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Fixed points of Bregman relatively nonexpansive mappings and solutions of variational inequality problems

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Abstract

In this paper, we propose an iterative scheme for finding a common point of the fixed point set of a Bregman relatively nonexpansive mapping and the solution set of a variational inequality problem for a continuous monotone mapping. We prove a strong convergence theorem for the sequences produced by the method. Our results improve and generalize various recent results.

Keywords: Bregman distance function, Bregman relatively nonexpansive mapping, fixed points of mappings, strong convergence, monotone mapping.
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1. Introduction.

Let E denote a real reflexive Banach space with norm $||.||$ and E^* stands for the (topological) dual of E endowed with the induced norm $||.||_*$. Let C be a nonempty subset of E. A mapping $A : C \to E^*$ is said to be *monotone* if for any $x, y \in C$, we have

$$\langle Ax - Ay, x - y\rangle \geq 0.$$

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We note that the class of monotone mappings includes the class of γ-inverse strongly monotone mappings, where a mapping $A : C \to E^*$ is called *γ-inverse strongly monotone* [7, 38] if there exists a positive real number γ such that,

$$\langle Ax - Ay, x - y\rangle \geq \gamma ||Ax - Ay||^2, \text{ for all } x, y \in C. \tag{1.1}$$

The monotone mapping A is called *maximal,* if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone mapping.

The variational inequality problem for a monotone mapping A is the problem of finding a point $x^* \in C$ satisfying

$$\forall x \in C, \ \langle Ax^*, x - x^*\rangle \geq 0. \tag{1.2}$$

We denote the solution set of this problem by $VI(C, A)$. We note that if A is a continuous monotone mapping then the solution set $VI(C, A)$ is always closed and convex.

The monotone variational inequalities were initially investigated by Kinderlehrer and Stampacchia in [9] and are related with the convex minimization problems, the zeros of monotone mappings and the complementarity problems. Consequently, many researchers have studied variational inequality problems for monotone mappings (see, e.g., [26, 27, 28, 31, 32]).

In this paper, $f : E \to (-\infty, +\infty]$ is always a proper, lower semi-continuous and convex function with $\text{dom}f = \{x \in E : f(x) < \infty\}$. For any $x \in \text{int}(\text{dom}f)$ and any $y \in E$, let $f^0(x, y)$ be the right-hand derivative of f at x in the direction of y, that is,

$$f^0(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{1.3}$$

The function f is said to be *Gâteaux differentiable* at x, if $\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^0(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. The function f is said to be *Frêchet differentiable* at $x \in E$ (see, for example, [4]), if for all $\epsilon > 0$, there exists $\delta > 0$ such that $||x - y|| \leq \delta$ implies that

$$|f(x) - f(y) - \langle x - y, \nabla f(y)\rangle| \leq \epsilon ||x - y||. \tag{1.4}$$

The function f is said to be *Frêchet differentiable*, if it is Frêchet differentiable everywhere. The function f is said to be *strongly coercive* if

$$\lim_{||x||\to\infty} \frac{f(x)}{||x||} = \infty. \tag{1.5}$$

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : \text{dom}f \times \text{int}(\text{dom}f) \to [0, +\infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle\nabla f(y), x - y\rangle,$$

is called the *Bregman distance with respect to* f [3]. A *Bregman projection* [3] of $x \in \text{int}(\text{dom}f)$ onto the nonempty closed and convex set $C \subset \text{dom}f$ is the unique vector $P^f_C(x) \in C$ satisfying

$$D_f(P^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

If E is a smooth Banach space, setting $f(x) = ||x||^2$ for all $x \in E$, we have $\nabla f(x) = 2Jx$, where J is the normalized duality mapping from E into 2^{E^*} defined by $Jx := \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2\}$ and hence $D_f(x, y)$ reduces to $\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2$ for all $x, y \in E$, which is the Lyapunov function introduced by Alber [1]. In this case, the Bregman projection P^f_C reduces to the *generalized projection,* Π_C (see [1]). If, in addition, $E = H$, a Hilbert space, then $D_f(x, y)$ becomes $\phi(x, y) = ||x - y||^2$ for $x, y \in H$ and the Bregman projection $P^f_C(x)$ reduces to the metric projection P_C from E onto C.

A point $x \in C$ is a *fixed point of* $T : C \to C$ if $Tx = x$ and we denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an *asymptotic fixed point of* T (see

[17]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$.

A mapping $T : C \to \mathrm{int}(\mathrm{dom} f)$ with $F(T) := \{x \in D(T) : Tx = x\} \neq \emptyset$ is called:

(i) *quasi-Bregman nonexpansive* [21] if,

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T);$$

(ii) *Bregman relatively nonexpansive* [21] if,

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T), \text{ and } \widehat{F}(T) = F(T).$$

When E is a smooth Banach space and $f(x) = ||x||^2$ for all $x \in E$, the above definitions reduce to the following definitions using Lyapunov function.

A mapping $T : C \to \mathrm{int}(\mathrm{dom} f)$ with $F(T) \neq \emptyset$ is called:

(i) *quasi-nonexpansive* [21] if,

$$\phi(p, Tx) \leq \phi(p, x), \forall x \in C, p \in F(T);$$

(ii) *relatively nonexpansive* [21] if,

$$\phi(p, Tx) \leq \phi(p, x), \forall x \in C, p \in F(T), \text{ and } \widehat{F}(T) = F(T).$$

Various methods have been introduced for approximating fixed points of relatively nonexpansive and quasi-nonexpansive mappings (see, e.g., [8, 10, 13, 15, 21, 24, 30]). In 2011, Zhang et al. [39] introduced an iteration method for finding fixed point of relatively nonexpansive mappings in a Banach space setting as follows.

Theorem 1.1 ([39]). *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : C \to C$ be a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)JTx_n), n \geq 1, \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. If the interior of $F(T)$ is nonempty, then they proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T.

In 2005, Matsushita and Takahashi [14] proposed the following hybrid iteration method for a relatively nonexpansive mapping T in a Banach space E. Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Define the sequences $\{x_n\}$ by

$$\begin{cases} x_0 \in C = C_1, \text{ chosen arbitrary}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n, \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), n \geq 1. \end{cases} \tag{1.7}$$

They proved that the sequence $\{x_n\}$ generated by (1.7) converges *strongly* to the point $\Pi_{F(T)}(x_0)$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

More recently, many authors have also considered the problem of finding a common element of the fixed point set of a relatively nonexpansive or a Bregman relatively nonexpansive mapping and the solution set of a variational inequality problem for $\gamma-$inverse strongly monotone mapping (see, e.g., [7, 11, 12, 26, 27, 28, 32, 33, 34, 35]). For other related results, we refer to [22, 23, 36, 37].

In 2009, Inoue et al. [8] proposed the following hybrid iteration method in a uniformly convex and uniformly smooth Banach space E for a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in C = C_1, \text{ chosen arbitrary}, \\ u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTJ_{r_n}x_n, \\ C_n = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), n \geq 1, \end{cases} \tag{1.8}$$

where $T : C \to C$ is a relatively nonexpansive mapping and $J_r = (J + rB)^{-1}J$, for $B : C \to E^*$ maximal monotone mapping and $r > 0$. They proved that the sequence $\{x_n\}$ converges *strongly* to the point $\Pi_{F(T)\cap B^{-1}(0)}(x_0)$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

In this paper, it is our purpose to investigate an iterative scheme for finding a common point of the fixed point set of a Bregman relatively nonexpansive mapping and the solution set of a variational inequality problem for a continuous monotone mapping in reflexive Banach spaces. We prove a strong convergence theorem for the sequence produced by the method. Our results improve and generalize various recent results (see, e.g., [8, 12]).

2. Preliminaries

Legendre function f from a general Banach space E into $(-\infty, +\infty]$ were defined in [2]. The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(y) = \sup\{\langle y, x\rangle - f(x) : x \in E\}$. If E is a reflexive Banach space and $f : E \to (-\infty, +\infty]$ is a Legendre function, then in view of [2],

$$\nabla f = (\nabla f^*)^{-1}, \text{ran } \nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom} f^*) \text{ and } \text{ran}\nabla f^* = \text{int}(\text{dom} f),$$

where $\text{ran}\nabla f$ denotes the range of ∇f. When E is a smooth and strictly convex Banach space, one important and interesting example of Legendre function is $f(x) := \frac{1}{p}||x||^p (1 < p < \infty)$. In this case the gradient ∇f of f coincides with the generalized duality mapping of E, i.e., $\nabla f = J_p (1 < p < \infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces.

Lemma 2.1 ([29])**.** *Let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(i) f is bounded on bounded subsets and uniformly smooth on bounded subsets of E;

(ii) f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*;

(iii) $\text{dom} f^* = E^*$, f^* is strongly coercive and uniformly convex on bounded subsets of E^*.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The *modulus of total convexity of f* at $x \in \text{dom} f$ is the function $\nu_f(x, .) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom} f, ||y - x|| = t\}.$$

The function f is called *totally convex* at x if $\nu_f(x, t) > 0$, whenever $t > 0$. The function f is called *totally convex* if it is totally convex at any point $x \in \text{int}(\text{dom} f)$ and is said to be *totally convex on bounded sets* if $\nu_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \text{int}(\text{dom} f) \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap \text{dom} f\}.$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [5], Theorem 2.10).

Let $B_r := \{z \in E : ||z|| \leq r\}$, for all $r > 0$ and $S_E = \{x \in E : ||x|| = 1\}$. Then a function $f : E \to \mathbb{R}$ is said to be *uniformly convex* on bounded subsets of E ([29], pp. 203) if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty]$ is defined by

$$\rho_r(t) := \inf_{x,y \in B_r, ||x-y||=t, \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}$$

for all $t \geq 0$.

In the sequel, we shall need the following lemmas.

Lemma 2.2 ([15]). *Let E be a Banach space, let $r > 0$ be a constant and let $f : E \to \mathbb{R}$ be a uniformly convex on bounded subsets of E. Then*

$$f(\sum_{k=0}^{n} \alpha_k x_k) \leq \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(||x_i - y_j||)$$

for all $i, j \in \{0, 1, 2, ..., n\}$, $x_k \in Br, \alpha_k \in (0, 1)$ and $k = 0, 1, 2, ..., n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 2.3 ([19]). *Let $f : E \to (-\infty, +\infty]$ be uniformly Fréchet differentiable and bounded on bounded sets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.4 ([18]). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty closed convex subset of* $\mathrm{int}(\mathrm{dom}f)$ *and $T : C \to C$ be a quasi-Bregman nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.5 ([4]). *The function $f : E \to (-\infty, +\infty)$ is totally convex on bounded subsets of E if and only if for any two sequences $\{x_n\}$ and $\{y_n\} \in \mathrm{int}(\mathrm{dom}f)$ and $\mathrm{dom}f$, respectively, such that the first one is bounded,*

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \implies \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Lemma 2.6 ([16]). *Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have*

$$D_f(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i))) \leq \sum_{i=1}^{N} t_i D_f(z, x_i).$$

Lemma 2.7 ([13]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable on* $\mathrm{int}(\mathrm{dom}f)$ *such that ∇f^* is bounded on bounded subsets of* $\mathrm{dom}f^*$. *Let $x \in E$ and $\{x_n\} \subset E$. If $\{D_f(x, x_n)\}$ is bounded, so is the sequence $\{x_n\}$.*

Lemma 2.8 ([5]). *Let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*

(i) $z = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \forall y \in C$.

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \forall y \in C$.

Let $f : E \to \mathbb{R}$ be a Legendre and Gâteaux differentiable function. Following [1] and [6], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \forall x \in E, x^* \in E^*. \tag{2.1}$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \text{ for all } x \in E \text{ and } x^* \in E^*. \tag{2.2}$$

Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*), \tag{2.3}$$

$\forall x \in E$ and $x^*, y^* \in E^*$ (see [10]).

Lemma 2.9 ([25])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim\limits_{n\to\infty} \alpha_n = 0, \sum\limits_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup\limits_{n\to\infty} \delta_n \leq 0$. Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.10 ([12])**.** *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, ..., k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

Following the agreement in [20] we have the following lemma.

Lemma 2.11. *Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function and C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be a continuous monotone mapping. For $r > 0$ and $x \in E$, define the mapping $F_r : E \to C$ as follows:*

$$F_r x := \{z \in C : \langle Az, y - z\rangle + \frac{1}{r}\langle\nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \forall\, y \in C\}$$

for all $x \in E$. Then the following hold:

(1) F_r is single- valued;

(2) $F(F_r) = VI(C, A)$;

(3) $D_f(p, F_r x) + D_f(F_r x, x) \leq \phi(p, x)$, for $p \in F(F_r)$;

(4) $VI(C, A)$ is closed and convex.

3. Main Results

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping and let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Then in what follows, for each n, let $F_{r_n} : E \to C$ be defined by

$$F_{r_n}(x) := \{z \in C : \langle Az, y - z\rangle + \frac{1}{r_n}\langle\nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \forall\, y \in C\},$$

for all $x \in E$, where $\{r_n\} \subset (a, \infty)$ for some $a > 0$.

We now prove the following theorem.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} f)$. *Let $T : C \to E$ be a Bregman relatively nonexpansive mapping and $A : C \to E^*$ be a continuous monotone mapping. Assume that $\mathcal{F} := F(T) \cap V(C, A)$ is nonempty. For $u, x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \nabla f^*\big(a_n \nabla f(x_n) + b_n \nabla f(F_{r_n}(x_n)) + c_n \nabla f(T(x_n))\big), \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)), \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subset [c, d] \subset (0, 1)$ such that $a_n + b_n + c_n = 1$ and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.

Proof. From Lemmas 2.4 and 2.11 we get that $\mathcal{F}$ is closed and convex. Thus, $P_{\mathcal{F}}^f$ is well-defined. Let $p = P_{\mathcal{F}}^f(u)$ and $u_n = F_{r_n}(x_n)$. Now, since f is bounded and uniformly smooth on bounded subsets of E by Lemma 2.1 we get that f^* is uniformly convex on bounded subsets of E^*. Then, from (3.1), (2.1), (2.2) and Lemmas 2.2, 2.11 together with the property of D_f we obtain

$$\begin{aligned} D_f(p, y_n) &= D_f(p, \nabla f^*(a_n \nabla f(x_n) + b_n \nabla f(u_n) + c_n \nabla f(T(x_n))) \\ &= V_f(p, a_n \nabla f(x_n) + b_n \nabla f(u_n) + c_n \nabla f(T(x_n))) \\ &\leq f(p) - \langle p, a_n \nabla f(x_n) + b_n \nabla f(u_n) + c_n \nabla f(T(x_n))\rangle \\ &\quad + f^*(a_n \nabla f(x_n) + b_n \nabla f(u_n) + c_n \nabla f(T(x_n)) \\ &\leq f(p) - a_n\langle p, \nabla f(x_n)\rangle - b_n\langle p, \nabla f(u_n)\rangle - c_n\langle p, \nabla f(T(x_n))\rangle \\ &\quad + a_n f^*(\nabla f(x_n)) + b_n \nabla f^*(f(u_n)) + c_n f^*(\nabla f(T(x_n)) \\ &\quad - a_n b_n \rho_r^*(||\nabla f(x_n) - \nabla f(u_n)||) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} D_f(p, y_n) \leq& a_n V_f(p, \nabla f(x_n)) + b_n V_f(p, \nabla f(u_n)) + c_n V_f(p, \nabla f(T(x_n))) \\ &- a_n b_n \rho_r^*(||\nabla f(x_n) - \nabla f(u_n)||) \\ =& a_n D_f(p, x_n) + b_n D_f(p, u_n) + c_n D_f(p, T(x_n)) \\ &- a_n b_n \rho_r^*(||\nabla f(x_n) - \nabla f(u_n)||) \\ \leq& a_n D_f(p, x_n) + b_n D_f(p, x_n) + c_n D_f(p, x_n) \\ &- a_n b_n \rho_r^*(||\nabla f(x_n) - \nabla f(u_n)||) \\ \leq& D_f(p, x_n) - a_n b_n \rho_r^*(||\nabla f(x_n) - \nabla f(u_n)||) \leq D_f(p, x_n). \end{aligned} \tag{3.3}$$

Similarly, we get that

$$D_f(p, y_n) \leq D_f(p, x_n) - a_n c_n \rho_r^*(||\nabla f(x_n) - \nabla f(T(x_n))||) \leq D_f(p, x_n). \tag{3.4}$$

In addition, from (3.1), (3.3) and Lemmas 2.6, 2.8 we have

$$\begin{aligned} D_f(p, x_{n+1}) =& D_f(p, P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)) \\ \leq& D_f(p, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)) \\ \leq& \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, y_n) \\ \leq& \alpha_n D_f(p, u) + (1-\alpha_n)\Big[D_f(p, x_n) \\ &\quad - a_n b_n \rho_r^*(||\nabla f(x_n) - \nabla f(u_n)||)\Big] \\ \leq& \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n). \end{aligned}$$

Thus, by induction,

$$D_f(p, x_{n+1}) \leq \max\{D_f(p, u), D_f(p, x_0)\}, \forall n \geq 0,$$

which implies that $\{x_n\}$ is bounded. Now, let $z_n = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n))$. Then we have that $x_{n+1} = P_C^f z_n$, for all $n \in \mathbb{N}$. Since f is strongly coercive, uniformly convex, uniformly Fréchet differentiable and bounded, by Lemmas 2.3 and 2.1 we get that ∇f and ∇f^* are bounded and hence $\{z_n\}$ and $\{y_n\}$ are bounded. Furthermore, using (2.2), (2.3) and property of D_f we obtain that

$$\begin{aligned} D_f(p, x_{n+1}) \leq & D_f(p, z_n) = D_f(p, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)) \\ = & V_f(p, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)) \\ = & V_f(p, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n) - \alpha_n(\nabla f(u) - \nabla f(p))) \\ & - \langle -\alpha_n(\nabla f(u) - \nabla f(p), z_n - p\rangle \\ = & V_f(p, \alpha_n \nabla f(p) + (1-\alpha_n)\nabla f(y_n)) + \alpha_n\langle \nabla f(u) - \nabla f(p), z_n - p\rangle \\ = & D_f(p, \nabla f^*(\alpha_n \nabla f(p) + (1-\alpha_n)\nabla f(y_n))) \\ & + \alpha_n\langle \nabla f(u) - \nabla f(p), z_n - p\rangle \\ \leq & D_f(p,p) + (1-\alpha_n)D_f(p, y_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), z_n - p\rangle \\ \leq & (1-\alpha_n)D_f(p, y_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), z_n - p\rangle. \end{aligned} \tag{3.5}$$

Thus, from (3.3), (3.4) and (3.5) we get

$$\begin{aligned} D_f(p, x_{n+1}) \leq & (1-\alpha_n)D_f(p, x_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), z_n - p\rangle \\ & - a_n b_n \rho_r^*(||\nabla f(x_n) - \nabla f(u_n)||) & (3.6) \\ \leq & (1-\alpha_n)D_f(p, x_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), z_n - p\rangle, & (3.7) \end{aligned}$$

or

$$\begin{aligned} D_f(p, x_{n+1}) \leq & (1-\alpha_n)D_f(p, x_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), z_n - p\rangle \\ & - a_n \delta_n \rho_r^*(||\nabla f(x_n) - \nabla f(T(x_n))||) \\ \leq & (1-\alpha_n)D_f(p, x_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), z_n - p\rangle. \end{aligned} \tag{3.8}$$

The rest of the proof is divided into two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(p, x_n)\}$ is non-increasing for all $n \geq n_0$. Thus, we get that $\{D_f(p, x_n)\}$ is convergent. Now, from (3.6) and (3.8) we have that

$$a_n b_n \rho_r^*(||\nabla f(x_n) - \nabla f(u_n)||) \to 0, \tag{3.9}$$

and

$$a_n c_n \rho_r^*(||\nabla f(x_n) - \nabla f(T(x_n))||) \to 0, \tag{3.10}$$

which give by the property of ρ_r^* that

$$\nabla f(x_n) - \nabla f(u_n) \to 0, \nabla f(x_n) - \nabla f(T(x_n)) \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Moreover, from (3.1) and (3.11) we have that

$$\begin{aligned} ||\nabla f(y_n) - \nabla f(x_n)|| \leq & a_n||\nabla f(x_n) - \nabla f(x_n)|| + b_n||\nabla f(u_n) - \nabla f(x_n)|| \\ & + c_n||\nabla f(T(x_n)) - \nabla f(x_n)|| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.12}$$

In addition, since f is strongly coercive and uniformly convex on bounded subsets of E we have that f^* is uniformly Fréchet differentiable on bounded subsets of E^* and by Lemma 2.1 we get that ∇f^* is uniformly continuous. Thus, this with (3.11) and (3.12) give that

$$x_n - u_n \to 0, x_n - T(x_n) \to 0, x_n - y_n \to 0 \text{ as } n \to \infty. \tag{3.13}$$

Furthermore, Lemma 2.6, property of D_f and the fact that $\alpha_n \to 0$ as $n \to \infty$, imply that

$$\begin{aligned} D_f(y_n, z_n) =& D_f(y_n, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)) \\ \leq& \alpha_n D_f(x_n, u) + (1-\alpha_n) D_f(y_n, y_n) \\ \leq& \alpha_n D_f(x_n, u) + (1-\alpha_n) D_f(y_n, y_n) \to 0 \text{ as } n \to \infty, \end{aligned} \tag{3.14}$$

and hence by Lemma 2.5 we get that

$$y_n - z_n \to 0 \text{ as } n \to \infty. \tag{3.15}$$

Now, since $\{z_n\}$ is bounded and E is reflexive, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and $\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), z_n - p\rangle = \lim_{i\to\infty} \langle \nabla f(u) - \nabla f(p), z_{n_i} - p\rangle$. Then, from (3.15) and (3.13) we get that

$$x_{n_i} \rightharpoonup z, \text{ as } i \to \infty. \tag{3.16}$$

Thus, from (3.13) and the fact that T is Bregman relatively nonexpansive we obtain that $z \in F(T)$.

Now, we show that $z \in VI(C, A)$. By definition we have that

$$\langle Au_n, y - u_n\rangle + \langle \frac{\nabla f(u_n) - \nabla f(x_n)}{r_n}, y - u_n\rangle \geq 0, \ \forall\, y \in C, \tag{3.17}$$

and hence

$$\langle Au_{n_i}, y - u_{n_i}\rangle + \langle \frac{\nabla f(u_{n_i}) - \nabla f(x_{n_i})}{r_{n_i}}, y - u_{n_i}\rangle \geq 0, \ \forall\, y \in C. \tag{3.18}$$

Set $v_t = ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in C$. Consequently, we get that $v_t \in C$. Now, from (3.18) it follows that

$$\begin{aligned} \langle Av_t, v_t - u_{n_i}\rangle &\geq \langle Av_t, v_t - u_{n_i}\rangle - \langle Au_{n_i}, v_t - u_{n_i}\rangle - \langle \frac{\nabla f(u_{n_i}) - \nabla f(x_{n_i})}{r_{n_i}}, v_t - u_{n_i}\rangle \\ &= \langle Av_t - Au_{n_i}, v_t - u_{n_i}\rangle - \langle \frac{\nabla f(u_{n_i}) - \nabla f(x_{n_i})}{r_{n_i}}, v_t - u_{n_i}\rangle. \end{aligned}$$

But, from (3.13) have that $\dfrac{\nabla f(u_{n_i}) - \nabla f(x_{n_i})}{r_{n_i}} \to 0$, as $i \to \infty$ and the monotonicity of A implies that $\langle Av_t - Au_{n_i}, v_t - u_{n_i}\rangle \geq 0$. Thus, it follows that

$$0 \leq \lim_{i\to\infty} \langle Av_t, v_t - u_{n_i}\rangle = \langle Av_t, v_t - z\rangle,$$

and hence

$$\langle Av_t, y - z\rangle \geq 0, \ \forall\, y \in C.$$

If $t \to 0$, the continuity of A implies that

$$\langle Az, y - z\rangle \geq 0, \ \forall\, y \in C.$$

This implies that $z \in VI(C, A)$ and hence $z \in \mathcal{F} = F(T) \cap VI(C, A)$.

Therefore, by Lemma 2.8, we immediately obtain that $\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), z_n - p\rangle = \lim_{i\to\infty} \langle \nabla f(u) - \nabla f(p), z_{n_i} - p\rangle = \langle \nabla f(u) - \nabla f(p), z - p\rangle \leq 0$. It follows from Lemma 2.9 and (3.7) that $D_f(p, x_n) \to 0$, as $n \to \infty$. Consequently, by Lemma 2.5 we obtain that, $x_n \to p$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_f(p, x_{n_i}) < D_f(p, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.10, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1})$ and $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$, for all $k \in \mathbb{N}$. Then from (3.6), (3.8) and the fact that $\alpha_n \to 0$ we obtain that

$$\rho_r^*(||\nabla f(x_{m_k}) - \nabla f(Tx_{m_k})||) \to 0 \text{ and } \rho_r^*(||\nabla f(x_{m_k}) - \nabla f(u_{m_k})||) \to 0,$$

as $k \to \infty$. Thus, following the method of proof in Case 1, we obtain that $x_{m_k} - Tx_{m_k} \to 0$, $x_{m_k} - u_{m_k} \to 0$, $x_{m_k} - y_{m_k} \to 0$, $y_{m_k} - z_{m_k} \to 0$ as $k \to \infty$, and hence we obtain that

$$\limsup_{k\to\infty} \langle \nabla f(u) - \nabla f(p), z_{m_k} - p\rangle \leq 0. \tag{3.19}$$

Now, from (3.7) we have that

$$D_f(p, x_{m_k+1}) \leq (1 - \alpha_{m_k}) D_f(p, x_{m_k}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), z_{m_k} - p\rangle, \tag{3.20}$$

and since $D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1})$, inequality (3.20) implies

$$\begin{aligned} \alpha_{m_k} D_f(p, x_{m_k}) &\leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), z_{m_k} - p\rangle \\ &\leq \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), z_{m_k} - p\rangle. \end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we get

$$D_f(p, x_{m_k}) \leq \langle \nabla f(u) - \nabla f(p), z_{m_k} - p\rangle.$$

Hence, from (3.19) we get $D_f(p, x_{m_k}) \to 0$ as $k \to \infty$. This together with (3.20) gives $D_f(p, x_{m_k+1}) \to 0$ as $k \to \infty$. But $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$ for all $k \in \mathbb{N}$, thus we obtain that $x_k \to p$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$ and the proof is complete. □

If, in Theorem 3.1, we assume that $T = I$, the identity mapping on C, we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of* $\mathrm{int}(\mathrm{dom} f)$. *Let $A : C \to E^*$ be a continuous monotone mapping. Assume that $V(C, A)$ is nonempty. For $u, x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \nabla f^*\big(a_n \nabla f(x_n) + (1 - a_n)\nabla f(F_{r_n}(x_n))\big), \forall n \geq 0, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(y_n)), \forall n \geq 0, \end{cases} \tag{3.21}$$

where $\{a_n\} \subset [c, d] \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P^f_{V(C,A)}(u)$.

If, in Theorem 3.1, we assume that $C = E$, the projection mapping P^f_C is not required and $VI(C, A) = A^{-1}(0)$ hence we get the following corollary.

Corollary 3.3. *Let $T : E \to E$ be a Bregman relatively nonexpansive mapping and $A : E \to E^*$ be a continuous monotone mapping. Assume that $\mathcal{F} := F(T) \cap A^{-1}(0)$ is nonempty. For $u, x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \nabla f^*\big(a_n \nabla f(x_n) + b_n \nabla f(F_{r_n}(x_n)) + c_n \nabla f(T(x_n))\big), \forall n \geq 0, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(y_n)), \forall n \geq 0, \end{cases} \tag{3.22}$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subset [c, d] \subset (0, 1)$ such that $a_n + b_n + c_n = 1$ and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$.

We also note that the method of proof of Theorem 3.1 provides the following theorem for approximating the minimum-norm common point of the fixed point set of a Bregman relatively nonexpansive mapping and the solution set of a variational inequality problem for a continuous monotone mapping.

Theorem 3.4. *Let C be a nonempty, closed and convex subset of* $\mathrm{int}(\mathrm{dom}f)$. *Let $T : C \to E$ be a Bregman relatively nonexpansive mapping and $A : C \to E^*$ be a continuous monotone mapping. Assume that $\mathcal{F} := F(T) \cap V(C, A)$ is nonempty. For $x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \nabla f^*\big(a_n \nabla f(x_n) + b_n \nabla f(F_{r_n}(x_n)) + c_n \nabla f(T(x_n))\big), \\ x_{n+1} = P_C^f \nabla f^*((1-\alpha_n)\nabla f(y_n)), \forall n \geq 0, \end{cases} \tag{3.23}$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subset [c, d] \subset (0, 1)$ such that $a_n + b_n + c_n = 1$ and $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to the minimum-norm point p of $\mathcal{F}$ with respect to the Bregman distance.

Remark 3.5. Theorem 3.1 improves and extends the corresponding results of Inoue *et al.* [8] to the class of Bregman relatively nonexpansive mappings and to the class of continuous monotone mappings in reflexive real Banach spaces.

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References

[1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., Dekker, New York, **178** (1996), 15–50. 1, 2

[2] J. F. Bonnans, A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer-Verlag, (2000). 2

[3] L. M. Bregman, *The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming*, USSR Comput. Math. Math. Phys., **7** (1967), 200–217. 1

[4] D. Butnariu, A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Springer Netherlands, Dordrecht, (2000). 1, 2.5

[5] D. Butnariu, E. Resmerita, *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces*, Abstr. Appl. Anal., **2006** (2006), 39 pages. 2, 2.8

[6] Y. Censor, A. Lent, *An iterative row-action method for interval convex programming*, J. Optim. Theory Appl., **34** (1981), 321–353. 2

[7] H. Iiduka, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal., **61** (2005), 341–350. 1, 1

[8] G. Inoue, W. Takahashi, K. Zembayashi, *Strong convergence theorems by hybrid method for maximal monotone operators and relatively nonexpansive mappings in Banach spaces*, J. Convex Anal., **16** (2009), 791–806. 1, 1, 1, 3.5

[9] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, (1980). 1

[10] F. Kohsaka, W. Takahashi, *Proximal point algorithms with Bregman functions in Banach spaces,* J. Nonlinear Convex Anal., **6** (2005), 505–523. 1, 2

[11] G. M. Korpelevich, *An extragradient method for finding saddle points and for other problems*, konom. i Mat. Metody, **12** (1976), 747–756. 1

[12] P. E. Mainge, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal., **16** (2008), 899–912. 1, 1, 2.10

[13] V. Martin-Marqueza, S. Reich, S. Sabach, *Bregman strongly nonexpansive operators in reflexive Banach spaces*, J. Math. Anal. Appl., **400** (2013), 597–614. 1, 2.7

[14] S. Matsushita, W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory, **134** (2005), 257–266. 1

[15] E. Naraghirad, J.-C. Yao, *Bregman weak relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., **2013** (2013), 43 pages. 1, 2.2

[16] R. R. Phelps, *Convex functions, monotone operators and differentiability*, Springer-Verlag, Berlin, (1993). 2.6
[17] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., Dekker, New York, **178** (1996), 313–318. 1
[18] S. Reich, S. Sabach, *Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces*, Fixed-point algorithms for inverse problems in science and engineering, Springer Optim. Appl., Springer, New York, **49** (2011), 301–316. 2.4
[19] S. Reich, S. Sabach, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal., **10** (2009), 471–485. 2.3
[20] S. Reich, S. Sabach, *Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces,* Nonlinear Anal., **73** (2010), 122–135. 2
[21] S. Reich, S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim., **31** (2010), 22–44. 1
[22] N. Shahzad, H. Zegeye, *Convergence theorem for common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings*, Fixed Point Theory Appl., **2014** (2014), 14 pages. 1
[23] N. Shahzad, H. Zegeye, A. Alotaibi, *Convergence results for a common solution of a finite family of variational inequality problems for monotone mappings with Bregman distance function*, Fixed Point Theory Appl., **2013** (2013), 13 pages. 1
[24] W. Takahashi, K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, Nonlinear Anal., **70** (2009), 45–57. 1
[25] H.-K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc., **65** (2002), 109–113. 2.9
[26] I. Yamada, *The hybrid steepest-descent method for variational inequality problems over the intersection of the fixed point sets of nonexpansive mappings,* Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, Edited by D. Butnariu, Y. Censor, and S. Reich, North-Holland, Amsterdam, Holland, (2001), 473–504. 1, 1
[27] Y. H. Yao, Y.-C. Liou, C.-L. Li, H.-T. Lin, *Extended extragradient methods for generalized varitional inequalities*, J. Appl. Math., **2012** (2012), 14 pages. 1, 1
[28] Y. H. Yao, H.-K. Xu, *Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications*, Optimization, **60** (2011), 645–658. 1, 1
[29] C. Zalinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing Co., River Edge, (2002). 2.1, 2
[30] H. Zegeye, E. U. Ofoedu, N. Shahzad, *Convergence theorems for equilibrium problem, variational inequality problem and countably infinite relatively quasi-nonexpansive mappings*, Appl. Math. Comput., **216** (2010), 3439–3449. 1
[31] H. Zegeye, N. Shahzad, *Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings*, Nonlinear Anal., **70** (2009), 2707–2716. 1
[32] H. Zegeye, N. Shahzad, *A hybrid approximation method for equilibrium, variational inequality and fixed point problems*, Nonlinear Anal., **4** (2010), 619–630. 1, 1
[33] H. Zegeye, N. Shahzad, *Strong convergence theorems for a finite family of nonexpansive mappings and semigroups via the hybrid method*, Nonlinear Anal., **72** (2010), 325–329. 1
[34] H. Zegeye, N. Shahzad, *A hybrid scheme for finite families of equilibrium, variational inequality and fixed point problems*, Nonlinear Anal., **74** (2011), 263–272. 1
[35] H. Zegeye, N. Shahzad, *Approximating common solution of variational inequality problems for two monotone mappings in Banach spaces*, Optim. Lett., **5** (2011), 691–704. 1
[36] H. Zegeye, N. Shahzad, *An algorithm for finding a common point of the solution set of a variational inequality and the fixed point set of a Bregman relatively nonexpansive mapping*, Appl. Math. Comput., **248** (2014), 225–234. 1
[37] H. Zegeye, N. Shahzad, *Convergence theorems for right Bregman strongly nonexpansive mappings in reflexive Banach spaces*, Abstr. Appl. Anal., **2014** (2014), 8 pages. 1
[38] H. Zegeye, N. Shahzad, *Strong convergence theorems for a common zero of a countably infinite family of α-inverse strongly accretive mappings*, Nonlinear Anal., **71** (2009), 531–538. 1
[39] C. Zhang, J. Li, B. Liu, *Strong convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, Comput. Math. Appl., **61** (2011), 262–276. 1, 1.1