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Almost Kenmotsu 3-h-manifolds with cyclic-parallel Ricci tensor

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Abstract

In this paper, we prove that the Ricci tensor of an almost Kenmotsu 3-h-manifold is cyclic-parallel if and only if it is parallel and hence, the manifold is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

Keywords: Almost Kenmotsu 3-manifold, almost contact metric manifold, Einstein-like metric, cyclic-parallel Ricci tensor.
2010 MSC: 53D15, 53C25.

1. Introduction

Let us recall the following notions defined by Gray [13]. A pseudo-Riemannian manifold (M, g) is said to belong to class $\mathcal{A}$ if its Ricci operator Q is cyclic-parallel, that is,

$$g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)Z, X) + g((\nabla_Z Q)X, Y) = 0 \tag{1.1}$$

for any vector fields X, Y, Z tangent to M. It is known that (1.1) is equivalent to requiring that the Ricci tensor is a Killing tensor, that is,

$$g((\nabla_X Q)X, X) = 0 \tag{1.2}$$

for any vector field X on M. Moreover, (M, g) is said to belong to class $\mathcal{B}$ if its Ricci operator Q is of Codazzi-type, that is,

$$(\nabla_X Q)Y = (\nabla_Y Q)X \tag{1.3}$$

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for any vector fields X, Y tangent to M. Here, we remark that equation (1.3) is also equivalent to requiring that the Riemannian curvature tensor is harmonic, that is, $\mathrm{div}R = 0$. In addition, (M, g) is said to belong to class $\mathcal{P}$ if its Ricci operator Q is parallel, that is,

$$\nabla Q = 0. \tag{1.4}$$

A. Gray in [13] obtained an interesting result, namely $\mathcal{E} \subset \mathcal{P} = \mathcal{A} \cap \mathcal{B}$, where $\mathcal{E}$ denotes the class of all Einstein manifolds. A semi-Riemannian metric whose Ricci tensor satisfying relations (1.1), (1.3) or (1.4) is called an Einstein-like metric.

Many authors studied equations (1.1)-(1.4) on some types of almost contact metric manifolds and some other manifolds. For examples, Gouli-Andreou and Xenos in [12] proved that a k-contact metric manifold of dimension $2n + 1$ satisfying equation (1.3) is locally isomeric to either an Einstein-Sasakian manifold or the product space $\mathbb{S}^n(4) \times \mathbb{R}^{n+1}$. Moreover, they proved that a contact metric 3-τ-manifold satisfying equation (1.3) is either flat or an Einstein-Sasakian manifold. Gouli-Andreou et al. in [11] proved that a complete three-dimensional (κ, μ, ν)-contact metric manifold satisfying (1.1) is either Sasakian or a (κ, μ)-contact metric manifold. De and Pathak [7] obtained that a three-dimensional Kenmotsu manifold has a cyclic-parallel Ricci tensor if and only if the manifold is of constant sectional curvature -1. Generalizing this result, the cyclic-parallel Ricci tensors on three-dimensional normal almost contact metric manifolds were also studied by De and Mondal [6]. For more results regarding Equations (1.1)-(1.4) on some semi-Riemannian manifolds and almost contact metric manifolds, we refer reader to De et al. [5, 8], Calvaruso [2], Wang [18, 20, 22] and the present author [19].

We remark that Cho [3] and Wang [18] recently obtained an interesting local classification of an almost Kenmotsu 3-manifold, namely any almost Kenmotsu 3-manifold is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. We observe that almost Kenmotsu 3-manifolds under some additional geometric conditions were also investigated by Cho [4] and Wang [20, 21], respectively. Note that (1.4) implies (1.1) trivially, however the converse is not necessarily true. Therefore, in this paper we aim to present an extension of the corresponding results shown in [3, 18] on a special class of three-dimensional almost Kenmotsu manifolds. Our main results is to show that the equations (1.1)-(1.4) are equivalent to each other on an almost Kenmotsu 3-h-manifold.

2. Almost Kenmotsu manifolds

A smooth manifold M^{2n+1} of dimensional $2n+1$ is called an almost contact metric manifold if it admits an almost contact structure (ϕ, ξ, η), that is, there exist a $(1, 1)$-type tensor field ϕ, a global vector field ξ and a 1-form η such that

$$\begin{aligned} &\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \end{aligned} \tag{2.1}$$

for any vector fields X, Y tangent to M^{2n+1}, where id denotes the identity map and ξ is called the Reeb vector field. On the product manifold $M^{2n+1} \times \mathbb{R}$ of an almost contact manifold M^{2n+1} and $\mathbb{R}$, one can define an almost complex structure J by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a smooth function. An almost contact structure is said to be normal if the above almost complex structure J is integrable, that is, J is a complex structure. According to Blair [1], the normality of an almost contact structure is given by $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

for any vector fields X, Y on M^{2n+1}. The fundamental 2-form Φ of an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y.

From [1] and [15], an almost contact metric manifold is called

(1) a contact metric manifold if $d\eta = \Phi$;

(2) an almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$;

(3) an almost cosymplectic manifold if $d\eta = 0$ and $d\Phi = 0$.

A normal contact metric (resp. almost Kenmotsu, almost cosymplectic) manifold is called a Sasakian (resp. Kenmotsu, cosymplectic) manifold.

We denote by $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on an almost Kenmotsu manifold M^{2n+1}. Following [9, 10], it is seen that both h and h' are symmetric operators and the following formulas are true.

$$h\xi = l\xi = 0,\ \mathrm{tr}h = \mathrm{tr}(h') = 0,\ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla\xi = h' + \mathrm{id} - \eta \otimes \xi, \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{2.5}$$

$$\mathrm{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \mathrm{tr}h^2, \tag{2.6}$$

where $l := R(\cdot\,, \xi)\xi$ is the Jacobi operator along the Reeb vector field and the Riemannian curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

tr and S denote the trace operator and the Ricci tensor, respectively.

3. Almost Kenmotsu 3-*h*-manifolds with cyclic-parallel Ricci tensors

We first give the following definition.

Definition 3.1. A three-dimensional almost Kenmotsu manifold is called an almost Kenmotsu 3-h-manifold if it satisfies $\nabla_\xi h = 0$.

It is known that on a three-dimensional Kenmotsu manifold there holds $h = 0$ and hence $\nabla_\xi h = 0$ holds trivially. However, Dileo and Pastore [9, Proposition 6] proved that even on a locally symmetric non-Kenmotsu almost Kenmotsu manifold there still holds $\nabla_\xi h = 0$. By using this condition, Wang [20, 21] gave some local classifications of three-dimensional almost Kenmotsu manifolds. He also presented some examples of three-dimensional almost Kenmotsu manifolds on which $\nabla_\xi h = 0$ but $h \neq 0$.

Example 3.2 ([21])**.** Let G be a three-dimensional non-unimodular Lie group (see [17]) with a left invariant local orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying

$$[e_1, e_2] = \alpha e_2 + \beta e_3,\ [e_2, e_3] = 0,\ [e_1, e_3] = \beta e_2 + (2 - \alpha)e_3$$

for $\alpha, \beta \in \mathbb{R}$. If either $\alpha \neq 1$ or $\beta \neq 0$, G admits a left invariant non-Kenmotsu almost Kenmotsu structure satisfying $\nabla_\xi h = 0$ and $h \neq 0$.

For almost Kenmotsu structures defined on three-dimensional non-unimodular Lie groups we refer the reader to Dileo and Pastore [10, Section 5]. From Example 3.2, although there exist many examples of almost Kenmotsu 3-h-manifolds, but not all non-Kenmotsu almost Kenmotsu 3-manifolds satisfy $\nabla_\xi h = 0$.

Example 3.3 ([20])**.** Let M be a (k, μ, ν)-almost Kenmotsu manifold, that is, its Reeb vector field ξ satisfies the (k, μ, ν)-nullity condition,

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)h'X - \eta(X)h'Y) \tag{3.1}$$

for any vector fields X, Y, Z and some smooth functions k, μ, ν. Then, if either $\mu \neq 0$ or $\nu \neq -2$, then we have $\nabla_\xi h = \mu h' - (\nu + 2)h \neq 0$ provided that $h \neq 0$ (or equivalently, $k < -1$).

Let us recall some useful formula shown in Cho [4]. Let $\mathcal{U}_1$ be the open subset of a three-dimensional almost Kenmotsu manifold M^3 such that $h \neq 0$ and $\mathcal{U}_2$ the open subset of M^3 defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Hence, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we set $he = \lambda e$ and hence $h\phi e = -\lambda\phi e$, where λ is a positive function on $\mathcal{U}_1$. The eigenvalue function λ is continuous on M^3 and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Lemma 3.4 ([4, Lemma 6])**.** *On $\mathcal{U}_1$ we have*

$$\begin{aligned}
&\nabla_\xi \xi = 0,\ \nabla_\xi e = a\phi e,\ \nabla_\xi \phi e = -ae,\\
&\nabla_e \xi = e - \lambda\phi e,\ \nabla_e e = -\xi - b\phi e,\ \nabla_e \phi e = \lambda\xi + be,\\
&\nabla_{\phi e}\xi = -\lambda e + \phi e,\ \nabla_{\phi e} e = \lambda\xi + c\phi e,\ \nabla_{\phi e}\phi e = -\xi - ce,
\end{aligned} \tag{3.2}$$

where a, b, c are smooth functions.

Applying Lemma 3.4 in the following Jacobi identity

$$[[\xi, e], \phi e] + [[e, \phi e], \xi] + [[\phi e, \xi], e] = 0,$$

we obtain

$$\begin{cases} e(\lambda) - \xi(b) - e(a) + c(\lambda - a) - b = 0,\\ \phi e(\lambda) - \xi(c) + \phi e(a) + b(\lambda + a) - c = 0. \end{cases} \tag{3.3}$$

Moreover, applying Lemma 3.4 we have

$$\begin{cases} Q\xi = -2(\lambda^2 + 1)\xi - (\phi e(\lambda) + 2\lambda b)e - (e(\lambda) + 2\lambda c)\phi e,\\ Qe = -(\phi e(\lambda) + 2\lambda b)\xi - (f + 2\lambda a)e + (\xi(\lambda) + 2\lambda)\phi e,\\ Q\phi e = -(e(\lambda) + 2\lambda c)\xi + (\xi(\lambda) + 2\lambda)e - (f - 2\lambda a)\phi e, \end{cases} \tag{3.4}$$

where $f = e(c) + \phi e(b) + b^2 + c^2 + 2$.

We also need the following well known result (see also [13]).

Lemma 3.5. *If the Ricci tensor of a Riemannian manifold is cyclic-parallel or of Codazzi-type, then the scalar curvature is a constant.*

We first give the following result for Kenmotsu 3-manifolds.

Proposition 3.6. *On a three-dimensional Kenmotsu manifold the following statements are equivalent.*

(1) *The Ricci tensor is parallel;*

(2) *The Ricci tensor is of Codazzi-type;*

(3) *The Ricci tensor is cyclic-parallel;*

(4) *The manifold is of constant sectional curvature -1.*

Proof. It is known that on a three-dimensional Riemannian manifold the curvature tensor R is given by

$$R(Y,Z)W = g(Z,W)QY - g(Y,W)QZ + g(QZ,W)Y - g(QY,W)Z - \frac{r}{2}(g(Z,W)Y - g(Y,W)Z)$$

for any vector fields Y, Z, W. Taking the covariant derivative of the above relation along arbitrary vector field X gives

$$\begin{aligned}(\nabla_X R)(Y,Z)W =& g(Z,W)(\nabla_X Q)Y - g(Y,W)(\nabla_X Q)Z + g((\nabla_X Q)Z,W)Y \\ &- g((\nabla_X Q)Y,W)Z - \frac{1}{2}X(r)(g(Z,W)Y - g(Y,W)Z)\end{aligned}$$

for any vector fields X, Y, Z and W, where r denotes the scalar curvature.

If a three-dimensional Kenmotsu manifold M^3 has a parallel Ricci tensor, then, the scalar curvature of M^3 is a constant and hence by the above relation we see that the manifold is locally symmetric. On the other hand, K. Kenmotsu [16, Corollary 6] proved that a locally symmetric Kenmotsu manifold is of constant sectional curvature -1. This means (1) $\Rightarrow$ (4).

If a three-dimensional Kenmotsu manifold M^3 is of constant sectional curvature -1, that is, $R(X,Y)Z = -g(Y,Z)X + g(X,Z)Y$ for any vector fields X, Y, Z, then we obtain easily that the Ricci operator is given by $Q = -2\mathrm{id}$. This means that the Ricci tensor is parallel (that is, $\nabla Q = 0$) and hence we obtain (4) $\Rightarrow$ (1), (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3).

It can be easily seen that a three-dimensional Riemannian manifold with a Codazzi-type or a cyclic-parallel Ricci tensor is of constant scalar curvature (see also Gray [13]). Moreover, J. Inoguchi in [14] proved that a three-dimensional Kenmotsu manifold having a constant scalar curvature is of constant sectional curvature -1. This means (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4). This completes the proof. □

Next we give our main result, stating that the equations (1.1)-(1.4) are equivalent to each other even on a special type of non-Kenmotsu almost Kenmotsu 3-manifolds, namely non-Kenmotsu almost Kenmotsu 3-h-manifolds.

Theorem 3.7. *On an almost Kenmotsu* 3-h-*manifold with* $h \neq 0$, *the following statements are equivalent.*

(1) *The Ricci tensor is parallel;*

(2) *The Ricci tensor is of Codazzi-type;*

(3) *The Ricci tensor is cyclic-parallel;*

(4) *The manifold is locally isometric to the product space* $\mathbb{H}^2(-4) \times \mathbb{R}$.

Proof. Recently, Wang [18] and Cho [3] independently obtained that any non-Kenmotsu almost Kenmotsu 3-manifold is locally symmetric if and only if it is locally isometric to the product space $\mathbb{H}^2(-4) \times \mathbb{R}$. Since on a locally symmetric non-Kenmotsu almost Kenmotsu manifold there holds $\nabla_\xi h = 0$, therefore, (1) $\Leftrightarrow$ (4) follows from [18] and [3].

Very recently, Wang [20] obtained that a three-dimensional almost Kenmotsu manifold satisfying $\nabla_\xi h = 0$ and having a harmonic curvature tensor is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product space $\mathbb{H}^2(-4) \times \mathbb{R}$. This means (2) $\Leftrightarrow$ (4).

Since the product space $\mathbb{H}^2(-4) \times \mathbb{R}$ is locally symmetric, then in what follows we need only to show that (3) $\Rightarrow$ (4).

Now, we suppose that M^3 is a non-Kenmotsu almost Kenmotsu 3-h-manifold whose Ricci tensor is cyclic-parallel. Firstly, by a direct calculation we obtain from Lemma 3.4 and relation (2.4) that

$$(\nabla_\xi h)e = \xi(\lambda)e + 2a\lambda\phi e \text{ and } (\nabla_\xi h)\phi e = -\xi(\lambda)\phi e + 2a\lambda e.$$

In view of Definition 3.1 and λ being a positive function, we have

$$\xi(\lambda) = a = 0. \tag{3.5}$$

Then, using (3.5) in (3.4) we obtain from Lemma 3.4 that

$$(\nabla_\xi Q)\xi = -\xi(\phi e(\lambda) + 2\lambda b)e - \xi(e(\lambda) + 2\lambda c)\phi e, \tag{3.6}$$

$$(\nabla_\xi Q)e = -\xi(\phi e(\lambda) + 2\lambda b)\xi - \xi(f)e, \tag{3.7}$$

$$(\nabla_\xi Q)\phi e = -\xi(e(\lambda) + 2\lambda c)\xi - \xi(f)\phi e, \tag{3.8}$$

$$\begin{aligned}(\nabla_e Q)\xi =& 2(\phi e(\lambda) - 3\lambda e(\lambda) + 2\lambda b - 2\lambda^2 c)\xi \\ &+ (f - 2 - e(\phi e(\lambda) + 2\lambda b) - b(e(\lambda) + 2\lambda c))e \\ &+ (2\lambda^3 + b(\phi e(\lambda) + 2\lambda b) - e(e(\lambda) + 2\lambda c) - \lambda f)\phi e,\end{aligned} \tag{3.9}$$

$$\begin{aligned}(\nabla_e Q)e =& (f - 2 - e(\phi e(\lambda) + 2\lambda b) - b(e(\lambda) + 2\lambda c))\xi \\ &- (e(f) + 2\phi e(\lambda))e + (e(\lambda) + \lambda\phi e(\lambda) + 2\lambda^2 b - 2\lambda c)\phi e,\end{aligned} \tag{3.10}$$

$$\begin{aligned}(\nabla_e Q)\phi e =& (2\lambda^3 - f\lambda + b(\phi e(\lambda) + 2\lambda b) - e(e(\lambda) + 2\lambda c))\xi \\ &+ (e(\lambda) + \lambda\phi e(\lambda) - 2\lambda c + 2\lambda^2 b)e \\ &+ (2\lambda(e(\lambda) + 2\lambda c) - e(f) - 4\lambda b)\phi e,\end{aligned} \tag{3.11}$$

$$\begin{aligned}(\nabla_{\phi e} Q)\xi =& 2(e(\lambda) - 3\lambda\phi e(\lambda) + 2\lambda c - 2\lambda^2 b)\xi \\ &+ (2\lambda^3 + c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b) - \lambda f)e \\ &+ (f - 2 - \phi e(e(\lambda) + 2\lambda c) - c(\phi e(\lambda) + 2\lambda b))\phi e,\end{aligned} \tag{3.12}$$

$$\begin{aligned}(\nabla_{\phi e} Q)e =& (2\lambda^3 - f\lambda + c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b))\xi \\ &- (\phi e(f) + 4\lambda c - 2\lambda(\phi e(\lambda) + 2\lambda b))e \\ &+ (\phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b)\phi e,\end{aligned} \tag{3.13}$$

$$\begin{aligned}(\nabla_{\phi e} Q)\phi e =& (f - 2 - \phi e(e(\lambda) + 2\lambda c) - c(\phi e(\lambda) + 2\lambda b))\xi \\ &+ (\phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b)e - (\phi e(f) + 2e(\lambda))\phi e.\end{aligned} \tag{3.14}$$

Since on M^3 the Ricci tensor is assumed to be cyclic-parallel, substituting X with e and ϕe, respectively, in (1.2) we have

$$\begin{cases} e(f) + 2\phi e(\lambda) = 0, \\ \phi e(f) + 2e(\lambda) = 0. \end{cases} \tag{3.15}$$

From (3.4) and (3.5) we get $r = -2\lambda^2 - 2 - 2f$. By Lemma 3.5 we know that the scalar curvature is a constant, then it follows that

$$e(f) = -2\lambda e(\lambda) \text{ and } \phi e(f) = -2\lambda\phi e(\lambda).$$

Using this in (3.15) we observe that either $\lambda = 1$ or λ is a positive constant not equal to 1, where we have used (3.5) and that λ is continuous.

Now let us consider the second case, that is, λ is a positive constant not equal to 1. Setting $Y = Z$ in Equation (1.1) and using the symmetry of the Ricci tensor give

$$g((\nabla_X Q)Y, Y) + 2g((\nabla_Y Q)Y, X) = 0. \tag{3.16}$$

Putting $X = e$ and $Y = \phi e$ in (3.16) and using (3.11) and (3.14) we have

$$b - \lambda c = 0. \tag{3.17}$$

Similarly, putting $X = \phi e$ and $Y = e$ in (3.16) and using (3.10) and (3.13) we have

$$c - \lambda b = 0. \tag{3.18}$$

Since $\lambda \neq 1$, it follows from (3.17) and (3.18) that

$$b = c = 0$$

and using this in (3.7), (3.11) and (3.12) gives

$$(\nabla_\xi Q)e = 0,\ (\nabla_e Q)\phi e = 2\lambda(\lambda - 1)\xi,\ (\nabla_{\phi e} Q)\xi = 2\lambda(\lambda - 1)e,$$

where we have used $f = 2$. Putting $X = e$, $Y = \phi e$ and $Z = \xi$ in equation (1.1) and using the above relation we obtain $\lambda = 1$, a contradiction. Then, based on the above analysis we conclude that $\lambda = 1$. Next we prove that in this context the cyclic-parallelism of the Ricci tensor implies the parallelism.

Putting $X = e$ and $Y = \phi e$ in (3.16) and using (3.11), (3.14) we get

$$b = c. \tag{3.19}$$

Using (3.5), $\lambda = 1$ and (3.19) in the first term of Relation (3.3) we have

$$\xi(b) = 0. \tag{3.20}$$

Similarly, using $X = \xi$ and $Y = e$ in (3.16) and applying (3.7), (3.10) we obtain

$$f - 2 - 2e(b) - 2b^2 = 0. \tag{3.21}$$

It follows from the above relation, (3.4), and (3.19) that

$$e(b) = \phi e(b). \tag{3.22}$$

Using (3.22) and putting $X = e$, $Y = \xi$, and $Z = \phi e$ in (1.1) we have

$$f - 2 + 2e(b) - 2b^2 = 0.$$

Comparing the above relation with (3.21) and making using of (3.20) and (3.22) we conclude that $b = c$ is a constant. Finally, applying $\lambda = 1$, $f = 2 + 2b^2$, and $b = c =$ constant in equations (3.6)-(3.14) it can be easily seen that the Ricci operator Q is parallel and hence the manifold is locally symmetric.

Because Wang [18] and Cho [3] proved that any almost Kenmotsu 3-manifold is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product space $\mathbb{H}^2(-4) \times \mathbb{R}$, then the proof follows. □

Remark 3.8. Proposition 3.4 and Theorem 3.7 can be regarded as some generalizations of the main results proved in [3], [7] and [18].

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