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Some integral inequalities of the Hermite–Hadamard type for log-convex functions on co-ordinates

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Abstract

In the paper, the authors establish some new integral inequalities for log-convex functions on co-ordinates. These newly-established inequalities are connected with integral inequalities of the Hermite–Hadamard type for log-convex functions on co-ordinates.

Keywords: Log-convex functions, co-ordinates, integral inequality, Hermite–Hadamard type.
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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I and let $a, b \in I$ such that $a < b$. Then the double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \,\mathrm{d}\, x \le \frac{f(a)+f(b)}{2}$$

holds. This double inequality is known in the literature as the Hermite–Hadamard integral inequality.

Definition 1.1. If a positive function $f : I \subseteq \mathbb{R} \to \mathbb{R}_+ = (0, \infty)$ satisfies

$$f(\lambda x + (1-\lambda)y) \le [f(x)]^\lambda [f(y)]^{1-\lambda}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$, then we say that f is a logarithmically convex (or simply, log-convex) function on I. If the above inequality is reversed, then we say that f is a log-concave function.

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Equivalently, a function f is log-convex on I if and only if f is positive and its logarithm $\ln f$ is convex on I. Moreover, if the second derivative f'' exists on I, then f is log-convex if and only if $ff'' - (f')^2 \ge 0$.

A corresponding version of the Hermite–Hadamard integral inequality for log-convex functions was given in [5] as follows.

Theorem 1.2 ([5])**.** *Suppose that $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_+$ is a log-convex function on $[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le L(f(a), f(b)),$$

where $L(x, y)$ is the logarithmic mean

$$L(x,y) = \begin{cases} \dfrac{y-x}{\ln y - \ln x}, & x \ne y, \\ x, & x = y. \end{cases}$$

In [3, 4], the so-called convex functions on co-ordinates were introduced as follows.

Definition 1.3 ([3, 4])**.** A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on co-ordinates on Δ with $a < b$ and $c < d$ if the partial mappings

$$f_y : [a, b] \to \mathbb{R}, \quad f_y(u) = f_y(u, y) \quad \text{and} \quad f_x : [c, d] \to \mathbb{R}, \quad f_x(v) = f_x(x, v)$$

are convex for all $x \in (a, b)$ and $y \in (c, d)$.

Definition 1.4 ([3, 4])**.** A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on co-ordinates on Δ with $a < b$ and $c < d$ if the inequality

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le t\lambda f(x, y) + t(1-\lambda)f(x, w) + (1-t)\lambda f(z, y) + (1-t)(1-\lambda)f(z, w)$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

An inequality of the Hermite–Hadamard type for convex function on co-ordinates on a rectangle from the plane $\mathbb{R}^2$ was established in [3, 4] as follows.

Theorem 1.5 ([3, 4, Theorem 2.2])**.** *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex on co-ordinates on Δ with $a < b$ and $c < d$. Then one has*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)\mathrm{d}\,x + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)\mathrm{d}\,y\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}\,y\,\mathrm{d}\,x \\ &\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b \big[f(x,c) + f(x,d)\big]\mathrm{d}\,x + \frac{1}{d-c}\int_c^d \big[f(a,y) + f(b,y)\big]\mathrm{d}\,y\right] \\ &\le \frac{1}{4}\big[f(a,c) + f(b,c) + f(a,d) + f(b,d)\big]. \end{aligned}$$

In [1], Alomari and Darus introduced a class of log-convex functions on co-ordinates as follows.

Definition 1.6 ([1])**.** A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ is called log-convex on co-ordinates on Δ with $a < b$ and $c < d$ if

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le [f(x, y)]^{t\lambda}[f(x, w)]^{t(1-\lambda)}[f(z, y)]^{(1-t)\lambda}[f(z, w)]^{(1-t)(1-\lambda)}$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

Remark 1.7. If f and g are both log-convex on co-ordinates on Δ, then their composite $f\circ g$ is also log-convex on co-ordinates on Δ.

An inequality of the Hermite–Hadamard type for log-convex functions on co-ordinates on a rectangle from the plane $\mathbb{R}^2$ was established by Alomari and Darus in [1] as follows.

Theorem 1.8 ([1, Theorem 3.3]). *Suppose that $f : \Delta = [a,b]\times[c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ is log-convex on co-ordinates on Δ for $a<b$ and $c<d$. Let*

$$A=\frac{f(a,c)f(b,d)}{f(b,c)f(a,d)},\quad B=\frac{f(a,d)}{f(b,d)},\quad \text{and}\quad C=\frac{f(b,c)}{f(b,d)}.$$

Then the inequality

$$I=\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y \le f(b,d)\times\begin{cases}1, & A=B=C=1,\\ \dfrac{B-1}{\ln B}\dfrac{C-1}{\ln C}, & A=1,\\ H(C), & B=1,\\ H(B), & C=1,\\ \dfrac{C-1}{\ln C}, & A=B=1,\\ \dfrac{B-1}{\ln B}, & A=C=1,\\ \dfrac{\gamma+\ln(-\ln A)+Ei(1,-\ln A)}{\ln A}, & B=C=1,\\ \dfrac{1}{2}\left[\dfrac{B-1}{\ln B}+\dfrac{AB-1}{\ln(AB)}\right], & A,B,C>0,\\ \displaystyle\int_0^1 C^\beta\frac{AB-1}{\ln(AB)}\,\mathrm{d}\,\beta, & \text{otherwise}\end{cases}$$

holds, where γ is the Euler constant,

$$H(x)=\frac{Ei(1,-\ln A)+\ln\ln x-Ei(1,-\ln(Ax))-\ln\ln(Ax)}{\ln A}+\begin{cases}\dfrac{2\ln\ln A-\ln(-\ln A)}{\ln A}, & -1<\dfrac{\ln x}{\ln A}<0,\\ 0, & \textit{otherwise};\end{cases}$$

and

$$Ei(x)=V.P.\int_{-x}^{\infty}\frac{e^{-t}}{t}\,\mathrm{d}\,t$$

is the exponential integral function.

For more and detailed information on this topic, please refer to the newly published papers [2, 6–25] and plenty of references therein.

2. Some new integral inequalities of the Hermite–Hadamard type

In this section, we prove some new inequalities of the Hermite–Hadamard type for log-convex functions on co-ordinates.

Theorem 2.1. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}_+$ for $a < b$ and $c < d$ be log-convex on co-ordinates on Δ. Then one has*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\quad\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b L\big(f(x,c),f(x,d)\big)\,\mathrm{d}\,x+\frac{1}{d-c}\int_c^d L(f(a,y),f(b,y))\,\mathrm{d}\,y\right]\\
&\quad\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b \big[f(x,c)+f(x,d)\big]\,\mathrm{d}\,x+\frac{1}{d-c}\int_c^d \big[f(a,y)+f(b,y)\big]\,\mathrm{d}\,y\right]\\
&\quad\le \frac{1}{4}\big[L(f(a,c),f(b,c))+L(f(a,d),f(b,d))+L\big(f(a,c),f(a,d)\big)+L\big(f(b,c),f(b,d)\big)\big]\\
&\quad\le \frac{1}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)],
\end{aligned}$$

where $L(u, v)$ is the logarithmic mean.

Proof. For all $x, y > 0$, it is known that $L(x, y) \le \frac{x+y}{2}$. Setting $y = \lambda c + (1 - \lambda)d$ for all $0 \le \lambda \le 1$, using the log-convexity of f, and by the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y &= \frac{1}{b-a}\int_0^1\int_a^b f(x,\lambda c+(1-\lambda)d)\,\mathrm{d}\,x\,\mathrm{d}\,\lambda\\
&\le \frac{1}{b-a}\int_a^b\int_0^1 [f(x,c)]^{\lambda}[f(x,d)]^{1-\lambda}\,\mathrm{d}\,\lambda\,\mathrm{d}\,x\\
&= \frac{1}{b-a}\int_a^b L(f(x,c),f(x,d))\,\mathrm{d}\,x\\
&\le \frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,d)]\,\mathrm{d}\,x.
\end{aligned}$$

Since $f(x, c) \le [f(a, c)]^t[f(b, c)]^{1-t}$ and $f(x, d) \le [f(a, d)]^t[f(b, d)]^{1-t}$ for each $t \in [0, 1]$, we have

$$\begin{aligned}
\frac{1}{2(b-a)}\int_a^b [f(x,c+f(x,d)]\,\mathrm{d}\,x &\le \frac{1}{2}\int_0^1 \big\{[f(a,c)]^t[f(b,c)]^{1-t}+[f(a,d)]^t[f(b,d)]^{1-t}\big\}\,\mathrm{d}\,x\\
&= \frac{1}{2}\big[L(f(a,c),f(b,c))+L(f(a,d),f(b,d))\big]\\
&\le \frac{1}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)].
\end{aligned}$$

By a similar argument, we can obtain

$$\begin{aligned}
\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y &\le \frac{1}{d-c}\int_c^d L(f(a,y),f(b,y))\,\mathrm{d}\,y\\
&\le \frac{1}{2(d-c)}\int_c^d [f(a,y)+f(b,y)]\,\mathrm{d}\,y\\
&\le \frac{1}{2}\big[L\big(f(a,c),f(a,d)\big)+L\big(f(b,c),f(b,d)\big)\big]\\
&\le \frac{1}{4}[f(a,c)+f(b,c)+f(a,d)+f(b,d)].
\end{aligned}$$

The proof of Theorem 2.1 is thus complete. □

Example 2.2. The function $f(x, y) = x^2y^2 + 1$ is log-convex on co-ordinates on $\Delta = [-1, 1]^2$. In Theorem 1.8, since $A = B = C = 1$, we have

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y = \frac{10}{9} < 2 = f(b,d).$$

By Theorem 2.1, we obtain

$$\begin{aligned}\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y &= \frac{10}{9} < \frac{4}{3}\\ &= \frac{1}{2}\left[\frac{1}{b-a}\int_a^b L\big(f(x,c),f(x,d)\big)\,\mathrm{d}\,x + \frac{1}{d-c}\int_c^d L(f(a,y),f(b,y))\,\mathrm{d}\,y\right] < 2.\end{aligned}$$

Theorem 2.3. *Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ with $a<b$ and $c<d$ be log-convex on co-ordinates on Δ. Then*

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left\{\frac{1}{b-a}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)f\left(a+b-x,\frac{c+d}{2}\right)\right]^{1/2}\mathrm{d}\,x\right.\\ &\quad\left.+\frac{1}{d-c}\int_c^d\left[f\left(\frac{a+b}{2},y\right)f\left(\frac{a+b}{2},c+d-y\right)\right]^{1/2}\mathrm{d}\,y\right\}\\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\ &\le \frac{1}{2(b-a)(d-c)}\int_c^d\int_a^b\big\{[f(x,y)f(x,c+d-y)]^{1/2}\\ &\quad+[f(x,y)f(a+b-x,y)]^{1/2}\big\}\,\mathrm{d}\,x\,\mathrm{d}\,y\\ &\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y.\end{aligned}\tag{2.1}$$

Proof. Utilizing the log-convexity of f leads to

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &= f\left(\frac{1}{2}[ta+(1-t)b+(1-t)a+tb],\frac{1}{2}\left(\frac{c+d}{2}+\frac{c+d}{2}\right)\right)\\ &\le\left[f\left(ta+(1-t)b,\frac{c+d}{2}\right)f\left((1-t)a+tb,\frac{c+d}{2}\right)\right]^{1/2}\end{aligned}\tag{2.2}$$

for all $0\le t\le 1$. On using the change of the variable $x=ta+(1-t)b$ for $0\le t\le 1$, integrating the inequality (2.2) over t on $[0,1]$, and by the arithmetic-geometric mean inequality, we procure

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\int_0^1\left[f\left(ta+(1-t)b,\frac{c+d}{2}\right)f\left((1-t)a+tb,\frac{c+d}{2}\right)\right]^{1/2}\mathrm{d}\,t\\ &= \frac{1}{b-a}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)f\left(a+b-x,\frac{c+d}{2}\right)\right]^{1/2}\mathrm{d}\,x\\ &\le \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x.\end{aligned}\tag{2.3}$$

Using the log-convexity of f, we find

$$f\left(x,\frac{c+d}{2}\right)\le\big[f(x,\lambda c+(1-\lambda)d)f(x,(1-\lambda)c+\lambda d)\big]^{1/2}\tag{2.4}$$

for all $0\le\lambda\le 1$ and $x\in[a,b]$.

Integrating the inequality (2.4) with respect to (x,λ) on $[a,b]\times[0,1]$ and using the inequality (2.3) give

$$\begin{aligned}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x &\le \frac{1}{b-a}\int_0^1\int_a^b\big[f(x,\lambda c+(1-\lambda)d)f(x,(1-\lambda)c+\lambda d)\big]^{1/2}\,\mathrm{d}\,x\,\mathrm{d}\,\lambda\\ &= \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\big[f(x,y)f\big(x,c+d-y\big)\big]^{1/2}\,\mathrm{d}\,x\,\mathrm{d}\,y\\ &\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y.\end{aligned}\tag{2.5}$$

Similarly, we obtain

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{d-c}\int_c^d \left[f\left(\frac{a+b}{2},y\right)f\left(\frac{a+b}{2},c+d-y\right)\right]^{1/2}\,\mathrm{d}\,y\\
&\le \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b \big[f(x,y)f\big(a+b-x,y\big)\big]^{1/2}\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}$$

A combination of (2.3), (2.5), and the last inequality gives the desired inequality (2.1). Theorem 2.3 is thus proved. □

Making use of Theorem 2.3, we derive the following corollary.

Corollary 2.4. *Let $f,g:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ with $a<b$ and $c<d$ be log-convex on co-ordinates on Δ. Then*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad\le \frac{1}{2}\left\{\frac{1}{b-a}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)f\left(a+b-x,\frac{c+d}{2}\right)g\left(a+b-x,\frac{c+d}{2}\right)\right]^{1/2}\mathrm{d}\,x\right.\\
&\qquad\left.+\frac{1}{d-c}\int_c^d\left[f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)f\left(\frac{a+b}{2},c+d-y\right)g\left(\frac{a+b}{2},c+d-y\right)\right]^{1/2}\mathrm{d}\,y\right\}\\
&\quad\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\
&\quad\le \frac{1}{2(b-a)(d-c)}\int_c^d\int_a^b\big\{[f(x,y)g(x,y)f(x,c+d-y)g(x,c+d-y)]^{1/2}\\
&\qquad+[f(x,y)g(x,y)f(a+b-x,y)g(a+b-x,y)]^{1/2}\big\}\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\quad\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}$$

Theorem 2.5. *Let $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ with $a<b$ and $c<d$ be log-convex on co-ordinates on Δ. Then*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)f\left(a+b-x,\frac{c+d}{2}\right)\right]^{1/2}\mathrm{d}\,x\right.\\
&\qquad\left.+\frac{1}{d-c}\int_c^d\left[f\left(\frac{a+b}{2},y\right)f\left(\frac{a+b}{2},c+d-y\right)\right]^{1/2}\mathrm{d}\,y\right]\\
&\quad\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b[f(x,y)f(x,c+d-y)f(a+b-x,y)f(a+b-x,c+d-y)]^{1/4}\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\quad\le \frac{1}{2(b-a)(d-c)}\int_c^d\int_a^b\big\{[f(x,y)f(x,c+d-y)]^{1/2}+[f(x,y)f(a+b-x,y)]^{1/2}\big\}\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\quad\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}$$

Proof. Since f is log-convex on co-ordinates on Δ, using the inequalities (2.3), (2.5), and the arithmetic-geometric inequality figures out

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{b-a}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)f\left(a+b-x,\frac{c+d}{2}\right)\right]^{1/2}\mathrm{d}\,x\\
&\le \frac{1}{b-a}\int_0^1\int_a^b\big[f(x,\lambda c+(1-\lambda)d)f(x,(1-\lambda)c+\lambda d)\\
&\quad\times f(a+b-x,\lambda c+(1-\lambda)d)f(a+b-x,(1-\lambda)c+\lambda d)\big]^{1/4}\mathrm{d}\,x\,\mathrm{d}\,\lambda\\
&= \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\big[f(x,y)f(x,c+d-y)f(a+b-x,y)\\
&\quad\times f(a+b-x,c+d-y)\big]^{1/4}\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{1}{2(b-a)(d-c)}\int_c^d\int_a^b\big\{[f(x,y)f(x,c+d-y)]^{1/2}+[f(x,y)f(a+b-x,y)]^{1/2}\big\}\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{d-c}\int_c^d\left[f\left(\frac{a+b}{2},y\right)f\left(\frac{a+b}{2},c+d-y\right)\right]^{1/2}\mathrm{d}\,y\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\big[f(x,y)f(x,c+d-y)f(a+b-x,y)\\
&\quad\times f(a+b-x,c+d-y)\big]^{1/4}\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{1}{2(b-a)(d-c)}\int_c^d\int_a^b\big\{[f(x,y)f(x,c+d-y)]^{1/2}+[f(x,y)f(a+b-x,y)]^{1/2}\big\}\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}$$

Hence, the proof of Theorem 2.5 is complete. □

Corollary 2.6. *Let $f,g:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}_+$ with $a<b$ and $c<d$ be log-convex on co-ordinates on Δ. Then*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)f\left(a+b-x,\frac{c+d}{2}\right)g\left(a+b-x,\frac{c+d}{2}\right)\right]^{1/2}\mathrm{d}\,x\right.\\
&\quad\left.+\frac{1}{d-c}\int_c^d\left[f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)f\left(\frac{a+b}{2},c+d-y\right)g\left(\frac{a+b}{2},c+d-y\right)\right]^{1/2}\mathrm{d}\,y\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\big[f(x,y)g(x,y)f(x,c+d-y)g(x,c+d-y)\\
&\quad\times f(a+b-x,y)g(a+b-x,y)f(a+b-x,c+d-y)g(a+b-x,c+d-y)\big]^{1/4}\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{1}{2(b-a)(d-c)}\int_c^d\int_a^b\big\{[f(x,y)g(x,y)f(x,c+d-y)g(x,c+d-y)]^{1/2}\\
&\quad+[f(x,y)g(x,y)f(a+b-x,y)g(a+b-x,y)]^{1/2}\big\}\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}$$

Theorems 2.1 and 2.3 can be improved as follows.

Corollary 2.7. *Under the conditions of Theorems* 2.1 *and* 2.3*, if* $f(x,y) = f_1(x)g_1(y)$ *for* $(x,y) \in \Delta$*, then*

$$\begin{aligned}
f_1\left(\frac{a+b}{2}\right)g_1\left(\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\left(\frac{1}{b-a}\int_a^b \left[f_1(x)f_1(a+b-x)\right]^{1/2}\,\mathrm{d}\,x\right)g_1\left(\frac{c+d}{2}\right)\right. \\
&\quad \left.+\left(\frac{1}{d-c}\int_c^d \left[g_1(x)g_1(c+d-y)\right]^{1/2}\,\mathrm{d}\,y\right)f_1\left(\frac{c+d}{2}\right)\right] \\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b \left[f_1(x)g_1(y)f_1(a+b-x)g_1(c+d-y)\right]^{1/2}\,\mathrm{d}\,x\,\mathrm{d}\,y \\
&\le \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f_1(x)g_1(y)\,\mathrm{d}\,x\,\mathrm{d}\,y \\
&\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b L\big(f_1(x)g_1(c), f_1(x)g_1(d)\big)\,\mathrm{d}\,x\right. \\
&\quad \left.+\frac{1}{d-c}\int_c^d L(f_1(a)g_1(y), f_1(b)g_1(y))\,\mathrm{d}\,y\right] \\
&\le \frac{1}{4}\left[\frac{g_1(c)+g_1(d)}{b-a}\int_a^b f_1(x)\,\mathrm{d}\,x + \frac{f_1(a)+f_1(b)}{d-c}\int_c^d g_1(y)\,\mathrm{d}\,y\right] \\
&\le \frac{1}{4}\big[L(f_1(a), f_1(b))[g_1(c)+g_1(d)] + [f_1(a)+f_1(b)]L(g_1(c), g_1(d))\big] \\
&\le \frac{1}{4}\big[[f_1(a)+f_1(b)][g_1(c)+g_1(d)]\big].
\end{aligned}$$

3. Conclusions

By the arithmetic-geometric inequality and other techniques, we establish some new integral inequalities for log-convex functions on co-ordinates. These newly-established inequalities are connected with integral inequalities of the Hermite–Hadamard type for log-convex functions on co-ordinates.

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