, m \in \mathbb{N}^+$ and $t > 0$,

then $\{x_n\}$ is a Cauchy sequence in X.

Proof. It is evident that the condition (1) implies that $\varphi^n(t) > 0$ for $n \in \mathbb{N}^+$ and $t > 0$. We now prove that

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \text{ for } t > 0.$$

By the condition (2), we have

$$M(x_n, x_{n+1}, \varphi(t)) \geq M(x_{n-1}, x_n, t) \text{ for } n \in \mathbb{N}^+ \text{ and } t > 0. \tag{3.7}$$

By noting that $M(x_0, x_1, t) \to 1$ as $t \to \infty$, for $\varepsilon \in (0, 1]$, there exists $t_1 > 0$ such that $M(x_0, x_1, t_1) > 1 - \varepsilon$. Since $\varphi \in \Phi_k$, there exist $t_0^{(1)} \in L_{t_1}$ and $N(t_0^{(1)}) \in \mathbb{N}^+$, such that $\varphi^n(t_0^{(1)}) \geq 0$ for $n \geq N(t_0^{(1)})$.

By Lemma 3.2, we have $\inf_{\{t_m\}\in T}\{t_0 | \lim_{m\to\infty} t_m = t_0\} = 0$. Then for each $t > 0$, there exists $\{t_m\}$ such that $\lim_{m\to\infty} t_m = t_0 < t$. Hence there exists $k \in \mathbb{N}^+$, such that $t_m < t$ for $m \geq k$. By $\varphi \in \Phi_k$, there exists $N(t_0^{(m)}) \in \mathbb{N}^+$, such that $\varphi^n(t_0^{(m)}) < t$ for $n \geq N(t_0^{(m)})$ and $m \geq k$. Since the distribution function is nondecreasing, by (3.7), we obtain

$$\begin{aligned}
1 - \varepsilon &< M(x_0, x_1, t_1) \leq M(x_0, x_1, t_0^{(1)}) \leq M(x_1, x_2, \varphi(t_0^{(1)})) \leq \cdots \leq M(x_{n_1}, x_{n_1+1}, \varphi^{N(t_0^{(1)})}(t_0^{(1)})) \\
&\leq M(x_{n_1}, x_{n_1+1}, t_0^{(2)}) \leq M(x_{n_1+1}, x_{n_1+2}, \varphi(t_0^{(2)})) \leq \cdots \leq M(x_{n_1+n_2}, x_{n_1+n_2+1}, \varphi^{N(t_0^{(2)})}(t_0^{(2)})) \\
&\leq M(x_{n_1+\cdots+n_{m-1}}, x_{1+n_1+\cdots+n_{m-1}}, t_0^{(m)}) \leq M(x_{n_1+\cdots+n_{m-1}+1}, x_{n_1+\cdots+n_{m-1}+2}, \varphi(t_0^{(m)})) \leq \cdots \\
&\leq M(x_{n_1+\cdots+n_m}, x_{n_1+\cdots+n_m+1}, \varphi^{N(t_0^{(m)})}(t_0^{(m)})) \leq M(x_n, x_{n+1}, t),
\end{aligned}$$

for $n \geq \sum_{i=1}^{m} n_i$, $m \geq k$. Hence $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ holds for $t > 0$.

Then by Lemma 3.3, there exists an $N'(r) \in \mathbb{N}^+$ such that $\varphi^{N'(r)}(r) < t \leq \varphi^{N'(r)-1}(r)$. Let $n \in \mathbb{N}$ be given, for any $k \in \mathbb{N}$ and $r \in L_t$, we have

$$\begin{aligned}
M(x_{n+1}, x_{n+k+1}, t) &= M(x_{n+1}, x_{n+k+1}, t - \varphi^{N'(r)}(r) + \varphi^{N'(r)}(r)) \\
&\geq \Delta(M(x_n, x_{n+1}, t - \varphi^{N'(r)}(r)), M(x_{n+1}, x_{n+k+1}, \varphi^{N'(r)}(r))) \\
&\geq \Delta(M(x_n, x_{n+1}, t - \varphi^{N'(r)}(r)), M(x_n, x_{n+k}, \varphi^{N'(r)-1}(r))) \\
&\geq \Delta(M(x_n, x_{n+1}, t - \varphi^{N'(r)}(r)), M(x_n, x_{n+k}, t)) \\
&\geq \Delta(M(x_n, x_{n+1}, t - \varphi^{N'(r)}(r)), \Delta^{k-1}(M(x_n, x_{n+1}, t - \varphi^{N'(r)}(r)))).
\end{aligned} \tag{3.8}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. For any $0 < \varepsilon < 1$, by the hypothesis, $\Delta^n(t)$ is equicontinuous at $t = 1$ and $\Delta^n(1) = 1$. So for any $0 < \varepsilon < 1$, there exists $\sigma > 0$, such that $\Delta^n(s) > 1 - \varepsilon$

for $s \in (1-\sigma, 1]$. Since $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for $t > 0$, we have $\lim_{n\to\infty} M(x_n, x_{n+1}, t - \varphi^{N'(r)}(r)) = 1$. Hence there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_{n+1}, t - \varphi^{N'(r)}(r)) > 1 - \sigma$ for $n \geq n_0$. Since $\Delta^n(s) > 1 - \varepsilon$ for $s \in (1-\sigma, 1]$ for $n \in \mathbb{N}$, by (3.8), we have $M(x_n, x_{n+k}, t) > 1 - \varepsilon$ for $n \geq n_0$ and $k \in \mathbb{N}$. This shows that $\{x_n\}$ is a Cauchy sequence. □

Lemma 3.5. Let (X, M, Δ) be a KM-fuzzy space and $x, y \in X$ satisfying $\lim_{n\to\infty} M(x, y, t) = 1$ for $x, y \in X$. If there exists a function $\varphi \in \Phi_k$ such that

$$M(x, y, \varphi(t)) \geq M(x, y, t) \text{ for } t > 0, \tag{3.9}$$

then we have $x = y$.

Proof. It is easy to show that (3.9) implies that $\varphi(t) > 0$ for $t > 0$. In fact, suppose that there exists some $t_0 > 0$, such that $\varphi(t_0) = 0$. Then it follows from (3.9) that $0 = M(x, x, \varphi(t_0)) \geq M(x, x, t_0) = 1$, which is a contradiction. Hence $\varphi^n(t) > 0$ for $n \in \mathbb{N}$.

In order to prove that $x = y$, we need to show that $M(x, y, t) = 1$ for $t > 0$. Owing to $M(x, y, t) \to 1$ as $t \to \infty$, for $\varepsilon \in (0, 1]$, there exists $t_1 > 0$ such that $M(x, y, t_1) > 1 - \varepsilon$. Since $\varphi \in \Phi_k$, there exists $t_0^{(1)} \in L_{t_1}$ and $N(t_0^{(1)}) \in \mathbb{N}^+$, such that $0 < \varphi^n(t_0^{(1)}) < t_1$ for $n \geq N(t_0^{(1)})$.

By Lemma 3.2, we have $\inf_{\{t_m\}\in T}\{t_0 | \lim_{m\to\infty} t_m = t_0\} = 0$. Then for each $t > 0$, there exists $\{t_m\}$ such that $\lim_{m\to\infty} t_m < t$. So there exists $k \in \mathbb{N}^+$ such that $t_m < t$ for $m \geq k$. By $\varphi \in \Phi_k$, there exists $N(t_0^{(m)}) \in \mathbb{N}$ such that $\varphi^{N(t_0^{(m)})}(t_0^{(m)}) < t$ for $m \geq k$. Since the distribution function is nondecreasing, by (3.9), we obtain

$$\begin{aligned}
1 - \varepsilon < M(x, y, t_1) &\leq M(x, y, t_0^{(1)}) \leq M(x, y, \varphi(t_0^{(1)})) \leq \cdots \leq M(x, y, \varphi^{N(t_0^{(1)})}(t_0^{(1)})) \\
&\leq M(x, y, t_0^{(2)}) \leq M(x, y, \varphi(t_0^{(2)})) \leq \cdots \leq M(x, y, \varphi^{N(t_0^{(2)})}(t_0^{(2)})) \leq \cdots \\
&\leq M(x, y, \varphi^{N(t_0^{(m-1)})}(t_0^{(m-1)})) \leq M(x, y, t_0^{(m)}) \leq M(x, y, \varphi(t_0^{(m)})) \leq \cdots \\
&\leq M(x, y, \varphi^{N(t_0^{(m)})}(t_0^{(m)})) \leq M(x, y, t),
\end{aligned}$$

for $m \geq k$, which implies $M(x, y, t) = 1$ for $t > 0$, i.e., $x = y$. □

Theorem 3.6. Let (X, M, Δ) be a complete KM-fuzzy metric space with a t-norm Δ of H-type satisfying $\lim_{t\to\infty} M(x, y, t) = 1$ for $x, y \in X$. If $T : X \to X$ is a probabilistic φ-contraction, i.e., it satisfies condition

$$M(Tx, Ty, \varphi(t)) \geq M(x, y, t) \text{ for } x, y \in X \text{ and } t > 0, \tag{3.10}$$

where $\varphi \in \Phi_k$, then T has a unique fixed point $x_* \in X$ and $\{T^n(x_0)\}$ converges to $x_* \in X$ for $x_0 \in X$.

Proof. It is not hard to see that (3.10) implies that $\varphi(t) > 0$ for $t > 0$. In fact, if there exists some $t_0 > 0$ such that $\varphi(t_0) = 0$. Then we have $0 = M(Tx, Tx, \varphi(t_0)) \geq M(x, x, t_0) = 1$, which is a contradiction. Let $x_0 \in X$ and $x_n = Tx_{n-1}$ for $n \in \mathbb{N}^+$. By (3.10), we have

$$M(x_n, x_m, \varphi(t)) = M(Tx_{n-1}, Tx_{m-1}, \varphi(t)) \geq M(x_{n-1}, x_{m-1}, t) \text{ for } n, m \in \mathbb{N}^+ \text{ and } t > 0.$$

By Lemma 3.4, we conclude that $\{x_n\}$ is Cauchy sequence of (X, M, Δ). Since X is complete, we assume that $x_n \to x_* \in X$.

Now we prove that x_* is a fixed point of T. By (FM-4), we have

$$M(x_*, Tx_*, t) \geq \Delta(M(x_*, x_{n+1}, \frac{t}{2}), M(Tx_n, Tx_*, \frac{t}{2})) \geq \Delta(a_n, a_n), \tag{3.11}$$

where $a_n = \min\{M(x_*, x_{n+1}, \frac{t}{2}), M(Tx_n, Tx_*, \frac{t}{2})\}$. Note that $a_n \to 1$ as $n \to \infty$ and $\Delta(t, t)$ is continuous at $t = 1$. By letting $n \to \infty$ in (3.11), we get $M(x_*, Tx_*, t) = 1$ for $t > 0$. Hence $Tx_* = x_*$. Finally, we show the uniqueness of the fixed point. Suppose that y_* is a fixed point of T, i.e., $Ty_* = y_*$. By (3.10), we have

$$M(x_*, y_*, \varphi(t)) = M(Tx_*, Ty_*, \varphi(t)) \geq M(x_*, y_*, t),$$

for $t > 0$. From Lemma 3.5, we conclude that $x_* = y_*$. □

4. An example

Example 4.1. Let $X = [0, \infty)$ and define $M : X \times X \times [0, \infty) \to [0, 1]$ as follows:

$$M(x, y, t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } |x-y| \geq t, \\ 1, & \text{if } |x-y| < t. \end{cases} \tag{4.1}$$

Then (X, M, Δ_M) is a complete KM-fuzzy metric space (see the example in [3]). It is easy to verify that (X, M, Δ_M) is complete and $\lim_{t\to\infty} M(x, y, t) = 1$ for $x, y \in X$. Let $T : X \to X$ be defined by $Tx = \frac{x}{1+x}$, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by (3.4). It is easy to prove that φ satisfies (3.3), i.e., $\varphi \in \Phi_k$, since $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t \in (0, 1) \cup (1, 2)$ and $\lim_{n\to\infty} \varphi^n(t) = 1$ for $t \in [2, +\infty) \cup \{1\}$.

If $|Tx - Ty| < \varphi(t)$, then we have $M(Tx, Ty, \varphi(t)) = 1 \geq M(x, y, t)$, i.e., (3.10) holds. Suppose that $|Tx - Ty| \geq \varphi(t)$. From (3.4), it is easy to see that

$$\varphi(t) \geq \frac{t}{1+t} \text{ for } t > 0.$$

So we get $|Tx - Ty| \geq \frac{t}{1+t}$. By noting that

$$|Tx - Ty| = \frac{|x-y|}{1+x+y+xy} = \frac{|x-y|}{1+|x-y|+2\min\{x,y\}+xy} \leq \frac{|x-y|}{1+|x-y|},$$

we have $\frac{t}{1+t} \leq \frac{|x-y|}{1+|x-y|}$, which implies that $|x - y| \geq t$, since the function $f(u) = \frac{u}{1+u}$ is strictly increasing on $[0, \infty)$. By (4.1), we have $M(x, y, t) = \frac{t}{t+|x-y|}$. Thus,

$$M(Tx, Ty, \varphi(t)) = \frac{\varphi(t)}{\varphi(t) + |Tx - Ty|} \geq \frac{\frac{t}{1+t}}{\frac{t}{1+t} + \frac{|x-y|}{1+|x-y|}} \geq \frac{t}{t+|x-y|} = M(x, y, t),$$

i.e., (3.10) holds. This shows that all the conditions of Theorem 3.6 are satisfied. By Theorem 3.6, we conclude that T has a unique fixed point $x_* \in X$. Indeed, $x = 0$ is the unique fixed point of T.

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