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On common fixed points for (α, ψ)-contractions and generalized cyclic contractions in b-metric-like spaces and consequences

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Abstract

In this paper, using the concept of α-admissible pairs of mappings, we prove several common fixed point results in the setting of b-metric-like spaces. We also introduce the notion of generalized cyclic contraction pairs and establish some common fixed results for such pairs in b-metric-like spaces. Some examples are presented making effective the new concepts and results. Moreover, as consequences we prove some common fixed point results for generalized contraction pairs in partially ordered b-metric-like spaces.

Keywords: Common fixed point, b-metric-like space, cyclic contraction, admissible mapping.
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1. Introduction and Preliminaries

The concept of b-metric spaces and related fixed point theorems have been investigated by a number of authors; see for example [5, 8, 11, 12, 14, 15, 23, 28]. In 2013, Alghamdi et al. [2] generalized the notion of a b-metric by introduction of the concept of a b-metric-like and proved some related fixed point results. After that, Chen et al. [13] and Hussain et al. [16] proved some fixed point theorems in the setting of b-metric-like spaces.

First, we recall some basic concepts and notations on b-metric-like concept.

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Definition 1.1. Let X be a non-empty and $s \geq 1$. Let $d : X \times X \to [0, \infty)$ be a function such that:
$(d1)$ $d(x, y) = 0$ implies $x = y$,
$(d2)$ $d(x, y) = d(y, x)$,
$(d3)$ $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then, d is called a b-metric-like and the pair (X, d) is called a b-metric-like space. The number s is called the coefficient of (X, d).

In the following, some examples of a b-metric-like which is nor a b-metric neither a metric-like.

Example 1.2. Let $X = \{0, 1, 2\}$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(0, 0) = 0, \quad d(1, 1) = d(2, 2) = 2,$$

$$d(0, 1) = 4, \quad d(1, 2) = 1 \text{ and } d(2, 0) = 2,$$

with $d(x, y) = d(y, x)$ for all $x, y \in X$. Then, (X, d) is a $b-$metric-like space with coefficient $s = 2$, but is nor a b-metric, neither a metric-like since $d(0, 1) = 4 > 3 = d(0, 2) + d(2, 1) = 2 + 1$.

Example 1.3. Let $X = \mathbb{R}$ and $p > 1$ be a real number. Define the function $d : X \times X \to [0, \infty)$ by

$$d(x, y) = (|x| + |y|)^p \quad \forall x, y \in X.$$

Then, (X, d) is a b-metric-like space with coefficient $s = 2^{p-1}$, but is neither a b-metric space since $d(1, 1) = 2^p$ nor a metric-like space since $d(-1, 1) = 2^p > 2 = 1 + 1 = d(-1, 0) + d(0, 1)$.

Example 1.4. Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(x, y) = (x^3 + y^3)^2, \quad \forall x, y \in X.$$

Then (X, d) is a b-metric-like space with coefficient $s = 2$, but is nor a b-metric space since $d(1, 1) = 4$ neither a metric-like space since $d(1, 2) = 81 > 65 = 1 + 64 = d(1, 0) + d(0, 2)$.

Definition 1.5. Let (X, d) be a b-metric-like space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} d(x_n, x) = d(x, x). \tag{1.1}$$

Remark 1.6. In a b-metric-like space, the limit for a convergent sequence is not unique in general.

Definition 1.7. Let (X, d) be a b-metric-like space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} d(x_n, x_m)$ exists and is finite.

Definition 1.8. Let (X, d) be a b-metric-like space. We say that (X, d) is complete if and only if each Cauchy sequence in X is convergent.

Lemma 1.9. *Let (X, d) be a b-metric-like space and $\{x_n\}$ be a sequence that converges to u with $d(u, u) = 0$. Then, for each $z \in X$ one has*

$$\frac{1}{s} d(u, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(u, z).$$

Lemma 1.10. *Let (X, d) be a b-metric-like space and $T : X \to X$ be a given mapping. Suppose that T is continuous at $u \in X$. Then, for all sequence $\{x_n\}$ in X such that $x_n \to u$, we have $Tx_n \to Tu$, that is,*

$$\lim_{n\to\infty} d(Tx_n, Tu) = d(Tu, Tu).$$

Let (X, d) be a b-metric-like space. We need in the sequel the following trivial inequality:

$$d(x, x) \leq 2s d(x, y), \quad \text{for all } x, y \in X. \tag{1.2}$$

In 2012, Samet *et al.* [27] introduced the concept of α-admissible maps.

Definition 1.11 ([27])**.** For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that the self-mapping T on X is α-admissible if for all $x, y \in X$, we have,

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{1.3}$$

Many papers dealing with above notion have been considered to prove some (common) fixed point results, for example see [1, 3, 6, 9, 17, 18, 19, 20, 21, 24, 26].

Very recently, Aydi [4] generalized Definition 1.11 to a pair of mappings.

Definition 1.12. For a nonempty set X, let $A, B : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that (A, B) is an α-admissible pair if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Ax, By) \geq 1 \quad \text{and } \alpha(By, Ax) \geq 1.$$

The following examples illustrate Definition 1.12.

Example 1.13. Let $X = \mathbb{R}$ and $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in [0, 1], \\ 0 \text{ otherwise.} \end{cases}$$

Consider the mappings $A, B : X \to X$ given by

$$Ax = \frac{x}{2} \quad \text{and} \quad Bx = x^2, \quad \forall x \in X.$$

Then, (A, B) is an α-admissible pair. In fact, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By definition of α, this implies that $x, y \in [0, 1]$. Thus,

$$\alpha(Ax, By) = \alpha(\frac{x}{2}, y^2) = 1 \quad \text{and} \quad \alpha(By, Ax) = \alpha(y^2, \frac{x}{2}) = 1.$$

Then, (A, B) is an α-admissible pair.

Example 1.14. Let $X = \mathbb{R}$ and $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = e^{xy} \quad \forall\, x, y \in X.$$

Consider the mappings $A, B : X \to X$ given by

$$Ax = x^3 \quad \text{and} \quad Bx = x^5, \quad \forall x \in X.$$

Then, (A, B) is an α-admissible pair. In fact, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By definition of α, this implies that $xy \geq 0$. Thus,

$$\alpha(Ax, By) = \alpha(By, Ax) = e^{x^3 y^5} \geq 1,$$

because $x^3 y^5 = x^2 y^4 xy \geq 0$. Then, (A, B) is an α-admissible pair.

Take $s \geq 1$. Denote $\mathbb{N}$ the set of positive integers and Ψ_s the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:

(ψ_1) ψ is nondecreasing;

(ψ_2) $\sum_n s^n \psi^n(t) < \infty$ for each $t \in \mathbb{R}^+$, where ψ^n is the nth iterate of ψ.

Remark 1.15. It is easy to see that if $\psi \in \Psi_s$, then $\psi(t) < t$ for any $t > 0$.

In this paper, we provide some common fixed point results for generalized contractions (including cyclic contractions and contractions with a partial order) via α-admissible pair of mappings on b-metric-like spaces. As consequences of our obtained results, we prove some existing known fixed point results on metric-like spaces and on b-metric spaces. Our results will be illustrated by some concrete examples.

2. Fixed Point Theorems for (α, ψ)-contractions

First, we introduce the concept of α-contractive pair of mappings in the setting of b-metric-like spaces.

Definition 2.1. Let (X, d) be a b-metric-like space, $\psi \in \Psi_s$ and $\alpha : X \times X \to [0, \infty)$. A pair $A, B : X \to X$ is called an (α, ψ)-contraction pair if

$$d(Ax, By) \leq \psi(M(x, y)), \tag{2.1}$$

for all $x, y \in X$ satisfying $\alpha(x, y) \geq 1$, where

$$M(x, y) = \max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4s}\}. \tag{2.2}$$

Our first main result is

Theorem 2.2. *Let (X, d) be a complete b-metric-like space and $A, B : X \to X$ be an (α, ψ)-contraction pair. Suppose that*

(i) *(A, B) is an $\alpha-$admissible pair;*
(ii) *there exists $x_0 \in X$ such that $\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1$;*
(iii) *A and B are continuous on (X, d);*
(iv) *$\alpha(z, z) \geq 1$ for every z satisfying the conditions*

$$d(z, z) = 0,\ d(z, Az) \leq sd(Az, Az) \leq s^2 d(z, Az) \text{ and } d(z, Bz) \leq sd(Bz, Bz) \leq s^2 d(z, Bz); \tag{2.3}$$

(v) *$\psi(t) < \frac{t}{2s^2}$ for each $t > 0$.*

Then, A and B admit a common fixed point, i.e. there exists $u \in X$ such that

$$Au = u = Bu. \tag{2.4}$$

Proof. Choose $x_1 = Ax_0$ and $x_2 = Bx_1$. By induction, we can construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Ax_{2n} \text{ and } x_{2n+2} = Bx_{2n+1}, \tag{2.5}$$

for all $n \geq 0$. We split the proof into several steps.
Step 1: $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \geq 0$.

By condition (ii) and the fact that the pair (A, B) is α-admissible,

$$\alpha(x_0, x_1) \geq 1 \Rightarrow \begin{cases} \alpha(x_1, x_2) = \alpha(Ax_0, Bx_1) \geq 1 \text{ and} \\ \alpha(x_2, x_1) = \alpha(Bx_1, Ax_0) \geq 1. \end{cases}$$

Again

$$\alpha(x_2, x_1) \geq 1 \Rightarrow \begin{cases} \alpha(x_3, x_2) = \alpha(Ax_2, Bx_1) \geq 1 \text{ and} \\ \alpha(x_2, x_3) = \alpha(Bx_1, Ax_2) \geq 1. \end{cases}$$

By induction, we may obtain $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \geq 0$.
Step 2: We will show that

$$\text{if for some } \quad n, \quad d(x_{2n}, x_{2n+1}) = 0, \quad \text{then} \quad Ax_{2n} = x_{2n} = Bx_{2n} \tag{2.6}$$

and

$$\text{if for some } \quad n, \quad d(x_{2n+1}, x_{2n+2}) = 0, \quad \text{then} \quad Ax_{2n+1} = x_{2n+1} = Bx_{2n+1}. \tag{2.7}$$

Suppose for some n that $d(x_{2n}, x_{2n+1}) = 0$. We shall prove that $d(x_{2n+1}, x_{2n+2}) = 0$. We argue by contradiction. For this, assume that

$$d(x_{2n+1}, x_{2n+2}) > 0.$$

Then, by Step 1 and (2.1),

$$d(x_{2n+1}, x_{2n+2}) = d(Ax_{2n}, Bx_{2n+1}) \leq \psi(M(x_{2n}, x_{2n+1})),$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Ax_{2n}), d(x_{2n+1}, Bx_{2n+1}), \\ &\quad \frac{d(x_{2n}, Bx_{2n+1}) + d(x_{2n+1}, Ax_{2n})}{4s}\} \\ &= \max\{0, d(x_{2n+1}, x_{2n+2}), \frac{1}{4s}(d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}))\} \\ &= d(x_{2n+1}, x_{2n+2}), \end{aligned}$$

because

$$\begin{aligned} d(x_{2n+1}, x_{2n+1}) &\leq 2sd(x_{2n+1}, x_{2n+2}) \quad \text{and} \\ d(x_{2n}, x_{2n+2}) &\leq sd(x_{2n}, x_{2n+1}) + sd(x_{2n+1}, x_{2n+2}) = sd(x_{2n+1}, x_{2n+2}). \end{aligned}$$

Consequently,

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n+1}, x_{2n+2})).$$

Since $\psi(t) < t$, so we get

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}),$$

a contradiction. Thus, if $d(x_{2n}, x_{2n+1}) = 0$, then $d(x_{2n+1}, x_{2n+2}) = 0$. We deduce that $x_{2n} = x_{2n+1} = x_{2n+2}$, so that

$$\begin{aligned} x_{2n} &= x_{2n+1} = Ax_{2n} \quad \text{and} \\ x_{2n} &= x_{2n+2} = Bx_{2n+1} = B(Ax_{2n}) = Bx_{2n}, \end{aligned}$$

that is x_{2n} is a common fixed point of A and B.

Similarly, one shows that

$$d(x_{2n+1}, x_{2n+2}) = 0 \Rightarrow d(x_{2n+2}, x_{2n+3}) = 0,$$

and so $x_{2n+1} = x_{2n+2} = x_{2n+3}$, which implies

$$\begin{aligned} x_{2n+1} &= x_{2n+2} = Bx_{2n+1} \quad \text{and} \\ x_{2n+1} &= x_{2n+3} = Ax_{2n+2} = A(Bx_{2n+1}) = Ax_{2n+1}, \end{aligned}$$

that is x_{2n+1} is a common fixed point of A and B.

By (2.6) and (2.7), the proof is completed in the case when $d(x_k, x_{k+1}) = 0$ for some $k \geq 0$. From now on, we assume that

$$d(x_n, x_{n+1}) > 0, \quad \forall n \geq 0. \tag{2.8}$$

Step 3. We will show that

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \quad \text{for all } \ n \geq 0. \tag{2.9}$$

By Step 1, $\alpha(x_{2n}, x_{2n-1}) \geq 1$, then

$$d(x_{2n+1}, x_{2n}) = d(Ax_{2n}, Bx_{2n-1}) \leq \psi(M(x_{2n}, x_{2n-1}))$$

where

$$\begin{aligned} M(x_{2n}, x_{2n-1}) =& \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \\ & \frac{d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})}{4s}\} \\ =& \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \frac{1}{4s}(d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n}))\} \\ =& \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\}, \end{aligned}$$

because

$$\begin{aligned} d(x_{2n}, x_{2n}) \leq 2sd(x_{2n}, x_{2n+1}) \quad \text{and} \\ d(x_{2n-1}, x_{2n+1}) \leq sd(x_{2n-1}, x_{2n}) + sd(x_{2n}, x_{2n+1}). \end{aligned}$$

If $\max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n+1})$ for some $n \geq 1$, then

$$0 < d(x_{2n+1}, x_{2n}) \leq \psi(d(x_{2n}, x_{2n+1})).$$

Taking into account $\psi(t) < t$, one obtains a contradiction. It follows that

$$\max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\} = d(x_{2n}, x_{2n-1})$$

for all $n \geq 1$. Then

$$d(x_{2n}, x_{2n+1}) \leq \psi(d(x_{2n}, x_{2n-1})). \tag{2.10}$$

A similar reasoning shows that

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n}, x_{2n+1})). \tag{2.11}$$

Consequently, by (2.10) and (2.11),

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \quad \forall n \geq 1. \tag{2.12}$$

Therefore

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)), \quad \forall n \geq 1.$$

Step 4. We shall show that $\{x_n\}$ is a Cauchy sequence. Using $(d3)$, we have

$$\begin{aligned} d(x_n, x_{n+2}) \leq& sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) \\ \leq& sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_n, x_{n+3}) \leq& sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+3}) \\ \leq& sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}). \end{aligned}$$

By induction, we get for all $m > n$

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} s^{i-n+1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} s^i d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} s^i \psi^i(d(x_0, x_1)) \to 0 \quad \text{as} \quad n \to \infty,$$

which leads to

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0, \tag{2.13}$$

that is, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete b-metric-like space, then there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = d(u, u) = \lim_{n,m\to\infty} d(x_n, x_m) = 0. \tag{2.14}$$

Step 5. u satisfies the condition (2.3).

By the continuity of A, we have $Ax_n \to Au$ in (X, d), that is $\lim_{n\to\infty} d(x_n, Au) = d(Au, Au)$, so that

$$\lim_{n\to\infty} d(x_{2n+1}, Au) = \lim_{n\to\infty} d(Ax_{2n}, Au) = d(Au, Au).$$

On the other side, $\lim_{n\to\infty} d(x_n, u) = 0 = d(u, u)$ and so by Lemma 1.9,

$$\frac{1}{s} d(u, Au) \leq \lim_{n\to\infty} d(x_{2n+1}, Au) \leq s d(u, Au).$$

This yields that

$$\frac{1}{s} d(u, Au) \leq d(Au, Au) \leq s d(u, Au). \tag{2.15}$$

Similarly, one shows that

$$\frac{1}{s} d(u, Bu) \leq d(Bu, Bu) \leq s d(u, Bu). \tag{2.16}$$

Step 6. u is a common fixed point of A and B.

Suppose by contradiction that $d(Au, Bu) > 0$. Since u satisfies (2.3), it follows from (iv) that $\alpha(u, u) \geq 1$, so by (2.1),

$$d(Au, Bu) \leq \psi(M(u, u)),$$

where

$$\begin{aligned} M(u, u) &= \max\{d(u, u), d(u, Au), d(u, Bu), \frac{d(u, Bu) + d(u, Au)}{4s})\} \\ &= \max\{0, d(u, Au), d(u, Bu), \frac{d(u, Bu) + d(u, Au)}{4s})\} \\ &= \max\{d(u, Au), d(u, Bu)\}. \end{aligned}$$

By using (2.15) and (2.16), we get

$$M(u, u) \leq \max\{2s^2 d(Au, Bu), 2s^2 d(Au, Bu)\} = 2s^2 d(Au, Bu).$$

Again, by condition (v), we have

$$d(Au, Bu) \leq \psi(2s^2 d(Au, Bu)) < d(Au, Bu),$$

which is a contradiction. Thus, $d(Au, Bu) = 0$. In this case, the fact that $d(u, Au) \leq s d(Au, Au)$ implies

$$0 \leq d(u, Au) \leq s d(Au, Au) \leq 2s^2 d(Au, Bu) = 0,$$

and so $Au = u$. Therefore, $Bu = Au = u$. The proof is completed. □

In the following, we state some consequences and corollaries of our obtained result.

Corollary 2.3. *Let (X, d) be a complete b-metric-like space, $\psi \in \Psi_s$ and $A, B : X \to X$ be given mappings. Suppose there exists a function $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha(x, y)d(Ax, By) \leq \psi(M(x, y)), \tag{2.17}$$

for all $x, y \in X$, where $M(x, y)$ is defined by (2.2).

Also, Suppose that

(i) *(A, B) is an $\alpha-$admissible pair;*
(ii) *there exists $x_0 \in X$ such that $\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1$;*
(iii) *A and B are continuous on (X, d);*
(iv) *$\alpha(z, z) \geq 1$ for every z satisfying the conditions*

$$d(z, z) = 0,\ d(z, Az) \leq sd(Az, Az) \leq s^2 d(z, Az) \text{ and } d(z, Bz) \leq sd(Bz, Bz) \leq s^2 d(z, Bz); \tag{2.18}$$

(v) *$\psi(t) < \frac{t}{2s^2}$, for each $t > 0$.*

Then, A and B have a common fixed point.

Proof. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. Then, if (2.17) holds, we have

$$d(Ax, By) \leq \alpha(x, y)d(Ax, By) \leq \psi(M(x, y)).$$

Then, the proof is concluded by Theorem 2.2. □

Corollary 2.4. *Let (X, d) be a complete b-metric-like space, $\psi \in \Psi_s$ and $A, B : X \to X$ be continuous mappings satisfying*

$$d(Ax, By) \leq \psi(M(x, y)), \tag{2.19}$$

for all $x, y \in X$, where $M(x, y)$ is defined by (2.2).

If $\psi(t) < \frac{t}{2s^2}$ for each $t > 0$, then A and B have a common fixed point.

Proof. It suffices to take $\alpha(x, y) = 1$ in Corollary 2.3. □

Corollary 2.5. *Let (X, d) be a complete b-metric-like space and $A, B : X \to X$ be continuous mappings. Suppose there exists $k \in [0, \frac{1}{2s^2})$ such that*

$$d(Ax, By) \leq kM(x, y), \tag{2.20}$$

for all $x, y \in X$, where $M(x, y)$ is defined by (2.2). Then, A and B have a common fixed point.

Proof. It suffices to take $\psi(t) = kt$ for all $t \geq 0$ in Corollary 2.4. □

Corollary 2.6. *Let (X, d) be a complete b-metric-like space and $A, B : X \to X$ be continuous mappings. Suppose there exists $k \in [0, \frac{1}{2s^2})$ such that*

$$d(Ax, By) \leq kd(x, y), \tag{2.21}$$

for all $x, y \in X$. Then, A and B have a common fixed point.

In the setting of b-metric spaces, we have,

Corollary 2.7. *Let (X, d) be a complete b-metric space, $\psi \in \Psi_s$ and $A, B : X \to X$ be given mappings. Suppose there exists a function $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha(x, y)d(Ax, By) \leq \psi(M(x, y)), \tag{2.22}$$

for all $x, y \in X$, where $M(x, y)$ is defined by (2.2).

Also, Suppose that

(i) (A, B) is an $\alpha-$admissible pair;
(ii) there exists $x_0 \in X$ such that $\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1$;
(iii) A and B are continuous on (X, d).

Then, A and B have a common fixed point.

Proof. Following the proof of Theorem 2.2, we know that the sequence $\{x_n\}$ is Cauchy in (X, d) and converges to some $u \in X$. We show that u is a common fixed point of A and B. Using the continuity of A and B and Lemma 1.9, we obtain $Au = Bu = u$. □

In metric-like spaces (the case $s = 1$), we may state the following result.

Corollary 2.8. *Let (X, d) be a complete metric-like space, $\psi \in \Psi_1$ and $A, B : X \to X$ such that*

$$d(Ax, By) \leq \psi(\max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4}\}),$$

for all $x, y \in X$ satisfying $\alpha(x, y) \geq 1$.
Also, Suppose that

(i) (A, B) is an $\alpha-$admissible pair;
(ii) there exists $x_0 \in X$ such that $\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1$;
(iii) A and B are continuous on (X, d);
(iv) $\alpha(z, z) \geq 1$ for every z satisfying the conditions

$$d(z, z) = 0,\ d(z, Az) = d(Az, Az) \text{ and } d(z, Bz) = d(Bz, Bz); \tag{2.23}$$

(v) $\psi(t) < \frac{t}{2}$ for each $t > 0$.

Then, A and B have a common fixed point.

Theorem 2.2 remains true if we replace the continuity hypothesis by the following property:

(H) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ and $\alpha(x, x_{n(k)}) \geq 1$ for all k.

The statement is given as follows.

Theorem 2.9. *Let (X, d) be a complete b-metric-like space and $A, B : X \to X$ an (α, ψ)-contraction pair. Suppose that*

(i) (A, B) is an $\alpha-$admissible pair;
(ii) there exists $x_0 \in X$ such that $\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1$;
(iii) (H) holds;
(iv) $\psi(t) < \frac{t}{s}$ for each $t > 0$.

Then, A and B admit a common fixed point.

Proof. Following the proof of Theorem 2.2, we know that the sequence $\{x_n\}$ is Cauchy in (X, d) and converges to some $u \in X$. We show that u is a common fixed point of A and B.

Suppose on the contrary that $Au \neq u$ or $Bu \neq u$. Assume that $d(u, Au) > 0$.

By assumption *(iii)* (that is, $\alpha(u, x_{2n(k)-1}) \geq 1$), we have

$$d(Au, x_{2n(k)}) = d(Au, Bx_{2n(k)-1})) \leq \psi(M(u, x_{2n(k)-1})),$$

where

$$\begin{aligned} M(u, x_{2n(k)-1})) &= \max\{d(u, x_{2n(k)-1}), d(u, Au), d(x_{2n(k)-1}, x_{2n(k)-1}), \\ &\quad \frac{d(u, Bx_{2n(k)-1}) + d(x_{2n(k)-1}, Au)}{4s}\} \\ &= \max\{d(u, x_{2n(k)-1}), d(u, Au), d(x_{2n(k)-1}, x_{2n(k)}), \\ &\quad \frac{d(u, x_{2n(k)}) + d(x_{2n(k)-1}, Au)}{4s}\} \\ &\leq \max\{d(u, x_{2n(k)-1}), d(u, Au), d(x_{2n(k)-1}, x_{2n(k)}), \\ &\quad \frac{d(u, x_{2n(k)}) + sd(x_{2n(k)-1}, u) + sd(u, Au)}{4s}\}. \end{aligned}$$

We know that

$$\lim_{n\to\infty} d(u, x_{2n(k)-1}) = \lim_{n\to\infty} d(x_{2n(k)-1}, x_{2n(k)}) = \lim_{n\to\infty} d(u, x_{2n(k)}) = 0.$$

Then, there exists $N \in \mathbb{N}$ such that for all $k \geq N$,

$$M(u, x_{2n(k)-1})) \leq d(u, Au).$$

Then, by (ψ_1), we obtain for all $k \geq N$,

$$d(Au, x_{2n(k)}) \leq \psi(d(u, Au)). \tag{2.24}$$

On the other hand, we have

$$d(Au, u) \leq sd(Au, x_{2n(k)}) + sd(x_{2n(k)}, u), \quad \forall k \geq 0. \tag{2.25}$$

Combining (2.24) and (2.25), we get for all $k \geq N$,

$$d(Au, u) \leq s\psi(d(u, Au)) + sd(x_{2n(k)}, u). \tag{2.26}$$

Having in mind $\psi(t) < \frac{t}{s}$, so letting $k \to \infty$ in (2.26), we get

$$0 < d(u, Au) \leq s\psi(d(u, Au)) < d(u, Au),$$

which is a contradiction. Similarly, if $d(u, Bu) > 0$ we get a contradiction. Hence, $Au = u = Bu$ and so u is a common fixed point of A and B. □

Analogously, we can derive the following results.

Corollary 2.10. *Let (X, d) be a complete b-metric-like space, $\psi \in \Psi_s$ and $A, B : X \to X$ be given mappings. Suppose there exists a function $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha(x, y)d(Ax, By) \leq \psi(M(x, y)), \tag{2.27}$$

for all $x, y \in X$, where $M(x, y)$ is defined by (2.2).

Also, Suppose that

(i) (A, B) *is an* α*-admissible pair;*

(ii) $\exists x_0 \in X$ *such that* $\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1$*;*

(iii) (H) *holds;*

(iv) $\psi(t) < \frac{t}{s}$ *for each* $t > 0$.

Then, A and B have a common fixed point.

Corollary 2.11. *Let* (X, d) *be a complete b-metric-like space,* $\psi \in \Psi_s$ *and* $A, B : X \to X$ *be given mappings. Suppose that*

$$d(Ax, By) \leq \psi(M(x, y)), \tag{2.28}$$

for all $x, y \in X$*, where* $M(x, y)$ *is defined by* (2.2).

If $\psi(t) < \frac{t}{s}$ *for each* $t > 0$*, then A and B have a common fixed point.*

Corollary 2.12. *Let* (X, d) *be a complete b-metric-like space and* $A, B : X \to X$ *be given mappings. Suppose there exists* $k \in [0, \frac{1}{s})$ *such that*

$$d(Ax, By) \leq kM(x, y), \tag{2.29}$$

for all $x, y \in X$*, where* $M(x, y)$ *is defined by* (2.2)*. Then, A and B have a common fixed point.*

In the case $s = 1$, we have the two following corollaries.

Corollary 2.13. *Let* (X, d) *be a complete metric-like space,* $\psi \in \Psi_1$ *and* $A, B : X \to X$ *such that*

$$d(Ax, By) \leq \psi(\max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4}\}),$$

for all $x, y \in X$ *satisfying* $\alpha(x, y) \geq 1$.

Also, Suppose that

(i) (A, B) *is an* $\alpha-$*admissible pair;*

(ii) there exists $x_0 \in X$ *such that* $\min\{\alpha(x_0, Ax_0), \alpha(Ax_0, x_0)\} \geq 1$*;*

(iii) (H) *holds.*

Then, A and B have a common fixed point.

Corollary 2.14. *Let* (X, d) *be a complete metric-like space,* $\psi \in \Psi_1$ *and* $A, B : X \to X$ *such that*

$$d(Ax, By) \leq \psi(\max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4}\}),$$

for all $x, y \in X$*. Then, A and B have a common fixed point.*

We provide the following example.

Example 2.15. Take $X = [0, \infty)$ endowed with the complete b-metric-like $d(x, y) = (x^3 + y^3)^2$. Consider the mappings $A, B : X \to X$ given by

$$Ax = \begin{cases} \frac{x}{\sqrt[6]{3}} & \text{if } x \in [0, 1] \\ 2x - 2 & \text{if } \quad x > 1 \end{cases} \quad , \quad Bx = \begin{cases} \frac{x}{\sqrt[6]{3}} & \text{if } x \in [0, 1] \\ x & \text{if } x > 1. \end{cases}$$

Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{1}{3}t$. Note that (A, B) is an α-admissible pair. In fact, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By definition of α, this implies that $x, y \in [0, 1]$. Thus,

$$\alpha(Ax, By) = \alpha(\frac{x}{\sqrt[6]{3}}, \frac{y}{\sqrt[6]{3}}) = 1 \quad \text{and} \quad \alpha(By, Ax) = \alpha(\frac{y}{\sqrt[6]{3}}, \frac{x}{\sqrt[6]{3}}) = 1.$$

Then, (A, B) is an α-admissible pair.

Now, we show that (A, B) is an (α, ψ)-contraction. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. So, $x, y \in [0, 1]$. We have

$$\begin{aligned} d(Ax, By) &= ((Ax)^3 + (By)^3)^2 = ((\frac{x}{\sqrt[6]{3}})^3 + (\frac{y}{\sqrt[6]{3}})^3)^2 \\ &= (\frac{1}{\sqrt[6]{3}})^6 (x^3 + y^3)^2 = \psi(d(x, y)) \leq \psi(M(x, y)). \end{aligned}$$

Now, we show that (H) is verified. Let $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_{n+}) \geq 1$ for all n and $x_n \to u$ in (X, d). Then, $\{x_n\} \subset [0, 1]$ and $x_n \to u$ in $(X, |.|)$, where $|.|$ is the standard metric on X. Thus, $x_n, u \in [0, 1]$ and so $\alpha(x_n, u) = \alpha(u, x_n) = 1$ for all n. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \geq 1$ and $\alpha(Ax_0, x_0) \geq 1$. In fact, for $x_0 = 1$, we have $\alpha(1, A1) = \alpha(1, \frac{1}{\sqrt[6]{3}}) = 1$ and $\alpha(A1, 1) = \alpha(\frac{1}{\sqrt[6]{3}}, 1) = 1$.

Thus, all hypotheses of Theorem 2.9 are verified. Here, $\{0, 2\}$ is the set of common fixed points of A and B.

The mappings considered in above example have two common fixed points which are 0 and 2. Note that $\alpha(0, 2) = 0$, which is not greater than 1. So for the uniqueness, we need the following additional condition.

(U) For all $x, y \in CF(A, B)$, we have $\alpha(x, y) \geq 1$, where $CF(A, B)$ denotes the set of common fixed points of A and B.

Theorem 2.16. *Adding condition (U) to the hypotheses of* Theorem 2.2 *(resp.* Theorem 2.9*, with $\psi(t) < \frac{t}{2s}$ for all $t > 0$), we obtain that u is the unique common fixed point of A and B.*

Proof. In Theorem 2.2, mention that $\psi(t) < \frac{t}{2s^2}$ implies $\psi(t) < \frac{t}{2s}$. We argue by contradiction, that is, there exist $u, v \in X$ such that $u = Au = Bu$ and $v = Av = Bv$ with $u \neq v$. By assumption (U), we have $\alpha(u, v) \geq 1$. So by (2.1) and since $\psi(t) < \frac{t}{2s}$, we have

$$\begin{aligned} d(u, v) = d(Au, Bv) &\leq \psi(M(u, v))) \leq \psi(\max\{d(u, v), d(u, u), d(v, v), \frac{d(u, v)}{2s}\}) \\ &= \psi(\max\{d(u, v), d(u, u), d(v, v)\}) \\ &\leq \psi(\max\{d(u, v), 2sd(u, v)\}) = \psi(2sd(u, v)) < d(u, v), \end{aligned}$$

which is a contradiction. Hence, $u = v$. □

Corollary 2.17. *Let (X, d) be a complete b-metric-like space, $\psi \in \Psi_s$ and $A, B : X \to X$ be given mappings. Suppose that*

$$d(Ax, By) \leq \psi(M(x, y)), \tag{2.30}$$

for all $x, y \in X$, where $M(x, y)$ is defined by (2.2)*. If $\psi(t) < \frac{t}{2s}$ for all $t > 0$, then A and B have a unique common fixed point.*

Proof. It suffices to take $\alpha(x, y) = 1$ in Corollary 2.11. The uniqueness of u follows from Theorem 2.16. □

Corollary 2.18. *Let (X,d) be a complete b-metric-like space and $A,B : X \to X$ be given mappings. Suppose there exists $k \in [0, \frac{1}{2s})$ such that*

$$d(Ax,By) \le kM(x,y), \tag{2.31}$$

for all $x,y \in X$, where $M(x,y)$ is defined by (2.2). Then, A and B have a unique common fixed point.

Proof. It suffices to take $\psi(t) = kt$ in Corollary 2.17. The uniqueness of u follows from Theorem 2.16. □

The following example illustrates Theorem 2.2 where A and B have a unique common fixed point.

Example 2.19. Take $X = [0, \frac{3}{2}]$ endowed with the complete b-metric-like $d(x,y) = x^2 + y^2 + (x-y)^2$ with $s = 2$. Consider the mappings $A, B : X \to X$ given by

$$Ax = \begin{cases} \ln(1+\frac{x}{3}) & \text{if } x \in [0,1] \\ x - 1 + \ln\frac{4}{3} & \text{if } x \in (1, \frac{3}{2}] \end{cases} \quad , \quad Bx = \begin{cases} \ln(1+\frac{x}{3}) & \text{if } x \in [0,1] \\ x + \ln(1+\frac{x}{3}) - 1 & \text{if } x \in (1, \frac{3}{2}]. \end{cases}$$

Define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{1}{9}t$. It is obvious that

(i) (A,B) is an α-admissible pair;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Ax_0) \ge 1$ and $\alpha(Ax_0, x_0) \ge 1$;

(iii) A and B are continuous on (X,d);

(iv) $\psi(t) < \frac{t}{2s^2}$.

Now, we shall show that (A,B) is an (α,ψ)-contraction. Let $x,y \in X$ such that $\alpha(x,y) \ge 1$. So, $x,y \in [0,1]$.

We have

$$\begin{aligned} d(Ax,By) &= (Ax)^2 + (By)^2 + (Ax - By)^2 \\ &= (\ln(1+\frac{x}{3}))^2 + (\ln(1+\frac{y}{3}))^2 + (\ln(1+\frac{x}{3}) - \ln(1+\frac{y}{3}))^2 \\ &\le (\frac{x}{3})^2 + (\frac{y}{3})^2 + \frac{1}{9}(x-y)^2 = \frac{1}{9}[x^2+y^2+(x-y)^2] = \frac{1}{9}d(x,y) \le \psi(M(x,y)). \end{aligned}$$

Thus, all hypotheses of Theorem 2.2 are verified. Here, 0 is the unique common fixed points of A and B.

3. Fixed Point Theorems for generalized cyclic contractions

In 2003, Kirk *et al.* [22] introduced the concepts of cyclic mappings and cyclic contractions. For papers dealing with cyclic contractions, see [7, 10, 25]. We recall some definitions from [22].

Definition 3.1 ([22])**.** Let F and G be nonempty subsets of a space X. A mapping $T : F \cup G \to F \cup G$ is called cyclic if $T(F) \subset G$ and $T(G) \subset F$.

Definition 3.2 ([22])**.** Let F and G be nonempty subsets of a metric space (X,d). A mapping $T : F \cup G \to F \cup G$ is called a cyclic contraction if there exists $k \in [0,1)$ such that

$$d(Tx,Ty) \le kd(x,y), \tag{3.1}$$

for all $x \in F$ and $y \in G$.

Now, we introduce the concept of new generalized cyclic contractive pairs in the setting of b-metric-like spaces.

Definition 3.3. Let F and G be nonempty closed subsets of a b-metric-like space (X,d), $\alpha : X \times X \to [0,\infty)$, $\psi \in \Psi_s$ and $A,B : X \to X$ be mappings. The pair (A,B) is called a cyclic (α,ψ,F,G)-contraction pair if

(i) $F \cup G$ has a cyclic representation w.r.t. the pair (A,B), that is, $A(F) \subset G$ and $B(G) \subset F$;

(ii)
$$d(Ax,By) \le \psi(M(x,y)), \tag{3.2}$$
for all $x \in F$ and $y \in G$ satisfying $\alpha(x,y) \ge 1$ or $\alpha(y,x) \ge 1$, where
$$M(x,y) = \max\{d(x,y), d(x,Ax), d(y,By), \frac{d(x,By)+d(y,Ax)}{4s}\}.$$

Now, we state and prove the following results.

Theorem 3.4. *Let (X,d) be a complete b-metric-like space and F and G be nonempty closed subsets of X. Suppose that $A,B : X \to X$ is a cyclic (α,ψ,F,G)-contraction pair and the following conditions hold:*

(i) *$\alpha(Ax,BAx) \ge 1$ for all $x \in F$ and $\alpha(Bx,ABx) \ge 1$ for all $x \in G$;*

(ii) *A or B is continuous on (X,d);*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \ge 0$ and $x_n \to z$ as $n \to \infty$, then $\alpha(z,z) \ge 1$;*

(iv) *$\psi(t) < \frac{t}{2s^3+s}$ for each $t > 0$.*

Then, A and B have a common fixed point in $F \cap G$.

Proof. Let $x_0 \in F$ and $x_1 = Ax_0$. Since $A(F) \subset G$, then $x_1 \in G$. Also, let $x_2 = Bx_1 = BAx_0$. Since $B(G) \subset F$, then $x_2 \in F$. Continuing in this fashion, we can construct a sequence $\{x_n\}$ in X such that
$$x_{2n+2} = Bx_{2n+1} \in F, \quad x_{2n+1} = Ax_{2n} \in G, \quad \forall n \ge 0.$$

By condition (i), we have $\alpha(x_1,x_2) = \alpha(Ax_0,BAx_0) \ge 1$ and $\alpha(x_2,x_3) = \alpha(Bx_1,ABx_1) \ge 1$. Continuing this process, we get
$$\alpha(x_n,x_{n+1}) \ge 1, \quad \forall n \ge 0.$$

Following the proof of Theorem 2.2, we know that the sequence $\{x_n\}$ is Cauchy in (X,d) and converges to some $u \in X$ with $d(u,u) = 0$. We shall show that u is a common fixed point of A and B in $F \cap G$. Since $\{x_{2n}\}$ is a sequence in the closed set F and $\{x_{2n}\}$ converges to u, then $u \in F$. Also, $\{x_{2n+1}\}$ is a sequence in the closed set G and $\{x_{2n+1}\}$ converges to u, then $u \in G$. We deduce that $u \in F \cap G$.

First, assume that A is continuous on (X,d). Since $\{x_{2n}\}$ converges to u, so $\{x_{2n+1} = Ax_{2n}\}$ converges to Au.

On the other hand, $\lim_{n\to\infty} d(x_n,u) = 0 = d(u,u)$ and by Lemma 1.9, we have
$$\frac{1}{s}d(u,Au) \le d(Au,Au) \le sd(u,Au).$$

If $d(Au,Bu) = 0$, then $Au = Bu$. Moreover, the fact that $d(u,Au) \le sd(Au,Au)$ implies
$$0 \le d(u,Au) \le sd(Au,Au) \le 2s^2 d(Au,Bu) = 0,$$
and so $Au = u$. Then, $Bu = Au = u$ and so u is a common fixed point of A and B.

Suppose by contradiction that $d(Au,Bu) > 0$. Since $u \in F \cap G$ and by (iii), it follows that $\alpha(u,u) \ge 1$, so that
$$d(Au,Bu) \le \psi(M(u,u)),$$

where

$$\begin{aligned}
M(u,u) &= \max\{d(u,u), d(u,Au), d(u,Bu), \frac{d(u,Bu)+d(u,Au)}{4s})\}\\
&= \max\{0, d(u,Au), d(u,Bu), \frac{d(u,Bu)+d(u,Au)}{4s})\}\\
&= \max\{d(u,Au), d(u,Bu)\} \le \max\{d(u,Au), sd(u,Au)+sd(Au,Bu)\}\\
&= sd(u,Au)+sd(Au,Bu) \le 2s^3 d(Au,Bu) + sd(Au,Bu) = (2s^3+s)d(Au,Bu).
\end{aligned}$$

Then

$$d(Au,Bu) \le \psi((2s^3+s)d(Au,Bu)) < d(Au,Bu),$$

which is a contradiction.

The proof is similar when B is assumed to be continuous on (X,d).

□

Theorem 3.5. *Let (X,d) be a complete b-metric-like space and F and G be nonempty closed subsets of X. Suppose that $A,B : X \to X$ is a cyclic (α,ψ,F,G)-contraction pair and the following conditions hold:*

(i) $\alpha(Ax, BAx) \ge 1$ *for all* $x \in F$ *and* $\alpha(Bx, ABx) \ge 1$ *for all* $x \in G$;

(ii) A *and* B *are continuous on* (X,d);

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \ge 1$ *for all* $n \ge 0$ *and* $x_n \to z$ *as* $n \to \infty$, *then* $\alpha(z,z) \ge 1$;

(iv) $\psi(t) < \frac{t}{2s^2}$ *for each* $t > 0$.

Then, A and B have a common fixed point in $F \cap G$.

Proof. The proof is similar to the proofs of Theorem 3.4 and Theorem 2.2.

□

Theorem 3.4 and Theorem 3.5 can be proved without assuming the continuity of A or the continuity of B. For this instance, we suppose that X has the following property:

(R) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \ge 1$ for all k.

This statement is given as follows.

Theorem 3.6. *Let (X,d) be a complete b-metric-like space and F and G be nonempty closed subsets of X. Suppose that $A,B : X \to X$ is a cyclic (α,ψ,F,G)-contraction pair and the following conditions hold:*

(i) $\alpha(Ax, BAx) \ge 1$ *for all* $x \in F$ *and* $\alpha(Bx, ABx) \ge 1$ *for all* $x \in G$;

(ii) (R) *holds;*

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \ge 1$ *for all* $n \ge 0$ *and* $x_n \to z$ *as* $n \to \infty$, *then* $\alpha(z,z) \ge 1$;

(iv) $\psi(t) < \frac{t}{s}$ *for each* $t > 0$.

Then, A and B have a common fixed point in $F \cap G$.

Proof. The proof is similar to that of Theorem 3.4 and Theorem 2.9. □

Taking $A = B$ in Theorem 3.5 and Theorem 3.6, we state the followings results.

Corollary 3.7. *Let (X, d) be a complete b-metric-like space and F and G be nonempty closed subsets of X. Suppose that $\psi \in \Psi_s$, $\alpha : X \times X \to X$ and $A : X \to X$ such that*

$$d(Ax, Ay) \leq \psi(\max\{d(x, y), d(x, Ax), d(y, Ay), \frac{d(Ax, y) + d(x, Ay)}{4s}\})$$

for all $x \in F$ and $y \in G$ satisfying $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$.

Also, suppose the following conditions hold:

- *(i) $\alpha(Ax, AAx) \geq 1$ for all $x \in F \cap G$;*
- *(ii) A is a cyclic mapping;*
- *(ii) A is continuous on (X, d);*
- *(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to z$ as $n \to \infty$, then $\alpha(z, z) \geq 1$;*
- *(iv) $\psi(t) < \frac{t}{2s^2}$ for each $t > 0$.*

Then, A has a fixed point in $F \cap G$.

Corollary 3.8. *Let (X, d) be a complete b-metric-like space and F and G be nonempty closed subsets of X. Suppose that $\psi \in \Psi_s$, $\alpha : X \times X \to X$ and $A : X \to X$ a mapping such that*

$$d(Ax, Ay) \leq \psi(\max\{d(x, y), d(x, Ax), d(y, Ay), \frac{d(Ax, y) + d(x, Ay)}{4s}\}),$$

for all $x \in F$ and $y \in G$ satisfying $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$.

Also, suppose the following conditions hold:

- *(i) $\alpha(Ax, AAx) \geq 1$ for all $x \in F \cap G$;*
- *(ii) A is a cyclic mapping;*
- *(ii) (R) holds;*
- *(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to z$ as $n \to \infty$, then $\alpha(z, z) \geq 1$;*
- *(iv) $\psi(t) < \frac{t}{s}$ for each $t > 0$.*

Then, A has a fixed point in $F \cap G$.

Now, we give an example to illustrate Theorem 3.6.

Example 3.9. Let $X = \{0, 1, 2\}$ and $d : X \times X \to [0, \infty)$ defined by

$$d(0,0) = 9,\ d(1,1) = 0,\ d(2,2) = 0,\ d(0,1) = d(1,0) = 16,$$
$$d(0,2) = d(2,0) = 9 \ \text{ and } \ d(1,2) = d(2,1) = 49.$$

Then, (X, d) is a complete $b-$metric-like space with coefficient $s = 2$. Let $F = \{0, 1\}$ and $G = \{1, 2\}$. Note that F and G are nonempty closed subsets of X. Consider the mappings $A, B : X \to X$ and $\alpha : X \times X \to X$ as follows:

$$A0 = 2,\ A1 = 1,\ A2 = 0, \quad B0 = 0,\ B1 = 1 \ \text{ and } \ B2 = 1$$

and

$$\begin{cases} \alpha(1,1) = \alpha(2,1) = 1; \\ \alpha(x,y) = 0 \ \text{ otherwise.} \end{cases}$$

Now, we show that all the conditions of Theorem 3.6 are satisfied.

We show that condition (i) of Theorem 3.6 is verified. Let $x \in F$, then

$$\alpha(Ax, BAx) = \begin{cases} \alpha(2,1) = 1 \text{ if } x = 0; \\ \alpha(1,1) = 1 \text{ if } x = 1. \end{cases}$$

Also, let $x \in G$, then

$$\alpha(Bx, ABx) = \begin{cases} \alpha(1,1) = 1 \text{ if } x = 1; \\ \alpha(1,1) = 1 \text{ if } x = 2. \end{cases}$$

Then, $\alpha(Ax, BAx) \geq 1$ for all $x \in F$ and $\alpha(Bx, ABx) \geq 1$ for all $x \in G$.

It is clear that $A(F) \subset G$ and $B(G) \subset F$.

Now, we sow that (A, B) is a cyclic (α, ψ, F, G)-contraction pair.

Let $x \in F$ and $y \in G$ such that $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$. It follows from definition of α that $(x = y = 1)$ or $(x = 1, y = 2)$. We have for $(x = y = 1)$ or $(x = 1, y = 2)$

$$d(Ax, By) = d(1, 1) = 0 \leq \psi(M(x, y)),$$

for all $\psi \in \Psi_s$ such that $\psi(t) < \frac{t}{s}$ for all $t > 0$. Then, (A, B) is a cyclic (α, ψ, F, G)-contraction pair.

It is easy to show that X satisfies the property (R). Moreover, condition (iii) of Theorem 3.6 holds. Hence, all conditions of Theorem 3.6 are verified. Here, 1 is the unique common fixed point of A and B.

4. Fixed Point Theorems for generalized contractions in partially ordered *b*-metric-like spaces

Now, we give some fixed points results on partially ordered b-metric-like spaces as consequences of our results presented in the last section.

Definition 4.1. Let X be a nonempty set. We say that $(X, d, \preceq)$ is a partially ordered b-metric-like space if (X, d) is a b-metric-like space and $(X, \preceq)$ is a partially ordered set.

Definition 4.2. Let F and G be nonempty closed subsets of a partially ordered b-metric-like space $(X, d, \preceq)$, $\psi \in \Psi_s$ and $A, B : X \to X$ be mappings. The pair (A, B) is called a cyclic (ψ, F, G)-contraction pair if

(i) $F \cup G$ has a cyclic representation w.r.t. the pair (A, B);

(ii)
$$d(Ax, By) \leq \psi(M(x, y)), \tag{4.1}$$
for all $x \in F$ and $y \in G$ satisfying $x \preceq y$ or $y \preceq x$, where
$$M(x, y) = \max\{d(x, y), d(x, Ax), d(y, By), \frac{d(x, By) + d(y, Ax)}{4s}\}.$$

Definition 4.3. Let $(X, d, \preceq)$ a partially ordered b-metric-like space and F, G be nonempty closed subsets of X with $X = F \cup G$. Let $A, B : X \to X$ be mappings. We say that the pair (A, B) is (F, G)-weakly increasing if $Ax \preceq BAx$ for all $x \in F$ and $Bx \preceq ABx$ for all $x \in G$.

Now, we state and prove the following results.

Theorem 4.4. *$(X, d, \preceq)$ be a complete partially ordered b-metric-like space and F, G be nonempty closed subsets of X. Suppose that $A, B : X \to X$ is a cyclic (ψ, F, G)-contraction pair and the following conditions hold:*

(i) *(A, B) is (F, G)-weakly increasing;*

(ii) *A or B is continuous on (X, d);*

(iii) *$\psi(t) < \frac{t}{2s^3+s}$ for each $t > 0$.*

Then, A and B have a common fixed point in $F \cap G$.

Proof. Let the function $\alpha : X \times X \to X$ such that

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x \preceq y; \\ 0 \text{ otherwise.} \end{cases}$$

Then, all hypotheses of Theorem 3.4 are satisfied and hence A and B have a common fixed point in $F \cap G$. □

Also, by using the same technique, we have the following results.

Theorem 4.5. *$(X, d, \preceq)$ be a complete partially ordered b-metric-like space and F, G be nonempty closed subsets of X. Suppose that $A, B : X \to X$ is a cyclic (ψ, F, G)-contraction pair and the following conditions hold:*

(i) (A, B) is (F, G)-weakly increasing;
(ii) A and B are continuous on (X, d);
(iii) $\psi(t) < \frac{t}{2s^2}$ for each $t > 0$.

Then, A and B have a common fixed point in $F \cap G$.

Theorem 4.6. *$(X, d, \preceq)$ be a complete partially ordered b-metric-like space and F, G be nonempty closed subsets of X. Suppose that $A, B : X \to X$ is a cyclic (ψ, F, G)-contraction pair and the following conditions hold:*

(i) (A, B) is (F, G)-weakly increasing;
(ii) for a sequence $\{x_n\} \subset X$ with $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to z$ in (X, d), then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq z$, for all $k \in \mathbb{N}$;
(iv) $\psi(t) < \frac{t}{s}$ for each $t > 0$.

Then, A and B have a common fixed point in $F \cap G$.

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